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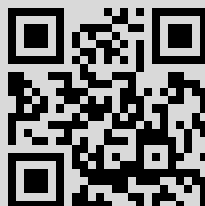
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DILOGARITHM IDENTITIES, PARTITIONS, AND SPECTRA IN CONFORMAL FIELD THEORY

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Abstract. We prove new identities for the values of Rogers' dilogarithm function and describe a connection between these identities and spectra in conformal field theory.

Introduction

The dilogarithm function $Li_2(x)$ defined for $0 \leq x \leq 1$ by

$$Li_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^x \frac{\log(1-t)}{t} dt,$$

is one of the less known transcendental functions. Nonetheless, it has many intriguing properties and has appeared in various branches of mathematics and physics such as number theory (the study of asymptotic behaviour of partitions, e.g., [RS, AL]; the values of ζ -functions in some special points [Za]) and algebraic K -theory (the Bloch group and a torsion in $K_3(\mathbb{R})$ —A. Beilinson, S. Bloch [Bl], A. Goncharov), geometry of hyperbolic three-manifolds [Th, NZ, Mi, Y], representation theory of Virasoro and Kac-Moody algebras [Ka, KW, FS] and conformal field theory (CFT).

In physics, the dilogarithm appears at first from a calculation of magnetic susceptibility in the XXZ model at small magnetic field [KR1, KR2, Ki1, BR]. More recently [Z], the dilogarithm identities (through the Thermodynamic Bethe Ansatz (TBA)) appear in the context of investigation of UV limit or the critical behaviour of integrable 2-dimensional quantum field theories and lattice models [Z, DR, K-M, KBP, Kl, KP, ...]. Evenmore, it was shown (e.g., [NRT]) using a method of Richmond and Szekeres [RS], that the dilogarithm identities may be derived from an investigation of the asymptotic behaviour of some characters of $2d$ CFT. Thus, it seems a very interesting problem to lift the dilogarithm identity in question to some identity between the characters of

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certain conformal field theory. A partial solution of this problem (without any proofs!) is contained in [Te] and [KKMM].

One aim of this paper is to prove some new identities between the values of the Rogers dilogarithm function using the analytical methods (see [Le, Ki1]). Based on such identities we give an answer to one question of W. Nahm [Na]: how big may be abelian group

$$\mathcal{W} := \left\{ \sum_i \frac{n_i L(\alpha_i)}{L(1)} \mid n_i \in \mathbb{Z}, \alpha_i \in \overline{\mathbb{Q}} \cap \mathbb{R} \text{ for all } i \right\} \cap \mathbb{Q}?$$

Theorem ♣. *The abelian group \mathcal{W} coincides with \mathbb{Q} i.e., any rational number can be obtained as the value of some dilogarithm sum.*

A proof of Theorem ♣ follows from Proposition 4.5. We also give a proof (and different generalizations) of the identity (3.1) from [NRT] (see our Theorem 3.1 and Proposition 5.4). Note the following “reciprocity law” for dilogarithm sums (see Theorem 3.2):

$$s(j, n, r) + s(j, r, n) = nr - 1.$$

This is a consequence of the corresponding reciprocity law for the Dedekind sums [Ra]. Note also that the author does not know any CFT interpretation for the dilogarithm identities from Propositions 4.4 and 5.4.

§1. Definition and the basic properties of Rogers’ dilogarithm

Let us remind the definition of the Rogers dilogarithm function $L(x)$ for $x \in (0, 1)$:

$$L(x) = -\frac{1}{2} \int_0^x \left[\frac{\log(1-x)}{x} + \frac{\log x}{1-x} \right] dx = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \log x \cdot \log(1-x). \quad (1.1)$$

The following two classical results (see, e.g., [Le2, GM, Ki1]) contain the basic properties of the function $L(x)$.

Theorem A. *The function $L(x) \in C^\infty((0, 1))$ and satisfies the functional equations*

$$1) \quad L(x) + L(1-x) = \frac{\pi^2}{6}, \quad 0 < x < 1, \quad (1.2)$$

$$2) \quad L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right), \quad (1.3)$$

where $0 < x, y < 1$.

Theorem B. *Let $f(x)$ be a function of class $C^3((0, 1))$ that satisfies relations (1.2) and (1.3). Then we have*

$$f(x) = \text{const} \cdot L(x)$$

We extend the function $L(x)$ to the whole real axis $\mathbb{R} = \mathbb{R}^1 \cup \{\pm\infty\}$ by the following rules:

$$L(x) = \frac{\pi^2}{3} - L(x^{-1}), \quad \text{if } x > 1, \quad (1.4)$$

$$L(x) = L\left(\frac{1}{1-x}\right) - \frac{\pi^2}{6}, \quad \text{if } x < 0, \quad (1.5)$$

$$L(0) = 0, \quad L(1) = \frac{\pi^2}{6}, \quad L(+\infty) = \frac{\pi^2}{3}, \quad L(-\infty) = -\frac{\pi^2}{6}.$$

The present work will deal with relations between the values of the Rogers dilogarithm function at certain algebraic numbers. More exactly, let us consider an abelian subgroup \mathcal{W} in the field of rational numbers \mathbb{Q} ,

$$\mathcal{W} = \left\{ \sum_i \frac{n_i L(\alpha_i)}{L(1)} \mid n_i \in \mathbb{Z}, \alpha_i \in \overline{\mathbb{Q}} \cap \mathbb{R} \text{ for all } i \right\} \cap \mathbb{Q}. \quad (1.6)$$

According to a conjecture of W. Nahm [Na], the abelian group \mathcal{W} "coincides" with the spectra in rational conformal field theory. Thus it seems a very interesting task to obtain a more explicit description of the group \mathcal{W} (e.g., to find a system of generators for \mathcal{W}) and also to connect the already known results about the spectra in conformal field theory (see, e.g., [BPZ, FF, GKO, FQS, Bi, Ka, KP]) with suitable elements in \mathcal{W} . One of the main results of this paper allows us to describe some part of a system of generators for abelian group \mathcal{W} . As a corollary, we shall show that the spectra of unitary minimal models [BPZ, GKO] and some others are really contained in \mathcal{W} . But at first we remind some already known relations between the values of the Rogers dilogarithm function.

§2. Some dilogarithm relations

It is easy to see from (1.2) and (1.3) that

$$L\left(\frac{1}{2}\right) = \frac{\pi^2}{12}, \quad L\left(\frac{1}{2}(\sqrt{5}-1)\right) = \frac{\pi^2}{10}, \quad L\left(\frac{1}{2}(3-\sqrt{5})\right) = \frac{\pi^2}{15}. \quad (2.1)$$

Proof. Let us put $\alpha := \frac{1}{2}(\sqrt{5}-1)$. It is clear that $\alpha^2 + \alpha = 1$. So we have $L(\alpha^2) = L(1-\alpha) = L(1) - L(\alpha)$. Now we use Abel's formula

$$L(x^2) = 2L(x) - 2L\left(\frac{x}{1+x}\right), \quad (2.2)$$

which may be obtained from (1.3) in the case $x = y$. From (2.2) we find that

$$L(\alpha^2) = 2L(\alpha) - 2L\left(\frac{\alpha}{1+\alpha}\right) = 2L(\alpha) - 2L(\alpha^2).$$

So, $3L(\alpha^2) = 2L(\alpha)$. But as we have already seen,

$$L(1) = L(\alpha) + L(\alpha^2) = \frac{5}{3}L(\alpha).$$

This proves (2.1). •

Apparently, there is no other algebraic point from the interval $(0, 1)$ at which there is such an elementary evaluation of the Rogers dilogarithm function. However, there are many identities relating the values of dilogarithm function at various powers of algebraic numbers. It is interesting to write down some of these identities in order to compare the elements of the abelian group W obtained by such manner with the spectra of known conformal models. As for the identities between the values of the dilogarithm function, we follow [Le1, Le2, Lo, RS].

