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This article contains a discussion of some of the motivations for doing quantum physics in arithmetic contexts such as $p$-adic and adelic spaces, a brief survey of some of the results obtained so far, and some indications of directions of future research, including some open problems and speculations at the end.

1. INTRODUCTION

The remarks that I wish to make on the occasion of the First International Conference on $p$-adic mathematical physics (Moscow, 2003) are general and do not contain much that is new. However, on such an occasion as this, it is not only necessary to talk about new discoveries but also to review the origins and achievements of the subject so far. This brief article is of the latter genre. I have discussed these matters elsewhere [1] with detailed references.

2. SPACE–TIME AT SMALL DISTANCES AND THE POINT OF VIEW OF RIEMANN AND HIS SUCCESSORS

Riemann (1854). The discovery that geometry and physics are related at a fundamental level is due to Riemann. In his celebrated Inaugural lecture [2] delivered in Göttingen in 1854, he discussed the so-called problem of space, namely, the question of determining the nature of the geometry of physical space. In this lecture, Riemann outlined his theory of Riemannian geometry, which naturally led to the point of view that space should be viewed as a manifold with a Riemannian structure on it. But in a radical departure from his predecessors, he took the view that this Riemannian structure was determined not a priori but by the phenomena themselves. Moreover, and this is one of the most interesting aspects of his lecture, he emphasized that his hypothesis, namely, that space is a Riemannian manifold, should not be extrapolated to very small distances.

Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Riemann’s ideas were forgotten for a long time except for some resonances in the writings of Clifford [3] till Einstein resurrected the Riemannian view. Einstein developed his ideas independently of Riemann and also introduced a gigantic conceptual change, namely, that not space but space–time is the fundamental object, and that space–time is not a Riemannian manifold but a pseudo-Riemannian manifold of signature $(1,3)$. Moreover, and it is here that his point of view coincided with that of Riemann, Einstein definitely postulated that the metric of space–time was in fact a dynamic object, determined by phenomena.
The ideas of Riemann and Einstein were examined and combined in a great synthesis by Weyl. In his great book *Space, Time, and Matter* [4], Weyl discusses the first thesis of Riemann, namely, that the geometry of space is determined by phenomena, as follows.

**Weyl (1930).** *If we disregard the first possibility, “that the reality which underlies space forms a discrete manifold”—although we do not by this in any way mean to deny finally, particularly nowadays in view of the results of the quantum theory, that the ultimate solution of the problem of space may after all be found in just this possibility—we see that Riemann rejects the opinion that had prevailed up to his own time, namely, that the metrical structure of space is fixed and inherently independent of the physical phenomena for which it serves as a background....*

In spite of the achievements of Riemann, Einstein, Weyl, and their successors, the second thesis of Riemann, namely, that, at small distances, space, or more canonically, space–time, need not be a manifold, lay unexplored until the middle of the second half of the twentieth century when the physicists, guided by experiments involving elementary particles in extremely small regions of space and time, began to construct models to explain what happens at very small distances. After the explosive developments in high-energy physics that led to the discovery of the standard model, it was natural to ask whether a single quantum theory could be developed that included gravitation also. It was realized that, in a quantum theory of gravitation, the structure of space–time must be allowed to have quantum fluctuations. Indeed, measurements in small regions meant an infusion of great amounts of energy in very small space–time regions, which led to uncertainty relations even for the measurement of the position of the space–time point. These could be written as

\[ \Delta x \geq \ell_{\text{Pl}}, \]

where

\[ \ell_{\text{Pl}} = \left( \frac{\hbar G}{c^3} \right)^{1/2} \approx 10^{-33} \text{ cm} \]

is the so-called *Planck length*. Thus, at distances below the Planck length, there was no possibility of measurement of any physical quantity so that conventional models of space–time would no longer be adequate.

