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ON SUPER-REPLICATION IN DISCRETE TIME UNDER TRANSACTION COSTS

1. Introduction and main result. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, \(T\) a finite horizon and \(\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}\) a family of increasing sub \(\sigma\)-algebras. \(\mathcal{F}_t\) represents the information available at time \(t\). We assume as usual that \(\mathcal{F}_0\) is trivial and \(\mathcal{F}_T = \mathcal{F}\). We fix throughout \(p \in [1, \infty)\) and we denote as usual by \(L^p\) the set of all random variables \(X\) with \(E[|X|^p] < \infty\) and by \(\| \cdot \|_p\) the associated norm. We denote by \(q\) the conjugate of \(p\) (i.e., \(1/p + 1/q = 1\)); as usual, \(q = \infty\) whenever \(p = 1\).

The financial market consists in one riskless asset whose price is assumed to be equal to one (without loss of generality) and a risky asset with positive \(\mathcal{F}\)-adapted price process \(S = \{S_t, t = 0, \ldots, T\}\) with \(S_t \in L^p\) for all \(t = 0, \ldots, T\).

The risky asset is subject to proportional transaction costs. Let us introduce the sublinear function defined on \(\mathbb{R}\) by:

\[
\tau(z) = \begin{cases} 
(1 + \lambda)z & \text{if } z > 0, \\
(1 - \mu)z & \text{if } z \leq 0,
\end{cases}
\]

where \((\lambda, \mu) \in (0, \infty) \times (0, 1)\) are, respectively, the transaction cost rates for purchasing and selling. This means that the algebraic cost (respectively, gain) in units of nonrisky asset (cash) induced by a position of \(z\) units of risky asset at time \(t\) is \(\tau(z) S_t\) (respectively, \(-\tau(-z) S_t\)).

Remark. A trivial but useful property of the function \(\tau\) is:

\[-\tau(-z) \leq rz \leq \tau(z)\]

for any \((r, z) \in [1 - \mu, 1 + \lambda] \times \mathbb{R}\).

A trading strategy is a pair \((\eta, \theta) = \{(\eta_t, \theta_t), t = 0, \ldots, T - 1\}\) of \(\mathcal{F}\)-adapted processes. Here, \(\eta_t\) and \(\theta_t\) are interpreted respectively, as the number of shares of nonrisky and risky asset held at time \(t\).

A trading strategy \((\eta, \theta)\) is said to be self-financing if:

\[
\eta_t - \eta_{t-1} + \tau(\theta_t - \theta_{t-1}) S_t \leq 0, \quad t = 1, \ldots, T - 1, \quad P\text{-a.s.}
\]

which means that, at every trading date, the cost induced by rebalancing the strategy is nonpositive; in other words, sales must finance purchases. We denote by \(\mathcal{A}\) the set of all self-financing trading strategies.
A contingent claim is a pair \( C = (C^0, C^1) \) of \( \mathcal{F}_T \)-measurable random variables, with \((C^0, C^1, ST) \in L^p \times L^p\), representing a time \( T \) portfolio consisting of \( C^0 \) units of the nonrisky asset \( S^0 \) and \( C^1 \) units of the risky one \( S \). We denote by \( \mathcal{L}_S^p \) the set of all contingent claims. This set is naturally endowed with the norm \( \|C\|_{\mathcal{L}_S^p} = \|C^0\|_p + \|C^1ST\|_p \).

We say that a trading strategy \((\eta, \theta)\) dominates some contingent claim \( C \) if:

\[
\eta_{T-1} - C^0 \geq \tau(C^1 - \theta_{T-1})ST \quad \text{P-a.s.}
\]

In this note, we are interested in a dual representation of the super-replication cost of a given contingent claim \( C \in \mathcal{L}_S^p \), given an initial holding in the risky asset \( \theta_{-1} \), defined by:

\[
\hat{\pi}(C, \theta_{-1}) = \inf \{ \eta_0 + \tau(\theta_0 - \theta_{-1})S_0 : (\eta, \theta) \in \mathcal{A} \text{ and dominates } C \}.
\]

We refer to [6] and [2] for a practical use of the dual representation.