$$6L\left(\frac{1}{3}\right) - L\left(\frac{1}{9}\right) = \frac{\pi^2}{3}, \quad (2.3)$$

$$\sum_{k=2}^n L\left(\frac{1}{k^2}\right) + 2L\left(\frac{1}{n+1}\right) = \frac{\pi^2}{6}, \quad (2.4)$$

consequently,

$$\sum_{k=2}^{\infty} L\left(\frac{1}{k^2}\right) = \frac{\pi^2}{6}.$$

This identities may be easily deduced from (1.2) and (2.2). Again, if $\alpha = \sqrt{2} - 1$, we have the relations

$$4L(\alpha) + L(1 - \alpha^2) = \frac{5\pi^2}{12}, \quad (2.5)$$

$$4L(\alpha) + 4L(\alpha^2) + L(1 - \alpha^4) = \frac{7\pi^2}{12}.$$

Proof. We use Abel's formula (2.2) and relation (1.3) with $y = \frac{1}{2}$

$$L\left(\frac{1}{2}\right) + L(x) = L\left(\frac{x}{2}\right) + L\left(\frac{x}{2-x}\right) + L\left(\frac{1-x}{2-x}\right). \quad (2.6)$$

We have

$$\begin{aligned} L(\alpha^2) &= 2L(\alpha) - 2L\left(\frac{\alpha}{1+\alpha}\right) = 2L(\alpha) - 2L\left(\frac{2-\sqrt{2}}{2}\right), \\ L\left(\frac{1}{2}\right) &= 2L\left(\frac{\sqrt{2}}{2}\right) - 2L(\alpha) = \frac{\pi^2}{3} - 2L\left(\frac{2-\sqrt{2}}{2}\right) - 2L(\alpha), \\ L(\alpha^4) &= 2L(\alpha^2) - 2L\left(\frac{\alpha^2}{1+\alpha^2}\right) = 2L(\alpha^2) - 2L\left(\frac{2-\sqrt{2}}{4}\right), \\ L\left(\frac{2-\sqrt{2}}{2}\right) + L\left(\frac{1}{2}\right) &= L\left(\frac{2-\sqrt{2}}{4}\right) + L(\alpha) + L(\alpha^2). \end{aligned}$$

Excluding successively $L(\frac{2-\sqrt{2}}{2})$ and $L(\frac{2-\sqrt{2}}{4})$ from these relations, we obtain (2.5). •

Watson [Wa] found three relations involving the roots of the cubic $x^3 + 2x^2 - x - 1$. Namely, if we take $\alpha = \frac{1}{2} \sec \frac{2\pi}{7}$, $\beta = \frac{1}{2} \sec \frac{\pi}{7}$, $\gamma = 2 \cos \frac{3\pi}{7}$, then α , $-\beta$, and $-\frac{1}{\gamma}$ are the roots of this cubic, and

$$\begin{aligned} L(\alpha) + L(1 - \alpha^2) &= \frac{4\pi^2}{21}, \\ 2L(\beta) + L(\beta^2) &= \frac{5\pi^2}{21}, \\ 2L(\gamma) + L(\gamma^2) &= \frac{4\pi^2}{21}. \end{aligned} \quad (2.7)$$

Lewin [Le1] and Loxton [Lo] found three relations involving the roots of the cubic $x^3 + 3x^2 - 1$. Namely, if we take $\delta = \frac{1}{2} \sec \frac{\pi}{9}$, $\varepsilon = \frac{1}{2} \sec \frac{2\pi}{9}$, $\zeta = 2 \cos \frac{4\pi}{9}$, then δ , $-\varepsilon$, and $-\frac{1}{\zeta}$ are the roots of this cubic, and

$$\begin{aligned} 3L(\delta) + 3L(\delta^2) + L(1 - \delta^3) &= \frac{17\pi^2}{18}, \\ 6L(\varepsilon) + 9L(1 - \varepsilon^2) + 2L(1 - \varepsilon^3) + L(\varepsilon^6) &= \frac{31\pi^2}{18}, \\ 6L(\zeta) + 9L(1 - \zeta^2) + 2L(1 - \zeta^3) + L(\zeta^6) &= \frac{35\pi^2}{18}. \end{aligned} \quad (2.8)$$

§3. Basic identities and conformal weights

In this section we present our main results dealing with the computation of the dilogarithm sum

$$\sum_{k=1}^{n-1} \sum_{m=1}^r L\left(\frac{\sin k\varphi \cdot \sin(n-k)\varphi}{\sin(m+k)\varphi \cdot \sin(m+n-k)\varphi}\right) := \frac{\pi^2}{6} s(j, n, r), \quad (3.1)$$

where $\varphi = \frac{(j+1)\pi}{n+r}$, $0 \leq j \leq n+r-2$.

It is clear that $s(j, n, r) = s(n+r-2-j, n, r)$, so we shall assume in the sequel that $0 \leq 2j \leq n+r-2$.

The dilogarithm sum (3.1) corresponds to the Lie algebra of type A_{n-1} . The case $j=0$ was considered in our previous paper [Ki1], where it was proved that

$$s(0, n, r) = \frac{(n^2 - 1)r}{n + r}.$$

It was stated in [Ki1] that this number coincides with the central charge of the $SU(n)$ level r WZNW model. Before formulating our result on the computation of the sum $s(j, n, r)$, let us remind the definition of Bernoulli polynomials. They are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.$$

We use also the modified Bernoulli polynomials

$$\overline{B}_n(x) = B_n(\{x\}), \quad \text{where } \{x\} = x - [x]$$

is a fractional part of $x \in \mathbb{R}$. It is well known that

$$\begin{aligned} \overline{B}_{2n}(x) &= (-1)^n \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}}, \\ \overline{B}_{2n+1}(x) &= (-1)^{n-1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n+1}}. \end{aligned} \quad (3.2)$$

Theorem 3.1. *We have*

$$s(j, n, r) = 6(r+n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left\{ \frac{1}{6} - \overline{B}_2((n-1-2k)\theta) \right\} - \frac{1}{4} \{2n^2 + 1 + 3(-1)^n\}, \quad (3.3)$$

where $\theta = \frac{j+1}{r+n}$ and $\gcd(j+1, r+n) = 1$.

Theorem 3.2 (level-rank duality [SA, KN1]).

$$s(j, n, r) + s(j, r, n) = nr - 1. \quad (3.4)$$

Corollary 3.3. *We have*

$$s(j, n, r) = c_r^{(n)} - 24h_j^{(r,n)} + 6 \cdot \mathbb{Z}_+, \quad (3.5)$$

where

$$c_r^{(n)} = \frac{(n^2 - 1)r}{n + r}, \quad h_j^{(r,n)} = \frac{n(n^2 - 1)}{24} \cdot \frac{j(j+2)}{r+n}, \quad 0 \leq j \leq r+n-2, \quad (3.6)$$

are the central charge and conformal dimensions of the $SU(n)$ level r WZNW primary fields, respectively.

Proof. Let us remind that

$$B_2(x) = x^2 - x + \frac{1}{6} \quad \text{and} \quad \overline{B}_2(x) = B_2(\{x\}).$$

Thus,

$$\begin{aligned}
 & 6(r+n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{1}{6} - \bar{B}_2((n-1-2k)\theta) \right) \\
 &= 6(r+n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-2k)\theta - 6(r+n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-2k)^2 \theta^2 \\
 &\quad + 6(r+n) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} [(n-1-2k)\theta]((n-1-2k)\theta - 1 + \{(n-1-2k)\theta\}) \\
 &=: 6\Sigma_1 - 6\Sigma_2 + 6\Sigma_3.
 \end{aligned}$$

Now if we take $\theta = \frac{j+1}{r+n}$, then it is clear that $\Sigma_3 \in \mathbb{Z}_+$. In order to compute Σ_1 and Σ_2 we use the summation formulas

$$\begin{aligned}
 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-2k) &= \frac{2n^2 - 1 + (-1)^n}{8} = \left[\frac{n}{2} \right] \cdot \left[\frac{n+1}{2} \right], \\
 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-1-2k)^2 &= \frac{n(n^2 - 1)}{6}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 s(j, n, r) &= \frac{3(2n^2 - 1 + (-1)^n)(j+1)}{4} - \frac{n(n^2 - 1)(j+1)^2}{r+n} - \frac{2n^2 + 1 + 3(-1)^n}{4} + 6\Sigma_3 \\
 &= \frac{(n^2 - 1)r}{r+n} - \frac{n(n^2 - 1)j(j+2)}{r+n} + 6j \left[\frac{n}{2} \right] \cdot \left[\frac{n+1}{2} \right] + 6\Sigma_3. \quad \bullet
 \end{aligned}$$

For small values of j we can compute the sum in (3.3) and thus find the corresponding positive integer in (3.5).