One of the radical ideas which led to new progress came from Volovich. In a famous paper [5] in 1987, he postulated that space–time geometry at the Planck scale may be *non-Archimedean*. More precisely:

**Volovich (1987).** *Under rather general assumptions there are only two new possibilities, namely, to consider, instead of real numbers, the p-adic numbers or to consider the theory on a finite Galois field.*

The first possibility is that the geometry of space–time is non-Archimedean at the Planck length. This is a natural suggestion because no measurements are possible at such distances according to the principles of General Relativity.

Let us recall what is meant by the *Archimedean* property of a geometry. As is well known, any geometry (at least in dimensions \( \geq 3 \)) can be coordinatized with the coordinates coming from a field, and so it is a question of defining the Archimedean property of a field. A field \( K \) is said to be *Archimedean* if it is linearly ordered and has the following property: If \( 0 < a, b \in K \) are distinct, there is an integer \( n > 0 \) such that \( n \cdot a > b \). In other words, one can use any magnitude, no matter how small, to measure any other magnitude, no matter how large. The idea of Volovich is now very clear: if no measurements are possible beyond the Planck length, clearly there is a violation of the Archimedean hypothesis.
In the course of his great work on the foundations of geometry [6], Hilbert first constructed non-Archimedean fields which were certain fields of real algebraic functions of a real variable $t$. Later on, the number theorists, starting from Hensel, discovered simpler examples which began to play a fundamental role in arithmetic. These are the so-called **local fields**, namely, $K = k_p$, the completion of the algebraic number field $k$ with respect to the valuation defined by a prime ideal $p$; for $k = \mathbb{Q}$, $p$ corresponds to a rational prime $p$ and $k_p$ becomes $\mathbb{Q}_p$, the field of $p$-adic numbers. For a prime characteristic $p$, $K = F((t))$, the field of Laurent series (with poles of finite order) over a finite field $F$ of characteristic $p$, should be added to this list. Although these fields are very special, they are locally compact and are closest to $\mathbb{R}$ and $\mathbb{C}$.

Volovich's suggestion clearly implied that there is, at a fundamental level, a deep arithmetic side to quantum physics. His new approach singled out two possibilities: quantum theory over $\mathbb{Q}_p$ or a finite extension of it, or a finite field. In either case, there is a choice of a prime which does not seem natural. Volovich himself made the suggestion that there should be no dependence on any prime (invariance of fundamental laws on the number field). But the principle was most forcefully formulated by Manin [7]:

**Manin (1989).** *The world is neither real nor $p$-adic; it is adelic. For some reasons, reflecting the physical nature of our kind of living matter (e.g., the fact that we are built of massive particles), we tend to project this adelic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate the most important things arithmetically. The relation between the “real” and the “arithmetical” pictures of the world is that of complementarity, like the relation between conjugate observables in quantum mechanics.*

In other words, there is only one reality, and that is adelic. The adelic hypothesis is that this reality has two faces: transcendental and arithmetic. It is up to the observer to choose the particular face to make the calculations. Each face by itself is an incomplete description of the reality.

Thus, the Manin point of view does not require the abandoning of the real field or the choice of a prime, or some ad hoc way to recover the conventional physics from the $p$-adic ones. Rather, it places all primes, including the infinite ones, on the same footing, thus reasserting in this context the famous Lefschetz principle of Harish-Chandra formulated in the theory of representations of algebraic groups defined over number fields [8].

To directly formulate quantum field theory in the adelic framework is a daunting task. Perhaps, one should do this after gaining experience with studies of physically motivated mathematical structures in $p$-adic and adelic contexts first. I shall give some examples of such structures a little further on. The basic point is that only *certain dynamical systems over $\mathbb{R}$* have meaning arithmetically. To discover these and perhaps encode them in some sort of *product formulae* is the fundamental problem in arithmetic physics.

Of course, there are other views about the nature of space–time geometry at very small distances. Here is a sampling of these.

**Supersymmetry (1970).** Space–time is a supermanifold whose odd part is spinorial. More precisely, space–time is a 2-step nilpotent super Lie group with the property that the odd part of its super Lie algebra carries a spinorial representation of the Poincaré group.