Let us introduce the set \( \mathcal{P}(\lambda, \mu) \) consisting of all pairs \((Q, \rho)\) satisfying:

1. \( Q \) is a probability measure equivalent to \( P \) with density \( dQ/dP \in L^q \);
2. \( \rho = \{ \rho_t, t = 0, \ldots, T \} \) is an \( \mathcal{F} \)-adapted process with values in the interval \([1 - \mu, 1 + \lambda] \);
3. the process \( \rho S = \{ \rho_t S_t, t = 0, \ldots, T \} \) is a martingale under \( Q \).

We then have the following theorem.

**Theorem.** Suppose that \( \mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset \) for some \( 0 < \varepsilon \leq \min\{\lambda, \mu\} \). Then for any \((C, \theta_{-1}) \in \mathcal{L}_S^p \times \mathbb{R} \), the super-replication cost of \( C \) given the initial holding \( \theta_{-1} \) is given by:

\[
\hat{\pi}(C, \theta_{-1}) = \sup_{(Q, \rho) \in \mathcal{P}(\lambda, \mu)} E^Q[ C^0 + \rho T(C^1 - \theta_{T-1})ST ].
\]

The set \( \mathcal{P}(\lambda, \mu) \) involved in the Theorem plays the same role in our model as the set of equivalent martingale measures in a financial market. It is proved in [3] that the condition \( \mathcal{P}(\lambda, \mu) \neq \emptyset \) is an equivalent characterization of the no free lunch condition which is stronger than the usual notion of no arbitrage. Therefore our condition \( \mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset \) is stronger than the no free lunch condition in our financial market and can be interpreted as a no free lunch condition in a financial market with smaller transaction costs.

The dual representation of the Theorem has been first established by Jouini and Kallal [3] under the weaker assumption \( \mathcal{P}(\lambda, \mu) \neq \emptyset \) (i.e., no arbitrage) but with a different, and somewhat unnatural, definition of the super-replication cost\(^2\). In the case of a finite probability space, Kusuoka [7] proved the dual representation of the Theorem under the assumption \( \mathcal{P}(\lambda, \mu) \neq \emptyset \). In the following proposition, we shall prove that Kusuoka's assumption is not weaker than ours under a nondegeneracy condition on the process \( \{S_t, t = 0, \ldots, T\} \).

**Proposition.** Let \( \Omega \) be finite and the conditional variance \( \mathbb{D}[S_t | \mathcal{F}_{t-1}] \neq 0 \) for all \( t = 1, \ldots, T \). Suppose that \( \mathcal{P}(\lambda, \mu) \neq \emptyset \). Then there exists \( \varepsilon \), with \( 0 < \varepsilon \leq \min\{\lambda, \mu\} \) such that \( \mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset \).

The proof of the Proposition is given in Appendix. Notice also that the dual representation of the Theorem has been established by Cvitanic and Karatzas [1] in a continuous time framework with diffusion price process. In the special case, where our price process \( S \) is a discrete record from a continuous time diffusion process as in [1], we clearly have \( \mathcal{P}(0, 0) \neq \emptyset \) which is a stronger requirement than our condition. Let us mention that while we were finishing the paper, we learned that Kabanov [5] extended the results of Cvitanic and Karatzas [1] to the general semi-martingale case which includes our discrete-time framework. But his dual representation formula is established under the same (stronger) assumption that in [1].

\(^2\) Namely, the lower-semicontinuous envelope of the \( \hat{\pi}(C, \theta_{-1}) \) with respect to the \( C \) variable.
2. Proof of the Theorem. In order to establish the required result, we provide some useful lemmas.

**Lemma 1.** Let \((Q, \rho) \in \mathcal{P}(\lambda, \mu)\). Then, for any self-financing trading strategy \((\eta, \theta) \in \mathcal{A}\), the process \(\{\eta_t + \rho_t \theta_t S_t, t = 0, \ldots, T - 1\}\) is a supermartingale under \(Q\).

**Proof.** The proof is similar to that of Proposition 3.1 in [6].

Next define the set:
\[
K = \left\{(U, V) \in \mathcal{L}^p_S : \exists (\eta, \theta) \in \mathcal{A}, \eta_0 + \tau(\theta_0) S_0 \leq 0 \text{ and } (\eta, \theta) \text{ dominates } (U, V) \right\}.
\]

The elements of \(K\) are interpreted as contingent claims which can be dominated by some self-financing strategy with nonpositive initial cost.