Corollary 3.4.

i) If $j \leq r$, then

$$s(j, 2, r) = c_r^{(2)} - 24h_j^{(r,2)} + 6j = \frac{3r}{r+2} + 6 \frac{j(r-j)}{r+2}. \quad (3.7)$$

ii) If $2j \leq r+1$, then

$$s(j, 3, r) = \frac{8r}{r+3} - 24 \frac{j(j+2)}{r+3} + 12j. \quad (3.8)$$

iii) If $(n-1)j < r+1$, then

$$s(j, n, r) = c_r^{(n)} - 24h_j^{(r,n)} + 6j \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor. \quad (3.9)$$

Proof. The assumption $(n-1)j < r+1$ is equivalent to the condition $\frac{(n-1)(j+1)}{n+r} < 1$. So the term Σ_3 (see the proof of Corollary 3.3) is equal to zero. •

It seems interesting to find the meaning of the positive integer in (3.5). Now we want to find a “dilogarithm interpretation” of the central charges and conformal dimensions for some well-known conformal models.

Corollary 3.5. *We have*

$$s(j_1, 2, k) + s(0, 2, 1) - s(j_2, 2, k+1) = c_k - 24h_{j_1+1, j_2+1} + 6(j_1 - j_2)(j_1 - j_2 + 1), \quad (3.10)$$

where

$$c_k = 1 - \frac{6}{(k+2)(k+3)}, \quad h_{r,s}^{(k)} = \frac{[(k+3)r - (k+2)s]^2 - 1}{4(k+2)(k+3)} \quad (3.11)$$

are the central charge and conformal dimensions of the primary fields for unitary minimal conformal models [BPZ, Ka, GKO].

Corollary 3.6.

$$s(j_1, n, k) + s(0, n, 1) - s(j_2, n, k+1) = c_{k,n} - 24h_{j_1+1, j_2+1}^{(n)} + 12\mathbb{Z}_+, \quad (3.12)$$

where

$$c_{k,n} = (n-1) \left\{ 1 - \frac{n(n+1)}{(k+n)(k+n+1)} \right\}, \quad (3.13)$$

$$h_{r,s}^{(n)}(k) = \frac{n(n^2-1)}{24} \cdot \frac{[(k+n+1)r - (k+n)s]^2 - 1}{(k+n)(k+n+1)}$$

are the central charge and conformal dimensions of the primary fields for W_n models [Bi, CR].

Corollary 3.7.

$$s(j_1, 2, k) + s\left(\frac{1}{2}(1 - (-1)^{j_1-j_2}), 2, 2\right) - s(j_2, 2, k+2)$$

$$= c(k) - 24\tilde{h}_{j_1+1, j_2+1} + 12 \left\lfloor \frac{j_1 - j_2 + 1}{2} \right\rfloor \cdot \left\lfloor \frac{j_1 - j_2 + 2}{2} \right\rfloor, \quad (3.14)$$

where

$$c(k) = \frac{3}{2} \left(1 - \frac{8}{(k+2)(k+4)} \right), \quad (3.15)$$

$$\tilde{h}_{r,s} := \tilde{h}_{r,s}(k) = \frac{[(k+4)r - (k+2)s]^2 - 4}{8(k+2)(k+4)} + \frac{1 - (-1)^{r-s}}{32},$$

are the central charge and conformal dimensions of the primary fields for unitary minimal $N = 1$ superconformal models [GKO, MSW].

We give a generalization of Corollaries 3.5–3.7 to the case of non-unitary minimal models.

Corollary 3.8. *If $p \geq q \geq 2$, then*

$$(p-q)s(j_1, 2, q-2) + s(0, 2, 1) - (p-q)s(j_2, 2, p-2) \\ = c - 24h_{j_1+1, j_2+1} + 6(j_1 - j_2)(p - q + j_1 - j_2), \quad (3.16)$$

where

$$c = 1 - \frac{6(p-q)^2}{pq}, \quad (3.17) \\ h_{r,s} := h_{r,s}(c) = \frac{(pr - qs)^2 - (p-q)^2}{4pq}, \quad r < q, s < p,$$

are the central charge and conformal dimensions of the primary fields for non-unitary (if $p - q \geq 2$) Virasoro minimal models [FQS]. Note that the “remainder term” in (3.16)

$$6(j_1 - j_2)(p - q + j_1 - j_2)$$

appears to be positive for all $0 \leq j_1 < q$, $0 \leq j_2 < p$ iff $p - q = 0$ (the trivial case) or $p - q = 1$ (the unitary case).

Corollary 3.9. *Let $p \geq q \geq 2$ and $p - q \equiv 0 \pmod{2}$. Then*

$$\frac{p-q}{2}s(j_1, 2, q-2) + s\left(\frac{1}{2}(1 - (-1)^{j_1-j_2}), 2, 2\right) - \frac{p-q}{2}s(j_2, 2, p-2) \\ = \tilde{c} - 24\tilde{h}_{j_1+1, j_2+1} + 6\left\{\frac{(j_1 - j_2)(j_1 - j_2 + p - q)}{2} + \frac{1 - (-1)^{j_1-j_2}}{4}\right\} \quad (3.18)$$

where

$$\tilde{c} = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right), \quad (3.19) \\ \tilde{h}_{r,s} = \frac{(pr - qs)^2 - (p-q)^2}{8pq} + \frac{1 - (-1)^{r-s}}{32}, \quad r < q, s < p,$$

are the central charge and conformal dimensions of the primary fields for the non-unitary (if $p - q > 2$) Neveu-Schwarz (if $r - s$ even) or Ramond (if $r - s$ odd) minimal models [FQS].

Corollary 3.10. *Let $p \geq q \geq n$, then*

$$(p-q)s(j_1, n, q-n) + s(0, n, 1) - (p-q)s(j_2, n, p-n) = c - 24h_{j_1+1, j_2+1}(c) + 6\mathbb{Z}, \quad (3.20)$$

where

$$c = (n-1) \left\{ 1 - \frac{n(n+1)(p-q)^2}{pq} \right\}, \quad (3.21)$$

$$h_{rs}(c) = \frac{n(n^2-1)(pr-qs)^2 - (p-q)^2}{24pq}, \quad r < q, s < p,$$

are the central charge and conformal dimensions of the primary fields for the non-unitary (if $p-q \geq 2$) W_n minimal models [Bi].

Finally we give a “dilogarithm interpretation” for the central charges and conformal weights of the restricted solid-on-solid (RSOS) lattice models and their fusion hierarchies [KP].

Corollary 3.11. *We have*

$$s(l, 2, N) + s(N-1, 2, N-2) - s(m-1, 2, N-2) = c + 1 - 24\Delta + 6(l - |m|), \quad m \in \mathbb{Z}, 0 \leq l \leq N, \quad (3.22)$$

where

$$c = \frac{2(N-1)}{N+2} \quad \text{and} \quad \Delta = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N} \quad (3.23)$$

are the central charge and conformal weights of \mathbb{Z}_N parafermion theories [FZ]. The members of (3.23) may be also realized as the central charge and conformal weights of the fusion $N+1$ -state RSOS(p, p) lattice models [DJKMO, BR] on the regime I/II critical line. Note that the physical constraints

$$|m| \leq l, \quad m \equiv l \pmod{2}$$

for the value of m in (3.23) are equivalent to the condition that the “remainder term” in (3.22), namely, $6(l - |m|)$, must belong to $12\mathbb{Z}_+$.

Corollary 3.12. *Let us fix the positive integers $k, p = 1, 2, \dots$ (the fusion level), j_1 and j_2 such that $0 \leq j_1 \leq k, 0 \leq j_2 \leq k+l$.*

Let $r_0 = p \left\{ \frac{j_1 - j_2}{p} \right\}$ be the unique integer determined by

$$0 \leq r_0 \leq p, \quad r_0 \equiv \pm(j_1 - j_2) \pmod{2p}. \quad (3.24)$$

Then we have

$$s(j_1, 2, k) + s(r_0, 2, p) - s(j_2, 2, k+p) = c - 24\Delta + 12 \frac{(j_1 - j_2)(p + j_1 - j_2) + r_0(p - r_0)}{2p}, \quad (3.25)$$

where

$$\begin{aligned} c &= \frac{3p}{p+2} \left(1 - \frac{2(p+2)}{(k+2)(k+p+2)} \right), \\ \Delta &= \frac{[(k+p+2)(j_1+1) - (k+2)(j_2+1)]^2 - p^2}{4p(k+2)(k+p+2)} + \frac{r_0(p-r_0)}{2p(p+2)} \end{aligned} \quad (3.26)$$

are the central charge and conformal weights of the fusion $(k+p+1)$ -state RSOS(p, p) lattice models [KP] on the regime III/IV critical line. It is easy to see that the "remainder term" in (3.25) belongs to $12\mathbb{Z}_+$. Note also that the fusion RSOS(p, q) lattice models, obtained by fusing $p \times q$ blocks of face weights together, are related to coset conformal fields theories obtained by the Goddard-Kent-Olive (GKO) construction [GKO]. Namely, c and Δ in (3.26) are the central charge and conformal dimensions of conformal field theory, which correspond to the coset pair [GKO]

$$\begin{array}{ccccc} & A_1 & \oplus & A_1 & \supset & A_1 \\ \text{levels} & k & & p & & k+p \end{array}$$

Thus the members of (3.26) are reduced to those of (3.11) if $p = 1$ and of (3.15) if $p = 2$.