**Doplicher et al.** Even for position coordinates there are commutation rules.

**Noncommutative geometry (Connes, 1990).** The coordinate ring of space–time is noncommutative. Indeed, the phenomenology of the standard model imposes on space–time the structure of a very specific noncommutative geometry.

The early papers in which unconventional rings and fields were introduced as a framework for quantum theory go back to Weyl (1930) where quantum systems over finite rings, for instance,
the ring $\mathbb{Z}/N\mathbb{Z}$ of integers mod $N$, were considered as approximations, for large $N$, to actual quantum systems. Weyl’s theme was taken up by Schwinger in the late 1950s, and continued by Digernes et al. in 1994, and by Albeverio et al. in 1998. But the actual consideration of $p$-adic quantum theory appears to have been first due to Ulam (1966), Beltrametti (1971), Beltrametti and Cassinelli (1972); Nambu (1987) considered systems over finite fields. A phase transition took place in 1987 with the seminal paper of Volovich after which there has been a flood of contributions, due to Aref’eva, Dragovich, Freund, Grossman, Khrennikov, Kochubei, Vladimirov, Witten, Zelenov, and many others (see [1] for detailed references).

3. EXAMPLES

We begin with some examples of product formulae.

**Product formula for volumes.** $G = \text{SL}(N)$, $G_\infty = \text{SL}(N, \mathbb{R})$, $G_p = \text{SL}(N, \mathbb{Q}_p)$, $\Gamma_\infty = \text{SL}(N, \mathbb{Z})$, $\Gamma_p = \text{SL}(N, \mathbb{Z}_p)$. Then,

$$\text{vol}(G_\infty/\Gamma_\infty) \prod_p \text{vol}(G_p/\Gamma_p) = 1,$$

which leads to

$$\text{vol}(G_\infty/\Gamma_\infty) = \zeta(2) \ldots \zeta(N).$$

Here, $\zeta$ is the Riemann zeta function. For $N = 2$, this yields Euler’s classic formula

$$\frac{\pi^2}{6} = \prod_p \left(1 - p^{-2}\right)^{-1}.$$

**Product formula for quaternion algebras.** For any field $k$ of characteristic $\neq 2$ and elements $a$ and $b$ both nonzero in $k$, define

$$(a, b)_k$$

to be 1 or $-1$ according as the algebra generated over $k$ by $i, j$ with

$$i^2 = a, \quad j^2 = b, \quad ij = -ji$$

is isomorphic to the $2 \times 2$ matrix algebra $M_2(k)$ or is a division algebra. If $k = \mathbb{Q}$, $a, b \in \mathbb{Q} \setminus \{0\}$,

$$(a, b)_\mathbb{R} = \prod_p (a, b)_{\mathbb{Q}_p}.$$

This is one of the many ways in which Gauss’s famous quadratic reciprocity law can be reformulated and goes back to Hilbert and Hasse.

**Product formula for Veneziano string amplitudes.** The Veneziano amplitude was generalized to the $p$-adic and adelic context and led to a remarkable product formula for the adelic amplitude. The $p$-adic amplitudes involve the $p$-adic beta function $B_p$

$$B_p(\alpha, \beta) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1} |1 - x|_p^{\beta-1} dx$$

and

$$B_\infty(\alpha, \beta)^{-1} = \prod_p B_p(\alpha, \beta).$$

The infinite product has to be interpreted carefully for this to make sense and $\alpha$ and $\beta$ restricted suitably (Volovich, Freund–Witten, Vladimirov).
Weyl systems. One of the conventional quantum systems which has been investigated profoundly in the $p$-adic and adelic contexts is the Weyl system and the naturally associated quantum systems with quadratic Lagrangians. I begin with Weyl systems.