**Lemma 2.** Under the conditions of the Theorem the set \(K\) is closed in \(\mathcal{L}^p_S\).

**Proof.** Consider a sequence \((U_n, V_n)\) of elements of \(K\) converging to \((U, V)\) in \(\mathcal{L}^p_S\). Then it converges to \((U, V)\) almost surely (possibly along a subsequence). Moreover, there exists a sequence \((\hat{\eta}_n, \hat{\theta}_n)\) in \(\mathcal{A}\), i.e.,
\[
-\delta^n_t := \eta^n_t - \eta^n_{t-1} + \tau \left( \hat{\theta}_t^n - \hat{\theta}_{t-1}^n \right) S_t \leq 0 \quad \text{P-a.s.} \tag{2}
\]
with nonpositive initial cost, i.e.,
\[
-\delta^n_0 := \hat{\eta}_0^n + \tau(\hat{\theta}_0^n) S_0 \leq 0 \tag{3}
\]
and dominating \((U_n, V_n)\), i.e.,
\[
\hat{\eta}_T^n - U_n - \tau(V^n - \hat{\theta}_T^n) S_T \geq 0 \quad \text{P-a.s.} \tag{4}
\]
First notice that the strategy \((\hat{\eta}^n, \hat{\theta}^n)\) defined by \(\hat{\theta}^n = \hat{\theta}_t^n + \sum_{i=0}^t \delta_i^n\), \(t = 0, \ldots, T - 1\), is a self-financing trading strategy with:
\[
\eta_0^n + \tau(\theta_0^n) S_0 = 0, \tag{5}
\]
\[
\eta_t^n - \eta_{t-1}^n + \tau(\hat{\theta}_t^n - \hat{\theta}_{t-1}^n) S_t = 0 \quad \text{P-a.s.} \tag{6}
\]
which also dominates \((U_n, V_n)\), i.e.,
\[
\hat{\eta}_T^n - U_n - \tau(V^n - \hat{\theta}_T^n) S_T \geq 0 \quad \text{P-a.s.} \tag{7}
\]
Now, let \((Q^*, \rho^*) \in \mathcal{P}(\lambda - \epsilon, \mu - \epsilon)\). Then, the convergence of \((U_n, V_n)\) towards \((U, V)\) in \(\mathcal{L}^p_S\) implies that: \(U_n \to U\) and \(V^n S_T \to V S_T\) in \(L^1(Q^*)\), by the Hölder inequality.

In order to prove the required result, we intend to show the existence of a self-financing strategy \((\eta, \theta) \in \mathcal{A}\) with nonpositive initial cost and dominating \((U, V)\).

In the sequel, we denote by \(\mathcal{H}^1(Q^*)\) the set of \(\mathcal{F}\)-adapted processes \(\{Y_t, t = 0, \ldots, T - 1\}\) such that \(Y_t \in L^1(Q^*)\) for any \(t\). The linear space \(\mathcal{H}^1(Q^*)\) is naturally endowed with the product norm.

**First step.** We first prove that \(\{\eta^n_t, \theta^n_t S_t, t = 0, \ldots, T - 1\}\) is bounded in \(\mathcal{H}^1(Q^*) \times \mathcal{H}^1(Q^*)\) uniformly in \(n\). Define:
\[
X^n_T = E^Q^n \left[ U^n - \tau(-V^n) S_T \right] \tag{8}
\]
which is bounded in \(L^1(Q^*)\) from the convergence of \((U_n)\) and \((V^n S_T)\) in \(L^1(Q^*)\).

We also introduce:
\[
I^n_t = \sum_{u=0}^{t-1} \theta_u^n (S_{u+1} - S_u).
\]
By the self-financing condition (6), we have:
\[
\eta_t^n = \eta_0^n - \sum_{u=0}^{t-1} \tau(\theta_{u+1}^n - \theta_u^n) S_{u+1} \leq \eta_0^n + \tau \theta_0^n S_0 - \tau \theta^n T_s + r I^n_t \leq -r \theta^n S_t + r I^n_t, \quad \forall t \in [1 - \mu, 1 + \lambda], \tag{9}
\]
\(^3\) This part of the proof is inspired from Soner, Shreve and Cvitanić [8] and Cvitanić and Karatzas [1] who deal with a continuous time model with Brownian filtration.
where we used the Remark and (5). From the sublinearity of \( \tau \), (7) implies:

\[
\eta_{t-1}^n - \tau(-\theta_{t-1}^n) S_T \geq U^n - \tau(-V^n) S_T \quad \text{P-a.s.}
\]