§4. A_1 -type dilogarithm identities

As is well-known [Le2], the Rogers dilogarithm function $L(x)$ admits a continuation to the whole complex plane \mathbb{C} . Following [Le, KR], we define the function

$$\begin{aligned} L(x, \theta) &:= -\frac{1}{2} \int_0^x \frac{\log(1 - 2x \cos \theta + x^2)}{x} dx + \frac{1}{4} \log |x| \cdot \log(1 - 2x \cos \theta + x^2) \\ &= \operatorname{Re} L(xe^{i\theta}), \quad x, \theta \in \mathbb{R} \end{aligned} \quad (4.1)$$

Our proof of Theorem 3.1 is based on the study of the properties of the function $L(x, \theta)$.

Proposition 4.1. *For all real φ, θ we have*

$$\begin{aligned} L\left(\left(\frac{\sin \theta}{\sin \varphi}\right)^2\right) &= \pi^2 \left\{ \overline{B}_2\left(\frac{\theta + \varphi}{\pi}\right) - \overline{B}_2\left(\frac{\varphi}{\pi}\right) - \overline{B}_2\left(\frac{\theta}{\pi}\right) + \frac{1}{6} \right\} \\ &\quad + 2L\left(-\frac{\sin(\varphi - \theta)}{\sin \theta}, \varphi\right) - 2L\left(-\frac{\sin \varphi}{\sin \theta}, \varphi + \theta\right). \end{aligned} \quad (4.2)$$

Before proving Proposition 4.1, let us list other useful properties of the function (4.1) (compare with [Le2]).

Lemma 4.2.

$$(i) \quad L(x, 0) = L(x), \quad L(-x, \varphi) = L(-x, \pi - \varphi) \quad (4.3)$$

$$(ii) \quad L(x, \varphi) = L(x, 2\pi k \pm \varphi), \quad k \in \mathbb{Z} \quad (4.4)$$

$$\begin{aligned}
 \text{(iii)} \quad L(-1, \varphi) &= \pi^2 \overline{B}_2 \left(\frac{\varphi}{2\pi} + \frac{1}{2} \right), \\
 L(1, \varphi) &= \pi^2 \overline{B}_2 \left(\frac{\varphi}{2\pi} \right)
 \end{aligned} \tag{4.5}$$

$$\begin{aligned}
 \text{(iv)} \quad L(x, \varphi) + L(x^{-1}, \varphi) &= 2\pi^2 \overline{B}_2 \left(\frac{\varphi}{2\pi} \right), \quad x > 0 \\
 L(-x, \varphi) + L(-x^{-1}, \varphi) &= 2\pi^2 \overline{B}_2 \left(\frac{\varphi}{2\pi} + \frac{1}{2} \right), \quad x < 0
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 \text{(v)} \quad L(0, \varphi) &= 0, \quad L(+\infty, \varphi) = 2\pi^2 \overline{B}_2 \left(\frac{\varphi}{2\pi} \right), \\
 L(-\infty, \varphi) &= 2\pi^2 \overline{B}_2 \left(\frac{\varphi + \pi}{2\pi} \right)
 \end{aligned} \tag{4.7}$$

$$\text{(vi)} \quad L(2 \cos \varphi, \varphi) = \pi^2 \left\{ \overline{B}_2 \left(\frac{\varphi}{\pi} \right) + \frac{1}{12} \right\} \tag{4.8}$$

$$\begin{aligned}
 \text{(vii)} \quad L(x^n, n\varphi) &= n \sum_{k=0}^{n-1} L \left(x, \varphi + \frac{2k\pi}{n} \right), \quad x \in \mathbb{R}_+, \\
 L(x^n) &= n \sum_{k=0}^{n-1} L \left(x \cdot \exp \frac{2k\pi i}{n} \right), \quad x \in (0, 1).
 \end{aligned} \tag{4.9}$$

More generally (Rogers' identity [Ro]),

$$L(1 - y^n) = \sum_{k=1}^n \sum_{l=1}^n [L(\lambda_k / \lambda_l) - L(x_k \lambda_l)],$$

where $\{x_k\}_{k=1}^n$ are the roots of the equation

$$1 - y^n = \prod_{k=1}^n (1 - \lambda_k x).$$

Proof. At first let us remind some properties of the modified Bernoulli polynomials.

$$\begin{aligned}
 \frac{d\overline{B}_n(x)}{dx} &= n\overline{B}_{n-1}(x), \\
 \overline{B}_n(x) &= \overline{B}_n(x+1), \quad \overline{B}_n(-x) = (-1)^n \overline{B}_n(x), \\
 \overline{B}_p(nx) &= n \sum_{k=1}^n \overline{B}_p \left(x + \frac{k}{n} \right).
 \end{aligned} \tag{4.10}$$

Note that identities (4.5) follow from the Fourier expansion for $\overline{B}_2(x)$ (see (3.2)).

In order to prove the identity (4.2), let us differentiate LHS and RHS of (4.1) with respect to φ using the formula

$$\begin{aligned}
 dL(x, \varphi) &= \left\{ -\frac{1}{4} \frac{\log(1 - 2x \cos \varphi + x^2)}{x} + \frac{1}{2} \log |x| \frac{x - \cos \varphi}{1 - 2x \cos \varphi + x^2} \right\} dx \\
 &+ \left\{ -\tan^{-1} \left(\frac{x \sin \varphi}{1 - x \cos \varphi} \right) + \frac{1}{2} \log |x| \frac{x \sin \varphi}{1 - 2x \cos \varphi + x^2} \right\} d\varphi.
 \end{aligned} \tag{4.11}$$

Acting in this way we find

$$\frac{d}{d\varphi} L\left(-\frac{\sin \varphi}{\sin \theta}, \varphi + \theta\right) = \varphi + \frac{1}{2} \cot(\varphi + \theta) \log\left(\frac{\sin \varphi}{\sin \theta}\right) - \frac{1}{2} \cot \varphi \log\left(\frac{\sin(\varphi + \theta)}{\sin \theta}\right), \quad (4.12)$$

$$\frac{d}{d\varphi} L\left(-\frac{\sin \theta}{\sin \varphi}, \varphi + \theta\right) = \theta - \frac{1}{2} \cot(\varphi + \theta) \log\left(\frac{\sin \varphi}{\sin \theta}\right) - \frac{1}{2} \cot \varphi \log\left(\frac{\sin(\varphi + \theta)}{\sin \theta}\right), \quad (4.13)$$

$$\text{if } 0 < \varphi, \theta < \pi, \varphi + \theta < \pi; \quad (4.13a)$$

$$\begin{aligned} & \frac{d}{d\varphi} L\left(\frac{\sin \theta}{\sin \varphi} \cdot \frac{\sin(\theta + \psi)}{\sin(\varphi + \psi)}\right) \\ &= \frac{1}{2} [\cot \varphi + \cot(\varphi + \psi)] \log\left(\frac{\sin(\varphi - \theta) \sin(\varphi + \theta + \psi)}{\sin \theta \sin(\theta + \psi)}\right) \\ & \quad - \frac{1}{2} [\cot(\varphi - \theta) + \cot(\varphi + \theta + \psi)] \log\left(\frac{\sin \varphi \sin(\varphi + \psi)}{\sin \theta \sin(\theta + \psi)}\right), \end{aligned} \quad (4.14)$$

$$\text{if } 0 < \theta < \varphi < \pi, 0 < \psi < \pi, \varphi + \theta + \psi < \pi. \quad (4.14a)$$

Further, let us use the reduction rules (4.3), (4.4) and (4.6) (compare with (1.4) and (1.5)) if the angles φ, θ (or φ, θ, ψ) do not satisfy condition (4.13a) (or (4.14a)). As a result one can obtain that the derivative of difference between LHS and RHS of (4.2) with respect to φ is equal to

$$2\pi \left\{ \overline{B}_1\left(\frac{\varphi + \theta}{\pi}\right) - \overline{B}_1\left(\frac{\varphi}{\pi}\right) \right\}.$$

Integrating (see (4.10)), then taking $\varphi = 0$ and using (4.5) to determine the integration constant, we obtain equality (4.2).