Given a pair $A, B$ of abelian groups in duality with a pairing $(\cdot, \cdot)$ taking values in the circle group of complex numbers of absolute value 1, a Weyl system is a pair of unitary representations $U, V$, of $A, B$, respectively, related by

$$U(a)V(b) = (a, b)^{-1}V(b)U(a), \quad (a, b) \in A \times B.$$ 

If $A = B = \mathbb{R}^n$, this is the finite version of the Heisenberg commutation rules. It goes back to Weyl in 1930 and is often called the Weyl commutation rules. One may replace $A \times B$ by an abelian group $G$ with symplectic structure. In this form, a Weyl system appears as a quantization of the classical system on the phase space $G$.

The case when $G$ is an arbitrary locally compact group was investigated by Mackey, Segal, Shale, Weil, Digernes–Varadarajan, and many others. Weil’s work (1964–1965) is the definitive treatment of what from our point of view may be called the quantum theory of quadratic systems. Special cases were studied much later in the physical literature, as quantum harmonic oscillators, by Dragovic, and many others.

The case when $G$ is not locally compact arises in the study of the canonical commutation rules in quantum field theory. The first rigorous treatment goes back to Shale in the real case. The $p$-adic case where $G$ is an infinite dimensional symplectic vector space over a local field has been examined by Zelenov and other members of the Russian school [9–15].

The metaplectic representation. The central fact in the theory of Weyl systems is its uniqueness. In particular, if we change a given Weyl system by an automorphism of $G$ that preserves the symplectic structure on $G$, we obtain a new Weyl system that must be isomorphic to the original one. Hence, we obtain a unitary operator that implements this isomorphism, determined uniquely up to a phase factor. Let $\text{Sp}(G)$ be the symplectic group of $G$, namely, the group of automorphisms of $G$ preserving the symplectic structure. The above remark shows that we have obtained in a natural manner a projective unitary representation of $\text{Sp}(G)$ or a unitary representation of a central extension of $\text{Sp}(G)$. This is the metaplectic representation first introduced by Segal and Shale over $\mathbb{R}$, and by Weil over $p$-adics and adeles. If $A$ is a finite dimensional vector space over a local field or the adelization of a finite dimensional vector space over a global field of characteristic different from 2, and $G = A \oplus \mathbb{R}^+$, the metaplectic representation can be chosen to be a unitary representation of the two-fold cover of the symplectic group (but not the symplectic group itself). This two-fold cover is called the metaplectic group, and the corresponding representation, the metaplectic representation.

Harmonic oscillators. The Weil theory of metaplectic representation leads naturally to the theory of harmonic oscillators in the $p$-adic and adelic contexts. Let $K$ be a local field; $T$ in $\text{SL}(2, K)$ is a nonsplit torus and $T'$ is its lift in the metaplectic group. If $K = \mathbb{R}$, the group $T$ is the dynamical group of the classical harmonic oscillator. In the case of arbitrary $K$, the group $T$ may still be interpreted as the dynamical group of the classical harmonic oscillator over $K$. Let us now consider the metaplectic representation restricted to $T'$. If $K = \mathbb{R}$, this restriction can be verified to be the dynamical group of the quantum harmonic oscillator. So, one can think of the restriction in the case of arbitrary $K$ as the quantum harmonic oscillator over $K$, and its decomposition as the spectrum of this oscillator. This point of view allows a unified view of many papers in the physical literature on this theme and suggests many generalizations, such as coupled oscillators in higher dimensions, generalizations of the Schwinger theory of hydrogenic atoms, etc.\footnote{For the spectrum in the general case, see [16]. The calculations by physicists treated only special cases. Rogawski, whose work was independent of the physicists', calculates the spectrum in complete generality.}
Locally compact configuration spaces: finite systems and their continuum limits, spectral theory. Although this subject may be viewed as just quantum mechanics in the conventional sense, the fact that the configuration space could be non-Archimedean gives it a special flavor. For instance, the theory of pseudodifferential operators replaces the usual theory of differential operators; and the spectral theory, as well as the theory of finite quantum systems and their continuum limits, can be done in great depth. For the real case, this was done by Digernes, Varadarajan, and Varadhan. Albeverio et al. have done this over general locally compact abelian groups.