Now, from the Remark and the Lemma applied to the pair \((Q^*, \rho^*) \in \mathcal{P}(\lambda - \epsilon, \mu - \epsilon)\), we get:

\[
\eta_t^n + \rho_t^n \theta_t^n S_t \geq X_t^n \quad \text{P-a.s.}
\]

which implies, by plugging (9) (with \( r = 1 + \lambda \) and \( r = 1 - \mu \)) in (10):

\[
\begin{align*}
\theta_t^n S_t &\leq \frac{1 + \lambda}{1 + \lambda - \rho_t^n} I_t^n - \frac{1}{1 + \lambda - \rho_t^n} X_t^n, \\
\theta_t^n S_t &\geq \frac{\mu - 1}{\rho_t^n - (1 - \mu)} I_t^n + \frac{1}{\rho_t^n - (1 - \mu)} X_t^n,
\end{align*}
\]

and therefore, since \( \rho_t^n \in [1 - \mu + \epsilon; 1 + \lambda - \epsilon] \): \( |\theta_t^n S_t| \leq \epsilon^{-1}((2 + \lambda)|I_t^n| + 2|X_t^n|) \). Thus, in order to show that \( (\theta_t^n S_t)_n \) is bounded in \( L^1(Q^*) \) for all \( t = 0, \ldots, T - 1 \), it remains to prove that \( I_t^n \) is bounded in \( L^1(Q^*) \). To see this, notice that \( I_t^n = I_{t-1}^n + \theta_{t-1}^n (S_t - S_{t-1}) \) and therefore: \( E^{Q^*}[|I_t^n|] \leq E^{Q^*}[|I_{t-1}^n|] + E^{Q^*}[|\theta_{t-1}^n S_{t-1}|] + E^{Q^*}[|\theta_{t-1}^n S_t|] \). Moreover, \( E^{Q^*}[|\theta_{t-1}^n S_{t-1}|] \leq E^{Q^*}[|\theta_{t-1}^n \rho_t^n S_t|/(1 - \mu)] \), and using the martingale property of \( \rho^* S \) under \( Q^* \), we get

\[
E^{Q^*}[\theta_{t-1}^n S_{t-1}] \leq \frac{1}{1 - \mu} E^{Q^*}[\theta_{t-1}^n \rho_t^n S_t/(1 - \mu)] \leq \frac{1 + \lambda}{1 - \mu} E^{Q^*}[\theta_{t-1}^n S_{t-1}],
\]

where we used again the martingale property of \( \rho^* S \) under \( Q^* \). This provides:

\[
E^{Q^*}[|I_t^n|] \leq E^{Q^*}[|I_{t-1}^n|] + \left(1 + \frac{1 + \lambda}{1 - \mu}\right) E^{Q^*}[|\theta_{t-1}^n S_{t-1}|]
\]

\[
\leq E^{Q^*}[|I_{t-1}^n|] + \frac{2 + \lambda}{(1 - \mu)\epsilon} \left(2 + \lambda\right) E^{Q^*}[|I_{t-1}^n|] + 2E^{Q^*}[|X_{t-1}^n|],
\]

and therefore the boundedness of \( I_t^n \) in \( L^1(Q^*) \) follows from a simple induction argument (recall that \( I_0^n = 0 \)).

Finally, since the sequence \( (\theta_t^n S_t)_n \) is bounded in \( L^1(Q^*) \) for any \( t = 0, \ldots, T - 1 \), we deduce immediately from (5), (6) and (11) that the sequence \( (\eta_t^n)_n \) is also bounded in \( L^1(Q^*) \) for any \( t = 0, \ldots, T - 1 \).