It is easy to see that (4.8) follows from (4.2) when $\varphi + \theta = \pi$. In order to prove (4.9) let us differentiate LHS and RHS of (4.9) with respect to x and use the summation formula

$$\sum_{k=0}^{n-1} \frac{\exp i(\varphi + \frac{2\pi k}{n})}{1 - x \exp i(\varphi + \frac{2\pi k}{n})} = \frac{nx^{n-1} \exp(in\varphi)}{1 - x^n \exp(in\varphi)}.$$

Thus, the proofs of Proposition 4.1 and Lemma 4.2 are finished. •

Proof of Theorem 3.1 for the case $n = 2$. If we substitute $\theta = m\varphi$ in (4.2), then we obtain

$$\begin{aligned} L\left(\left(\frac{\sin m\varphi}{\sin \varphi}\right)^2\right) &= \pi^2 \left\{ \overline{B}_2\left(\frac{(m+1)\varphi}{\pi}\right) - \overline{B}_2\left(\frac{m\varphi}{\pi}\right) - \overline{B}_2\left(\frac{\varphi}{\pi}\right) + \frac{1}{6} \right\} \\ & \quad + 2L\left(-\frac{\sin((m-1)\varphi)}{\sin \varphi}, m\varphi\right) - 2L\left(-\frac{\sin m\varphi}{\sin \varphi}, (m+1)\varphi\right). \end{aligned} \quad (4.15)$$

Let us introduce the notation

$$f_m(\varphi) := 1 - \frac{Q_{m-1}(\varphi)Q_{m+1}(\varphi)}{Q_m^2(\varphi)} = \frac{1}{Q_m^2(\varphi)}.$$

Then using (4.15) we find

$$\begin{aligned} \sum_{m=1}^r L(f_m(\varphi)) &= -2 \left\{ L\left(-Q_r(\varphi), (r+2)\varphi\right) - \frac{\pi^2}{6} \right\} \\ &\quad + \pi^2 \left\{ \overline{B}_2\left(\frac{(r+2)\varphi}{\pi}\right) - \frac{1}{6} \right\} + (r+2)\pi^2 \left\{ \frac{1}{6} - \overline{B}_2\left(\frac{\varphi}{\pi}\right) \right\} - \frac{\pi^2}{2}. \end{aligned} \quad (4.16)$$

Now put $\varphi = \frac{(j+1)\pi}{r+2}$, $0 \leq j \leq r+1$. Then $Q_r(\varphi) = (-1)^j$ and it is clear from (4.3) and (4.4) that

$$L((-1)^{j+1}, (j+1)\pi) = L(1) = \frac{\pi^2}{6}. \quad \bullet$$

Note that the polynomials $Q_m := Q_m(\varphi)$ satisfy the recurrence relation

$$Q_m^2 = Q_{m-1}Q_{m+1} + 1, \quad Q_0 \equiv 1, \quad m \geq 1,$$

whereas the polynomials $y_m := y_m(\varphi) = Q_{m-1}(\varphi) \cdot Q_{m+1}(\varphi)$ satisfy

$$y_m^2 = (1 + y_{m-1})(1 + y_{m+1}), \quad y_0 \equiv 0, \quad m \geq 1.$$

Following [Le2], we define the function $W(x, \varphi)$ by

$$\begin{aligned} W(x, \varphi) &:= W(x, \varphi, \theta) = L\left(\frac{\sin^2 \theta}{\sin \varphi (\sin \varphi + x \sin(\varphi + \theta))}\right) \\ &\quad + L\left(-\frac{x^2 \sin \varphi + x \sin(\varphi - \theta)}{x \sin(\varphi + \theta) + \sin \varphi}\right) - L\left(\frac{x \sin(\varphi + \theta)}{x \sin(\varphi + \theta) + \sin \varphi}\right). \end{aligned} \quad (4.17)$$

Note the following particular cases:

$$\begin{aligned} W(0, \varphi) &= L\left(\left(\frac{\sin \theta}{\sin \varphi}\right)^2\right), \quad W(-2 \cos \theta, \varphi) = L\left(-\frac{\sin^2 \theta}{\sin \varphi \sin(\varphi + 2\theta)}\right), \quad (4.18) \\ W(1, \varphi) &= L\left(\frac{\sin \theta \sin \frac{1}{2}\theta}{\sin \varphi \sin(\varphi + \frac{1}{2}\theta)}\right) + L\left(\frac{\sin(\frac{1}{2}\theta - \varphi)}{\sin(\frac{1}{2}\theta + \varphi)}\right) - L\left(\frac{\sin(\varphi + \theta)}{2 \sin(\varphi + \frac{1}{2}\theta) \cos \frac{1}{2}\theta}\right), \\ W(-1, \varphi, \theta) &= W(1, \varphi, \pi + \theta). \end{aligned}$$

Proposition 4.3. *We have*

$$W(x, \varphi) = 2L(-x, \theta) + 2L(-x_1, \varphi) - 2L(-x_2, \varphi + \theta) + \pi^2 \left\{ \overline{B}_2 \left(\frac{\varphi + \theta}{\pi} \right) - \overline{B}_2 \left(\frac{\varphi}{\pi} \right) - \overline{B}_2 \left(\frac{\theta}{\pi} \right) + \frac{1}{6} \right\}, \quad (4.19)$$

where

$$x_1 = \frac{x \sin \varphi + \sin(\varphi - \theta)}{\sin \theta} \quad \text{and} \quad x_2 = \frac{x \sin(\varphi + \theta) + \sin \varphi}{\sin \theta}.$$

The proof of Proposition 4.3 can be obtained in the same manner as that of Proposition 4.1. •

Note that x_2 is obtained from x_1 by replacing φ by $\varphi + \theta$. The angular parameter in (4.19) also increases in this way, and the terms $L(-x, \varphi)$ and $L(-x_2, \varphi + \theta)$ have opposite signs. So if we substitute successively the angles $\varphi, \varphi + \theta, \dots, \varphi + r\theta$ instead of φ in (4.19) and add all the results together, we shall obtain the following generalisation of (4.16):

$$\sum_{k=0}^r W(x, \varphi + k\theta) = 2(r+1)L(-x, \theta) + 2L(-x_1, \varphi) - 2L(-x_{r+2}, \varphi + (r+1)\theta) + \pi^2 \left\{ \overline{B}_2 \left(\frac{\varphi + (r+1)\theta}{\pi} \right) - \overline{B}_2 \left(\frac{\varphi}{\pi} \right) \right\} + (r+1)\pi^2 \left\{ \frac{1}{6} - \overline{B}_2 \left(\frac{\theta}{\pi} \right) \right\}, \quad (4.20)$$

where

$$x_{n+1} := \frac{x \sin(\varphi + n\theta) + \sin(\varphi + (n-1)\theta)}{\sin \theta}, \quad 0 \leq n \leq r+2.$$

Now we take $\varphi = \theta$ in (4.20). Then $x_1 = x$, so that (4.20) becomes

$$\sum_{k=1}^{r+1} W(x, k\theta) = 2(r+2)L(-x, \theta) + (r+2)\pi^2 \left\{ \frac{1}{6} - \overline{B}_2 \left(\frac{\theta}{\pi} \right) \right\} - \frac{\pi^2}{3} + \pi^2 \left\{ \overline{B}_2 \left(\frac{(r+2)\theta}{\pi} \right) - \frac{1}{6} \right\} - 2 \left(L(-x_{r+2}, (r+2)\theta) - \frac{\pi^2}{6} \right), \quad (4.21)$$

where

$$x_{n+1} := x_{n+1}(x, \theta) = \frac{x \sin(n+1)\theta + \sin n\theta}{\sin \theta}.$$

Note the following particular cases ($0 \leq n \leq r+2$):

$$\begin{aligned} x_{n+1}(0, \theta) &= \frac{\sin n\theta}{\sin \theta}, & x_{n+1}(-2 \cos \theta, \theta) &= -\frac{\sin(n+2)\theta}{\sin \theta}, \\ x_{n+1}(1, \theta) &= \frac{\sin(\frac{1}{2}(2n+1)\theta)}{\sin \frac{1}{2}\theta}, & x_{n+1}(-1, \theta) &= x_{n+1}(1, \theta + \pi). \end{aligned}$$

One can show that identity (4.21) is reduced to (4.2) if $x = -2 \cos \theta$ (or $x = 0$). Now we assume that $x \neq -2 \cos \theta$ and take $\theta = \frac{(j+1)\pi}{r+2}$ in (4.21). Then $x_{r+2} = (-1)^j$, so that (4.21) becomes

$$\sum_{k=1}^{r+1} W(x, k\theta) = 2(r+2)L(-x, \theta) + \frac{\pi^2}{6}(1 + s(j, 2, r)), \quad (4.22)$$

where

$$s(j, 2, r) = \frac{3r}{r+2} + 6 \frac{j(r-j)}{r+2}$$

(see (3.7)).