For Hamiltonians of the form

\[ H = -\Delta + V, \]

where \( V \) is a potential and \( \Delta \) is the pseudodifferential operator with symbol \(-|\xi|^\alpha\), defined on finite dimensional vector spaces over a local field, a path integral representation for the propagator has been obtained by Varadarajan, at least when \( V \) is bounded below and \( H \) is essentially self-adjoint when restricted to the space of Schwartz–Bruhat functions. The imaginary time propagator

\[ e^{-tH}, \quad t > 0, \]

has a path integral representation with Wiener measure replaced by a measure on suitable spaces of discontinuous functions defined by a stochastic process with independent increments. A consequence of this representation is that, if a ground state exists, it is unique.\(^3\)

Quantum field theory. Recently, Kochubei and Mustafa Sait-Ametov [21] have constructed interaction measures in the space of \( p \)-adic distributions on \( \mathbb{Q}_p^n \), \( n \leq 4 \), with a cut off. This is a deep generalization of the Simon theory of Euclidean quantum fields and appears to suggest a very fertile ground of further research.

4. OPEN PROBLEMS AND SPECULATIONS

1. Scattering theory. The work of Harish-Chandra on the harmonic analysis of the continuous spectrum of semisimple Lie groups has revealed a deep analogy with scattering theory. This analogy needs to be explored further in the \( p \)-adic and adelic contexts.\(^4\)

2. Quantum fields with interaction. I have already mentioned the recent work of Kochubei and Mustafa Sait-Ametov. It is necessary to extend it further, in the many obvious directions. Perhaps, such generalizations will provide links to the representation theory of \( p \)-adic groups and, eventually, to the Langlands program. The program to generalize the structures coming out of quantum field theory to arithmetically defined ground fields and rings thus appears to be interesting and promising. One should perhaps search for general product formulae that describe some of the main formulae of quantum field theory and strings in terms of arithmetic objects.

3. Supersymmetry. Can we make sense of supersymmetry in arithmetic context?

4. \( p \)-Adic vs \( \ell \)-adic. Throughout this paper, I have restricted myself to quantum theory where the wave functions take complex values. However, there has been much activity in the case where the wave functions take \( p \)-adic values.\(^5\) In algebraic geometry, there are two cohomology theories: \( p \)-adic and \( \ell \)-adic, where \( \ell \) is a prime \( \neq p \). It is the \( \ell \)-adic theory that is related to the usual real and complex cohomology. Perhaps, at the algebraic level, \( C \)-quantum mechanics may be

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\(^3\)For comprehensive references to work on the \( p \)-adic aspects of quantum strings, quantum fields, and other topics, as well as an extensive bibliography, the book [17] is the best reference. See also [18–20].

\(^4\)For Harish-Chandra’s theory, see [22]. See also [23].

\(^5\)Khrennikov has many papers on \( p \)-adic-valued quantum theory, for instance, [24]. See also his book [25].
viewed as $\mathbb{Q}_p$-quantum mechanics over $p$-adically defined spaces. This seems worth looking into. Also, in $\mathbb{Q}_p$-quantum mechanics, one cannot avoid considering finite extensions of $\mathbb{Q}_p$; as the tower of finite extensions of $\mathbb{Q}_p$ is much richer than the single extension $\mathbb{R} \subset \mathbb{C}$, it is natural to expect that the theory of particles and antiparticles would be much richer in $\mathbb{Q}_p$-quantum theory and that a central role would be played by the Galois group $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. One should perhaps seek a principle, such as supersymmetry, to direct attention to quadratic extensions, and to relate these to the unique antiparticle in the real case.

5. CONCLUSION

The above discussion, as well as the discussions and lectures at the conference, indicate that $p$-adic mathematical physics is a very active and fertile ground for major discoveries that interface with many disciplines of mathematics and physics and other natural sciences. Much remains to be done. If I have to single out one theme that deserves much more attention than it has received so far, it must be the role of symmetry and supersymmetry in non-Archimedean and finite physics.

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