2nd step. From the boundedness of the processes \( (\eta^n)_n \) and \( (\theta^n S)_n \) in \( \mathcal{F}^1(Q^*) \), we deduce by the Komlos theorem (see [4, Theorem 7.3, p. 205]) the existence of two \( \mathcal{F} \)-adapted processes \( \tau^* \) and \( \omega^* \) and subsequences \( (\eta_{t/n}^n)_{n \in \mathbb{N}}, (\theta_{t/n}^n S_{t/n})_{n \in \mathbb{N}} \) such that: \( \eta_{t/n}^n \rightarrow \tau^* \) and \( \theta_{t/n}^n S_{t/n} \rightarrow \omega^* \) \( \text{P-a.s.} \), where \( \tau^* = (1/n) \sum_{k=1}^n \eta^n k \) and \( \omega^* = (1/n) \sum_{k=1}^n \theta^n k \). By the convexity of the function \( \tau \) and sending \( n \) to infinity, equations (5) and (6) imply

\[
\eta_{t-1} - \tau(\theta_{t-1}) S_t \leq 0 \quad \text{P-a.s.}
\]

and \( \eta_0 + \tau(\theta_0) S_0 \leq 0 \). Now, define \( \tilde{U}^n = (1/n) \sum_{k=1}^n U^n k \) and \( \tilde{V}^n = (1/n) \sum_{k=1}^n V^n k \). By Cesaro's theorem, \( (\tilde{U}^n, \tilde{V}^n) \) converge almost surely to \((U, V)\). By the convexity of \( \tau \) and sending \( n \) to infinity, the domination relation (7) imply \( \eta_{t-1} - \tau(U - \theta_{t-1}) S_T \geq 0 \) \( \text{P-a.s.} \) which proves that \((U, V) \in K \). Lemma 2 is proved.

For the proof of the Theorem, we need to introduce the set \( \mathcal{L}^1_{\mathcal{F}} \) consisting of all \( \mathcal{F} \)-measurable random variables \( Z = (Z^0, Z^1) \) such that \( Z^0 \in L^2 \) and \( Z^1/S_T \in L^2 \). \( \mathcal{L}^1_{\mathcal{F}} \) is the dual space of \( \mathcal{L}^2_{\mathcal{F}} \) when endowed with the norm \( \|Z\|_{\mathcal{L}^1_{\mathcal{F}}} = \|Z^0\|_2 + \|Z^1/S_T\|_2 \).

Proof of the Theorem. Since \( \mathcal{P}(\lambda - \epsilon, \mu - \epsilon) \neq \emptyset \), it is clear that we also have \( \mathcal{P}(\lambda, \mu) \neq \emptyset \). The proof of the inequality

\[
\mathcal{F}(C, \theta_1) \geq \sup_{(Q, \rho) \in \mathcal{P}(\lambda, \mu)} \mathcal{E}^{Q^*}[C^0 + \rho_T(C^1 - \theta_1) S_T]
\]

4) We are indebted to Yu. Kabanov for mentioning us the Komlor theorem.
is similar to that of Proposition 3.2 in [6]. Therefore, we only prove the converse inequality. Take an arbitrary \( b < \bar{\pi}(C, \theta_{-1}) \) and consider the convex cone \( K \) as well as the singleton \( A = \{ (C^0 - b, C^1 - \theta_{-1}) \} \). From Lemma 2, \( K \) is closed and by the definition of \( \bar{\pi}(C, \theta_{-1}) \) it is easily seen that \( K \cap A = \emptyset \). Therefore by the Hahn–Banach Separation Theorem and the Riesz Representation Theorem, there exists a pair \((Z^0, Z^1) \in \mathcal{S}_1^\mathcal{F} \setminus \{ 0 \}\) such that:

\[
\mathbf{E}[Z^0 U + Z^1 V] \leq 0 < \mathbf{E}\left[ Z^0 (C^0 - b) + Z^1 (C^1 - \theta_{-1}) \right] \quad \forall (U, V) \in K.
\] (12)

We first claim that \( Z^0 > 0 \) P-a.s. To see this consider the pair \((-1/Z^0, 0)\) which lies in \( K \), since it is dominated by the null strategy, and apply the left-hand-side inequality of (12). Next, consider an arbitrary element \((Q, \mu) \in \mathcal{T}(\lambda, \mu)\) and define \( Z^0 := dQ/dP \), \( Z^1 := Z^0 \rho_T S_T \) and the sequences

\[
Z^{n,0} = \frac{1}{n} Z^0 + \left( 1 - \frac{1}{n} \right) Z^0, \quad Z^{n,1} = \frac{1}{n} Z^1 + \left( 1 - \frac{1}{n} \right) Z^1.
\]

Finally, we define the pair \((Q^n, \rho^n)\) by:

\[
\frac{dQ^n}{dP} = Z^{n,0} \quad \text{and} \quad \rho_t^n = \frac{\mathbf{E}_t(Z_t^{n,1})}{S_t \mathbf{E}_t(Z_t^{n,0})}, \quad t = 0, \ldots, T.
\]

We now use the following lemma whose proof is carried out later.