Finally, let us take $x = \pm 1$ in (4.22). After some manipulations we obtain the following result.

Proposition 4.4. *Let functions $W(\pm 1, \theta)$ be defined by (4.18) and $\theta = \frac{(j+1)\pi}{r+2}$. Then we have*

$$\begin{aligned} \sum_{k=1}^{r+1} W(-1, k\theta) &= \frac{\pi^2}{6} \left\{ 2r + 2 - \frac{3(j+1)^2}{r+2} \right\}, \\ \sum_{k=1}^{r+1} W(+1, k\theta) &= \frac{\pi^2}{6} \left\{ 2 - r - \frac{3(j+1)^2}{r+2} + 6j \right\}. \end{aligned} \quad (4.23)$$

Now we propose a generalisation of (4.16). Let a rational number p and the decomposition of p into the continued fraction

$$p = [b_r, b_{r-1}, \dots, b_1, b_0] = b_r + \frac{1}{b_{r-1} + \frac{1}{\dots + \frac{1}{b_1 + \frac{1}{b_0}}}}. \quad (4.24)$$

be given. We shall assume that $b_i > 0$ if $0 \leq i < r$ and $b_r \in \mathbb{Z}$. Using the decomposition (4.24), we define the set of integers y_i and m_i :

$$\begin{aligned} y_{-1} &= 0, \quad y_0 = 1, \quad y_1 = b_0, \quad \dots, \quad y_{i+1} = y_{i-1} + b_i y_i, \quad 0 \leq i \leq r, \\ m_0 &= 0, \quad m_1 = b_0, \quad m_{i+1} = |b_i| + m_i, \quad 0 \leq i \leq r. \end{aligned} \quad (4.25)$$

It is clear that $p = \frac{y_{r+1}}{y_r}$ and

$$\frac{y_{i+1}}{y_i} = p_i := b_i + \frac{1}{b_{i-1} + \frac{1}{\dots + \frac{1}{b_1 + \frac{1}{b_0}}}}, \quad 0 \leq i \leq r.$$

The following sequences of integers were first introduced by Takahashi and Suzuki [TS]:

$$\begin{aligned} r(j) &= i \quad \text{for } m_i \leq j < m_{i+1}, \quad 0 \leq i \leq r, \\ n_j &= y_{i-1} + (j - m_i)y_i \quad \text{for } m_i \leq j < m_{i+1} + \delta_{i,r}, \quad 0 \leq i \leq r. \end{aligned}$$

Finally, we define the dilogarithm sum of "fractional level p ":

$$\sum_{j=1}^{m_{r+1}} (-1)^{r(j)} L \left(\left(\frac{\sin y_{r(j)} \theta}{\sin(n_j + y_{r(j)}) \theta} \right)^2 \right) := (-1)^r \frac{\pi^2}{6} s(k, 2, p), \quad (4.26)$$

where $\theta = \frac{(k+1)\pi}{y_{r+1} + 2y_r}$.

The dilogarithm sum (4.26) (in the case $k = 0$) was first considered in [KR], where its interpretation as a low-temperature asymptotics of the entropy for the XXZ Heisenberg model was given.

Proposition 4.5. *We have*

$$\begin{aligned} \text{(i)} \quad s(0, 2, p) &= \frac{3p}{p+2}, \\ \text{(ii)} \quad s(k, 2, p) &= \frac{3p}{p+2} - 6 \frac{k(k+2)}{p+2} + 6\mathbb{Z}. \end{aligned} \quad (4.27)$$

Proof. We start with

Lemma 4.6. *Given an integer σ such that $m_i < \sigma \leq m_{i+1}$, for any $\theta \in \mathbb{R}$ we have*

$$\begin{aligned} \sum_{j=m_i}^{\sigma-1} L \left(\left(\frac{\sin y_{r(j)} \theta}{\sin(n_j + y_{r(j)}) \theta} \right)^2 \right) &:= (\sigma - m_i) \pi^2 \left\{ \frac{1}{6} - \overline{B}_2 \left(\frac{y_i \theta}{\pi} \right) \right\} \\ &+ \pi^2 \left\{ \overline{B}_2 \left(\frac{(n_{\sigma-1} + 2y_i) \theta}{\pi} \right) - \overline{B}_2 \left(\frac{(y_{i-1} + y_i) \theta}{\pi} \right) \right\} \\ &+ 2L \left(-\frac{\sin y_{i-1} \theta}{\sin y_i \theta}, (y_i + y_{i-1}) \theta \right) - 2L \left(-\frac{\sin(n_{\sigma-1} + y_i) \theta}{\sin y_i \theta}, (n_{\sigma-1} + 2y_i) \theta \right). \end{aligned} \quad (4.28)$$

The proof of Lemma 4.6 follows from Proposition 4.1. •

From (4.28) one can easily deduce the following generalization of (4.16).

Corollary 4.7. *Let an integer σ , $\sigma < m_i \leq m_{i+1}$, be given. Then*

$$\begin{aligned} \sum_{j=0}^{\sigma-1} L\left(\left(\frac{\sin y_{r(j)}\theta}{\sin(n_j + y_{r(j)})\theta}\right)^2\right) &:= \pi^2 \sum_{j=0}^{i-1} (-1)^j b_j \left\{ \frac{1}{6} - \overline{B}_2\left(\frac{y_j\theta}{\pi}\right) \right\} \\ &+ (-1)^i (\sigma - m_i) \pi^2 \left\{ \frac{1}{6} - \overline{B}_2\left(\frac{y_i\theta}{\pi}\right) \right\} + (-1)^i \pi^2 \overline{B}_2\left(\frac{(n_{\sigma-1} + 2y_i)\theta}{\pi}\right) \\ &- 2\pi^2 \sum_{j=0}^{i-1} (-1)^j \overline{B}_2\left(\frac{(y_{j-1} + y_j)\theta}{\pi}\right) - \pi^2 \overline{B}_2\left(\frac{\theta}{\pi}\right) \\ &+ 2 \sum_{j=0}^{i-1} (-1)^{j+1} \left\{ L\left(-\frac{\sin y_{j-1}\theta}{\sin y_j\theta}, (y_j + y_{j-1})\theta\right) + L\left(-\frac{\sin y_j\theta}{\sin y_{j-1}\theta}, (y_j + y_{j-1})\theta\right) \right\} \\ &+ (-1)^{i+1} 2L\left(-\frac{\sin(n_{\sigma-1} + y_i)\theta}{\sin y_i\theta}, (n_{\sigma-1} + 2y_i)\theta\right). \end{aligned} \quad (4.29)$$

In order to go further we must compute the last sum in (4.29). This computation is based on the following result.

Lemma 4.8. *Given the real numbers φ and θ , we define*

$$\varepsilon(\varphi, \theta) = \left\{ \frac{\varphi}{2\pi} + \frac{1}{2} \right\} + \left\{ \frac{\theta}{2\pi} + \frac{1}{2} \right\} - \left\{ \frac{\varphi}{2\pi} \right\} - \left\{ \frac{\theta}{2\pi} \right\} \pmod{2}, \quad (4.30)$$

where $\{x\} = x - [x]$ is the fractional part of $x \in \mathbb{R}$.

Then we have

$$L\left(-\frac{\sin \varphi}{\sin \theta}, \varphi + \theta\right) + L\left(-\frac{\sin \theta}{\sin \varphi}, \varphi + \theta\right) = 2\pi^2 \overline{B}_2\left(\frac{\varphi + \theta + \varepsilon(\varphi, \theta)\pi}{2\pi}\right). \quad (4.31)$$

The proof of Lemma 4.8 follows from (4.6). •

Now we are ready to complete the proof of Proposition 4.5. Namely, from (4.29) and (4.31) it follows that $s(k, 2, p) \in \mathbb{Q}$, $(y_{r+1} + 2y_r) \cdot s(k, 2, p) \in \mathbb{Z}$, and

$$s(0, 2, p) = \frac{3p}{p+2}.$$

Finally, we observe that (4.27) follows from (4.29) and (4.31) if we replace all modified Bernoulli polynomials by the ordinary ones. •

Proposition 4.9. *For all positive $p \in \mathbb{Q}$ the remainder term in (4.27) lies in $6\mathbb{Z}_+$. More exactly, given a positive $p \in \mathbb{Q}$ we define a set of integers $\{s_k\}$, $k = 0, 1, 2, \dots$, such that*

$$\left\lfloor \frac{j+1}{p+2} \right\rfloor = k \quad \text{iff} \quad s_k \leq j < s_{k+1}, \quad s_0 := 0.$$

Further, let us define the function

$$t(j) := t(j, p) = (2k+1)j + k - 2 \sum_{a=0}^k s_a \quad \text{iff } s_k \leq j < s_{k+1}.$$

Then we have

$$s(j, 2, p) = \frac{3p}{p+2} - \frac{6j(j+2)}{p+2} + 6t(j, p). \quad \bullet$$

It is clear that $t(j, p) \in \mathbb{Z}_+$.

Corollary 4.10. *Let us fix the positive integers $l = 1, 2, 3, \dots$ (the fusion level), $p > q$, j_1 , and j_2 . Then*

$$s\left(j_1, 2, \frac{ql}{p-q} - 2\right) + s(r_0, 2, l) - s\left(j_2, 2, \frac{pl}{p-q} - 2\right) = c - 24\Delta + 6\mathbb{Z}, \quad (4.32)$$

where $r_0 = l \cdot \left\{ \frac{j_1 - j_2}{l} \right\}$ and

$$c = \frac{3l}{l+2} \left(1 - \frac{2(l+2)(p-q)^2}{l^2 pq} \right), \quad (4.33)$$

$$\Delta = \frac{[p(j_1+1) - q(j_2+1)]^2 - (p-q)^2}{4lpq} + \frac{r_0(l-r_0)}{2l(l+2)}$$

are the central charge and conformal dimensions of RCFT, which correspond to the coset pair [GKO]

$$\text{levels} \quad \frac{A_1}{\frac{ql}{p-q} - 2} \oplus \frac{A_1}{l} \supset \frac{A_1}{\frac{pl}{p-q} - 2}$$

§5. Proof of Theorem 3.1

We start with a generalization of identity (4.2).

Proposition 5.1. *For all real φ , ψ , and θ we have*

$$\begin{aligned} & L\left(\frac{\sin \theta}{\sin \varphi} \cdot \frac{\sin(\theta + \psi)}{\sin(\varphi + \psi)}\right) \\ &= \pi^2 \left\{ \overline{B}_2\left(\frac{\theta + \varphi + \psi}{\pi}\right) - \overline{B}_2\left(\frac{2\theta + \psi + \overline{\varepsilon}(\theta, \theta + \psi)\pi}{2\pi}\right) \right. \\ & \quad \left. - \overline{B}_2\left(\frac{2\varphi + \psi + \overline{\varepsilon}(\varphi, \varphi + \psi)\pi}{2\pi}\right) + \frac{1}{6} \right\} \\ &+ L\left(-\frac{\sin(\varphi - \theta)}{\sin \theta}, \varphi\right) + L\left(-\frac{\sin(\varphi - \theta)}{\sin(\theta + \psi)}, \varphi + \psi\right) \\ &- L\left(-\frac{\sin \varphi}{\sin(\theta + \psi)}, \varphi + \theta + \psi\right) \\ &- L\left(-\frac{\sin(\varphi + \psi)}{\sin \theta}, \varphi + \theta + \psi\right), \end{aligned} \quad (5.1)$$

where $\bar{\varepsilon}(\varphi, \theta) = 1 - \varepsilon(\varphi, \theta)$ and $\varepsilon(\varphi, \theta)$ is defined by (4.31).

Proof. First of all we consider the case when $0 < \varphi + \theta + \psi < \pi$ and $\varphi, \theta, \psi > 0$. In this case $\bar{\varepsilon}(\theta, \theta + \psi) = \bar{\varepsilon}(\varphi, \varphi + \psi) = 0$ and one can use identities (4.12)–(4.14) in order to show that the derivative of the difference between LHS and RHS of (5.1) with respect to φ is equal to

$$2\pi \left\{ B_1 \left(\frac{\theta + \varphi + \psi}{\pi} \right) - B_1 \left(\frac{2\varphi + \psi}{2\pi} \right) \right\} = 2\theta + \psi.$$

Integrating (see (4.10)) we find that the difference between LHS and RHS of (5.1) is a function $c(\theta, \psi)$ which does not depend on φ . In order to find $c(\theta, \psi)$ let us take $\varphi = \theta$ in (5.1). After this substitution we obtain the relation

$$\frac{\pi^2}{6} = \frac{1}{2}(2\theta + \psi)^2 + c(\theta, \psi) - L \left(-\frac{\sin \theta}{\sin(\theta + \psi)}, 2\theta + \psi \right) - L \left(-\frac{\sin(\theta + \psi)}{\sin \theta}, 2\theta + \psi \right).$$

Comparing the last equality with (4.31) (in our case $\varepsilon(\theta, \theta + \psi) = 1$) we find $c(\theta, \psi) = 0$. In the general case we use the reduction rules (4.3), (4.4), and (4.6) and the following properties of function $\varepsilon(\varphi, \theta)$:

$$\begin{aligned} \varepsilon(\theta, \theta) &= 1, & \varepsilon(\theta, \varphi) &= \varepsilon(\varphi, \theta), \\ \varepsilon(\theta, \pi + \theta) &= 0, & \varepsilon(\varphi, -\theta) &= \varepsilon(\varphi, \pi + \theta), \\ \varepsilon(\theta, \pi - \theta) &= 1, & \varepsilon(-\varphi, -\theta) &= \varepsilon(\varphi, \theta). \end{aligned} \quad \bullet$$

Corollary 5.2. *We have*

$$L \left(\frac{\sin(\varphi + \theta)}{\sin \theta}, \varphi \right) + L \left(\frac{\sin(\varphi + \theta)}{\sin \varphi}, \theta \right) = 2\pi^2 \left\{ \bar{B}_2 \left(\frac{\varphi + \theta + \bar{\varepsilon}(\varphi, \theta)\pi}{2\pi} \right) + \frac{1}{12} \right\}. \quad (5.2)$$

Proof. Take $\psi = -\theta - \varphi$ in (5.1). \bullet

Let us continue the proof of Theorem 3.1 and take a specialization $\theta \rightarrow k\varphi$, $\varphi \rightarrow (m + k)\varphi$, and $\psi \rightarrow (n - 2k)\varphi$ in (5.1). Then we obtain

$$\begin{aligned} & L \left(\frac{\sin k\varphi}{\sin(m + k)\varphi} \cdot \frac{\sin(n - k)\varphi}{\sin(m + n - k)\varphi} \right) \\ &= \pi^2 \left\{ \bar{B}_2 \left(\frac{(m + n)\varphi}{\pi} \right) - \bar{B}_2 \left(\frac{n\varphi + \bar{\varepsilon}(k\varphi, (n - k)\varphi)\pi}{2\pi} \right) \right. \\ & \quad \left. - \bar{B}_2 \left(\frac{(n + 2m)\varphi + \bar{\varepsilon}((m + k)\varphi, (n - k)\varphi)\pi}{2\pi} \right) + \frac{1}{6} \right\} \\ &+ L \left(-\frac{\sin m\varphi}{\sin k\varphi}, (m + k)\varphi \right) + L \left(-\frac{\sin m\varphi}{\sin(n - k)\varphi}, (m + n - k)\varphi \right) \\ &- L \left(-\frac{\sin(m + k)\varphi}{\sin(n - k)\varphi}, (m + n)\varphi \right) - L \left(-\frac{\sin(m + n - k)\varphi}{\sin k\varphi}, (m + n)\varphi \right). \end{aligned}$$