**Lemma 3.** For all \( n > 0 \), we have \((Q^n, \rho^n) \in \mathcal{T}(\lambda, \mu)\) and

\[
\lim_{n \to \infty} \mathbf{E}^Q^n \left[ C^0 + \rho_T^n S_T (C^1 - \theta_{-1}) \right] = \mathbf{E} \left[ Z^0 C^0 + Z^1 (C^1 - \theta_{-1}) \right].
\]

Now, by the right-hand-side inequality of (12), we have: \( b < \mathbf{E}^Q [C^0 + \rho_T (C^1 - \theta_{-1}) S_T] \). Using the limit result of Lemma 3, this proves that, for some sufficiently large \( n \), we have

\[
b < \mathbf{E}^Q^n \left[ C^0 + \rho_T^n (C^1 - \theta_{-1}) S_T \right] \leq \sup_{(Q, \rho) \in \mathcal{T}(\lambda, \mu)} \mathbf{E}^Q \left[ C^0 + \rho_T (C^1 - \theta_{-1}) S_T \right];
\]

the required result follows from the arbitrariness of \( b < \bar{\pi}(C, \theta_{-1}) \).

**Proof of Lemma 3.** (i) We first prove that \((Q^n, \rho^n) \in \mathcal{T}(\lambda, \mu)\). Since \( Z^0 > 0 \) and \( Z^0 \geq 0 \) P-a.s., the measure \( Q^n \) satisfies condition (P1). It is easily checked that \((Q^n, \rho^n)\) satisfies (P3) by definition. It remains to prove that \((Q^n, \rho^n)\) satisfies (P2).

Fix an arbitrary nonnegative bounded \( \mathcal{F}_T \)-measurable random variable \( \xi \) and consider the self-financing strategy:

\[
\eta_s = \theta_s = 0 \quad \text{for} \quad s = 0, \ldots, t - 1, \quad \eta_s = -(1 + \lambda) \xi S_t \quad \text{and} \quad \theta_s = \xi \quad \text{for} \quad s = t, \ldots, T - 1.
\]

Then it is clear that \((\eta_{T-1}, \theta_{T-1}) \in K\) (recall that \( S_t \in L^p \)) and, therefore, applying the left-hand-side inequality of (12) we get \( \mathbf{E}[Z^1 \xi] \leq \mathbf{E}[Z^0 (1 + \lambda) \xi S_T] \). By a similar argument, we get \( \mathbf{E}[Z^0 (1 + \xi) \xi S_T] \geq \mathbf{E}[Z^0 (1 + \mu) \xi S_T] \). By the arbitrariness of \( \xi \), this provides:

\[
(1 - \mu) S_t \mathbf{E}_t[Z^0] \leq \mathbf{E}_t[Z^1] \leq (1 + \lambda) S_t \mathbf{E}_t[Z^0] \quad \text{for all} \quad t = 0, \ldots, T - 1.
\]

Considering the contingent claims \(-(1 + \lambda) \xi S_T, \xi \) \( \in K \) and \((1 - \mu) \xi S_T, -\xi \) \( \in K \) for arbitrary \( \mathcal{F}_T \)-measurable bounded and nonnegative \( \xi \) (since they are dominated by the null strategy), we see that the last inequalities hold also for \( T \). Now, since \((\bar{Q}, \bar{\rho})\) satisfies (P2), it is easily checked that \((Z^0, Z^1)\) also satisfies the last inequalities and therefore:

\[
(1 - \mu) S_t \mathbf{E}_t[Z_t^{n,0}] \leq \mathbf{E}_t[Z_t^{n,1}] \leq (1 + \lambda) S_t \mathbf{E}_t[Z_t^{n,0}], \quad t = 0, \ldots, T,
\]

which proves that \( \rho^n \) satisfies (P2).

(ii) To prove the limit result of the lemma, it suffices to notice that

\[
\mathbf{E}^Q^n \left[ C^0 + \rho_T^n S_T (C^1 - \theta_{-1}) \right] = \mathbf{E} \left[ Z^0 C^0 + Z^1 (C^1 - \theta_{-1}) \right]
\]
3. Appendix: proof of the Proposition. Let $\Omega$ be finite and assume that there exists $(Q, \rho) \in \mathcal{P}(\lambda, \mu)$. We intend to prove that there exists $\varepsilon$, with $0 < \varepsilon < \min\{\lambda, \mu\}$ such that $\mathcal{P}(\lambda - \varepsilon, \mu - \varepsilon) \neq \emptyset$.

**Step 1.** Fix some $t \in \{0, \ldots, T - 1\}$ and let $\sigma_t$ be any atom of $\mathcal{F}_t$; to simplify notation, we denote $(\rho^0, S^0) = (\rho(\sigma_t), S(\sigma_t))$ and $(\rho^1, S^1), \ldots, (\rho^n, S^n)$ the values of those processes $\rho$ and $S$ at successors of $\sigma_t$. We assume without loss of generality that $S^1 \geq \cdots \geq S^n > 0$. We will also denote by $q^1, \ldots, q^n$ the transition probabilities between $\sigma_t$ and its successors induced by the probability measure $Q$. By definition of set $\mathcal{P}(\lambda, \mu)$, we have:

$$q^0 S^0 = \sum_{i=1}^{n} q^i \rho^i S^i, \quad 1 - \mu \leq \rho^i \leq 1 + \lambda \quad \text{and} \quad q^i > 0 \text{ for all } i = 0, \ldots, n. \quad (13)$$

(i) We first prove that it is possible to transform $(\rho^0, \ldots, \rho^n) \in [1 - \mu; 1 + \lambda]^{n+1}$ into $(\tilde{\rho}^0, \ldots, \tilde{\rho}^n) \in [1 - \mu + \varepsilon; 1 + \lambda - \varepsilon] \times [1 - \mu; 1 + \lambda]^{n}$, for some $\varepsilon > 0$, in such a way that equation (13) remains valid. To see this, notice that there are two cases.

1) Suppose that there exist $i_1$ and $i_2$ in $\{1, \ldots, n\}$ such that $\rho^{i_1} S^{i_1} \neq \rho^{i_2} S^{i_2}$, then the result is immediate by a slight modification of $q^{i_1}$ and $q^{i_2}$; recall that $q^i$ lies in the open interval $(0, 1)$.

2) Otherwise, from the nondegeneracy condition $D[S_{t+1}|\mathcal{F}_t] > 0$ $P$-a.s., we must have $\rho^1 < \rho^n$. This implies that $\rho^1 < 1 + \lambda$ and $\rho^n > 1 - \mu$. It is then possible to increase (respectively, decrease) $\rho^0$ by increasing $\rho^1$ (respectively, decreasing $\rho^n$).

(ii) Next, we prove that it is possible to transform $(\rho^1, \ldots, \rho^n) \in [1 - \mu; 1 + \lambda]^{n}$ into $(\tilde{\rho}^1, \ldots, \tilde{\rho}^n) \in [1 - \mu + \varepsilon; 1 + \lambda - \varepsilon]^{n}$, keeping fixed $\rho^0$, for some $\varepsilon > 0$, in such a way that equation (13) remains valid. To see this, notice that there are again two cases.

1) Suppose that $\rho^i = 1 - \mu$ for all $i = 1, \ldots, n$. Then it is sufficient to increase $q^n$ and to decrease $q^1$. The case $\rho^i = 1 - \mu$ for all $i = 1, \ldots, n$ is solved by a similar argument.

2) Otherwise, there exist $i_1$ and $i_2$ such that $\rho^{i_1} > \rho^{i_2}$. Then we can decrease $\rho^{i_1}$ and increase $\rho^{i_2}$. Therefore, for any $\rho^i \in \{1 - \mu; 1 + \lambda\}$, it is possible to change its value by changing either $\rho^{i_1}$ or $\rho^{i_2}$, keeping fixed $\rho^0$.

**Step 2.** We now argue by forward induction. At time $t = 0$, we apply the transformation described in (i) of the first step, and then the second transformation as in (ii) of the first step. Next, for all $t = 1, \ldots, T - 1$, apply transformation (ii) of the first step to any atom. Notice that each transformation defines some $\varepsilon_t > 0$. Since $\Omega$ is finite, the number of transformations is necessarily finite. Therefore, the required result is obtained for the minimum value of all $\varepsilon_t$.

REFERENCES


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