Consequently, after summation we have

$$\begin{aligned}
 & \sum_{k=1}^{n-1} \sum_{m=1}^r L\left(\frac{\sin k\varphi}{\sin(m+k)\varphi} \cdot \frac{\sin(n-k)\varphi}{\sin(m+n-k)\varphi}\right) \\
 &= 2 \sum_{k=1}^{n-1} \sum_{m=1}^r L\left(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\right) \\
 &\quad - 2 \sum_{k=1}^{n-1} \sum_{m=1}^r L\left(-\frac{\sin(m+n-k)\varphi}{\sin k\varphi}, (m+n)\varphi\right) + \pi^2 \Sigma_3 \\
 &= 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} L\left(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\right) \\
 &\quad - 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} L\left(-\frac{\sin(r+m)\varphi}{\sin k\varphi}, (m+r+k)\varphi\right) + \pi^2 \Sigma_3 \\
 &= 2\Sigma_1 - 2\Sigma_2 + \pi^2 \Sigma_3,
 \end{aligned}$$

where

$$\begin{aligned}
 \Sigma_3 := \sum_{k=1}^{n-1} \sum_{m=1}^r \left\{ \overline{B}_2\left(\frac{(m+n)\varphi}{\pi}\right) - \overline{B}_2\left(\frac{n\varphi + \bar{\varepsilon}(k\varphi, (n-k)\varphi)\pi}{2\pi}\right) \right. \\
 \left. - \overline{B}_2\left(\frac{(n+2m)\varphi + \bar{\varepsilon}((m+k)\varphi, (n-k)\varphi)\pi}{2\pi}\right) + \frac{1}{6} \right\}. \quad (5.3)
 \end{aligned}$$

At first, let us consider the sum

$$\begin{aligned}
 2\Sigma_1 &:= 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} L\left(-\frac{\sin m\varphi}{\sin k\varphi}, (m+k)\varphi\right) \\
 &= 2 \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} L(-1, 2p\varphi) + 2 \sum_{p=3}^n \sum_{k=1}^{\lfloor \frac{p-1}{2} \rfloor} \left\{ L\left(-\frac{\sin(p-k)\varphi}{\sin k\varphi}, p\varphi\right) + 2L\left(-\frac{\sin k\varphi}{\sin(p-k)\varphi}, p\varphi\right) \right\} \\
 &= 2 \sum_{p=3}^n \sum_{k=1}^{\lfloor \frac{p-1}{2} \rfloor} 2\pi^2 \overline{B}_2\left(\frac{p\varphi + \bar{\varepsilon}((p-k)\varphi, k\varphi)\pi}{2\pi}\right) + 2 \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \pi^2 \overline{B}_2\left(\frac{p\varphi}{\pi} + \frac{1}{2}\right). \quad (5.4)
 \end{aligned}$$

Secondly, in order to compute the sum $2\Sigma_2$, we recall that $\varphi = \frac{(j+1)\pi}{n+r}$. Hence $\sin(m+r)\varphi = \sin \frac{(m+r)(j+1)\pi}{n+r} = (-1)^j \sin(n-m)\varphi$, and consequently (see (4.3) and (4.4))

$$\begin{aligned}
 & L\left(-\frac{\sin(r+m)\varphi}{\sin k\varphi}, (m+r+k)\varphi\right) \\
 &= L\left((-1)^{j+1} \frac{\sin(n-m)\varphi}{\sin k\varphi}, (j+1)\pi - (n-m-k)\varphi\right) \\
 &= L\left(-\frac{\sin(m-n)\varphi}{\sin k\varphi}, (m+k-n)\varphi\right).
 \end{aligned}$$

So we have

$$\begin{aligned}
 2\Sigma_2 &:= 2 \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} L\left(-\frac{\sin(r+m)\varphi}{\sin k\varphi}, (m+r+k)\varphi\right) \\
 &= \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} L\left(-\frac{\sin(m-n)\varphi}{\sin k\varphi}, (m+k-n)\varphi\right) \\
 &= 2 \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} L(2 \cos p\varphi, p\varphi) + \frac{(n-1)\pi^2}{3} \\
 &\quad + 2 \sum_{p=3}^{n-1} \sum_{k=1}^{\lfloor \frac{p-1}{2} \rfloor} \left\{ L\left(\frac{\sin p\varphi}{\sin k\varphi}, (p-k)\varphi\right) + L\left(\frac{\sin p\varphi}{\sin(p-k)\varphi}, k\varphi\right) \right\} \\
 &= \frac{(n-1)\pi^2}{3} + 2 \sum_{p=3}^{n-1} \sum_{k=1}^{\lfloor \frac{p-1}{2} \rfloor} 2\pi^2 \left\{ \bar{B}_2\left(\frac{p\varphi + \bar{\varepsilon}(k\varphi, (p-k)\varphi)\pi}{2\pi}\right) + \frac{1}{12} \right\} \\
 &\quad + 2 \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \pi^2 \left\{ \bar{B}_2\left(\frac{p\varphi}{\pi}\right) + \frac{1}{12} \right\}. \tag{5.5}
 \end{aligned}$$

Let us sum up our computations. First of all we proved that $s(j, n, k) \in \mathbb{Q}$. Secondly, in order to compute the dilogarithm sum $s(j, n, k)$ modulo \mathbb{Z} we may replace all modified Bernoulli polynomials appearing in (5.3)–(5.5) by the ordinary ones. After some bulky calculations we obtain (3.5), except for positivity of the remainder term in (3.5). Finally, in order to obtain the exact formulas (3.3) and (3.4) we are based on the properties of Dedekind sums [Ra]. Details will appear elsewhere. •

Now we propose a generalization of (4.27). For this goal let us define the function

$$L_k(\theta, \varphi) := 2L\left(\frac{\sin \theta \cdot \sin k\theta}{\sin \varphi \cdot \sin(\varphi + (k-1)\theta)}\right) - \sum_{j=0}^{k-1} L\left(\left(\frac{\sin \theta}{\sin(\varphi + j\theta)}\right)^2\right). \tag{5.6}$$

Lemma 5.3. *We have*

$$L_k(\theta, \varphi) = 2L\left(-\frac{\sin(\varphi - \theta)}{\sin k\theta}, \varphi + (k-1)\theta\right) - 2L\left(-\frac{\sin \varphi}{\sin k\theta}, \varphi + k\theta\right) + \pi^2 \mathbb{Q}. \bullet$$

Now let $p \in \mathbb{Q}$ and consider the decomposition of p/k into a continued fraction

$$\frac{p}{k} = b_r + \frac{1}{b_{r-1} + \frac{1}{\dots + \frac{1}{b_1 + \frac{1}{b_0}}}} \tag{5.7}$$

where $b_i \in \mathbb{N}$, $0 \leq i \leq r-1$, and $b_r \in \mathbb{Z}$. Using the decomposition (5.7) we define (compare with (4.25)):

$$\begin{aligned} y_{-1} &= 0, \quad y_0 = 1, \quad y_1 = b_0, \quad \dots, \quad y_{i+1} = y_{i-1} + b_i y_i, \\ m_0 &= 0, \quad m_1 = b_0, \quad m_{i+1} = |b_i| + m_i, \\ r(j) &:= r_k(j) = i, \quad \text{if } km_i \leq j < km_{i+1} + \delta_{i,r}, \\ n_j &:= n_k(j) = ky_{i-1} + (j - km_i)y_i, \quad \text{if } km_i \leq j < km_{i+1} + \delta_{i,r}, \end{aligned}$$

where $0 \leq i \leq r$.

Finally, we consider the dilogarithm sum

$$\sum_{j=1}^{km_{r+1}} (-1)^{r(j)} L_k(y_{r(j)}\theta, (n_j + y_{r(j)})\theta) = (-1)^r \frac{\pi^2}{6} s(l, k+1, p), \quad (5.8)$$

where $\theta = \frac{(l+1)\pi}{ky_{r+1} + (k+1)y_r}$.

Proposition 5.4. *We have*

$$(i) \quad s(0, k+1, p) := c_k = \frac{3(p+1-k)}{p+k+1}, \quad k \geq 1, \quad (5.8)$$

$$(ii) \quad s(l, k+1, p) = c_k - \frac{6k l(l+2)}{p+k+1} + 6\mathbb{Z}, \quad (5.9)$$

(iii) if $k = 1$ or 2 , then the remainder term in (5.9) lies in $6\mathbb{Z}_+$.

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