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ROUGH BOUNDARY TRACE FOR SOLUTIONS OF $Lu = \psi(u)$

Dedicated to A. V. Skorokhod on his 70th birthday

Пусть $L$ — эллиптический дифференциальный оператор второго порядка на $\mathbb{R}^d$, и пусть $E$ — ограниченная область в $\mathbb{R}^d$ с гладкой границей $\partial E$. С каждым положительным решением полулинейного дифференциального уравнения $Lu = \psi(u)$ в $E$ связана пара $(\Gamma, \nu)$, где $\Gamma$ — замкнутое подмножество $\partial E$ и $\nu$ есть мера Радона на $O = \partial E \setminus \Gamma$. Мы называем эту пару грубым (rough) следом решения на $\partial E$. (В [6] был введен тонкий (thin) след, который позволяет различать решения с одинаковым грубым следом.)

Случай $\psi(u) = u^\alpha$ с $\alpha > 1$ был исследован с помощью различных методов Легаллом (Le Gall), Дынкиным и Кузнецовым, а также Маркусом и Вероном. В настоящей статье мы рассматриваем широкий класс функций $\psi$ и существенно упрощаем доказательства, содержащиеся в наших предыдущих работах.

Ключевые слова и фразы: след решения на границе, умеренные решения, выметание, устранимые и тонкие подмножества границы, стохастические граничные значения, диффузия, ранг супердиффузии.

1. Conditions on $L$ and $\psi$. Suppose that $E$ is a bounded domain of class $C^2,\lambda$ in $\mathbb{R}^d$ and

$$Lu = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_i b_i \frac{\partial u}{\partial x^i}$$

is a uniformly elliptic operator with bounded smooth coefficients. We assume that a function $\psi(x, u)$ on $\mathbb{R}^d \times \mathbb{R}_+$ has the following properties.

1.A. For every $x$, $\psi(x, \cdot)$ is convex and $\psi(x, 0) = 0$, $\psi(x, u) > 0$ for $u > 0$.
1.B. $\psi(x, u)$ is continuously differentiable.
1.C. $\psi$ is locally Lipschitz continuous in $u$ uniformly in $x$, i.e., for every $c > 0$, there exists a constant $q(c)$ such that

$$\left| \psi(x; u_1) - \psi(x; u_2) \right| \leq q(c)|u_1 - u_2| \quad \text{for all } x \in E, \ u_1, u_2 \in [0, c]. \quad (1.1)$$

1.D. There is a constant $a$ such that $\psi(x, 2u) \leq a\psi(x, u)$ for all $x$ and $u$.

We study the set $\mathcal{U}$ of all positive functions of class $C^2$ on $E$ such that

$$Lu = \psi(u) \quad \text{in } E. \quad (1.2)$$

In addition to 1.A–1.C, we suppose that all positive bounded solutions of (1.2) in any subdomain $D$ of $E$ are uniformly bounded. Keller [8] and Osserman [9] have shown that

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this property holds if a certain integral involving \( \psi \) is finite. All conditions listed in this section hold for \( \psi(x,u) = k(x) u^\alpha \) under mild conditions on \( k(x) \) if \( \alpha > 1 \).

We define a boundary trace of \( u \in \mathcal{U} \) by using two tools: (a) trace of moderate solutions; (b) sweeping of solutions.

2. Trace of moderate solutions. Two linear operators play a key role in our investigation: the Green operator \( G \) and the Poisson operator \( K \). They can be defined by probabilistic formulae

\[
Gf(x) = \Pi_x \int_0^\xi f(\xi_t) \, dt, \quad Kf(x) = \Pi_x f(\xi),
\]

where \( \xi = (\xi_t, \Pi_x) \) is the diffusion in \( E \) with the generator \( L \) and \( \xi \) is the first exit time of \( \xi \) from \( E \).

Denote by \( \mathcal{H} \) the set of all positive solutions of the linear equation \( Lu = 0 \) in \( E \). We call elements of \( \mathcal{H} \) harmonic functions.

A function \( u \in \mathcal{U} \) is called a moderate solution if \( u \leq h \) for some \( h \in \mathcal{H} \). Denote by \( \mathcal{U}_1 \) the set of all moderate solutions. We have proved in [4] that the formula

\[
u(x) = \int_{\partial E} k(x,y) \nu(dy),
\]

defines a one-to-one mapping \( i \) from \( \mathcal{U}_1 \) onto a subset \( \mathcal{H}_1 \) of \( \mathcal{H} \). We denote the inverse mapping by \( j \). For every \( u \in \mathcal{U}_1 \), \( h = j(u) \) is the minimal harmonic majorant of \( u \), and, for every \( h \in \mathcal{H}_1 \), \( u = i(h) \) is the maximal element of \( \mathcal{U}_1 \) dominated by \( h \). Clearly, mappings \( i \) and \( j \) are monotone. Moreover, \( i(h_1 + h_2) \leq i(h_1) + i(h_2) \) (see Section 4.3 in [6]).

Every \( h \in \mathcal{H} \) has a representation

\[
u(x) = \int_{\partial E} k(x,y) \nu(dy),
\]

where \( k(x,y) \) is the Poisson kernel and \( \nu \) is a finite measure on \( \partial E \). Formula (2.2) establishes a one-to-one correspondence between the set of all finite measures on \( \partial E \) and \( \mathcal{H} \).

If \( \nu \) and \( h \) are related by formula (2.2), then we write \( h = K \nu \). Put \( \nu \in \mathcal{M}_1 \) if \( K\nu \in \mathcal{H}_1 \).

We say that \( \nu \) is the trace of a moderate solution \( u \) if

\[
u = u + G\psi(u)
\]

Formula (2.3) defines a one-to-one correspondence between \( \mathcal{M}_1 \) and \( \mathcal{U}_1 \).

3. Sweeping. It is proved in [6] that \( \mathcal{U} \) with the partial order \( \leq \) is a complete lattice that is, for every subset \( C \) of \( \mathcal{U} \) there exist \( \text{Sup} C \) and \( \text{Inf} C \) in \( \mathcal{U} \) (writing \( v = \text{Sup} C \) means that: (i) \( v \in \mathcal{U} \); (ii) \( u \leq v \) for all \( u \in \mathcal{U} \); (iii) if \( w \in \mathcal{U} \) and \( u \leq w \) for all \( u \in \mathcal{U} \), then \( u \leq w \); writing \( v = \text{Inf} C \) has a similar meaning). For every pair \( u, v \in \mathcal{U} \), we put \( u \vee v = \text{Sup}\{u,v\}, u \wedge v = \text{Inf}\{u,v\} \). We proved in [6] that, if \( C \subset \mathcal{U} \) is closed under \( \vee \), then there exists a sequence \( u_n \in C \) such that \( u_n(x) \uparrow v(x) \) for all \( x \in E \). We have \( v(x) = \text{Sup}_C u(x) \) for all \( x \in E \).

We use this result to introduce a sweeping \( Q_B(u) \) of \( u \in \mathcal{U} \) to a closed subset \( B \) of \( \partial E \). We put

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u = u + G\psi(u)
\]
4. Rough trace. Now we are prepared to define a rough trace of an arbitrary \( u \in \mathcal{U} \). We say that a compact set \( B \subset \partial E \) is \textit{moderate} for \( u \) if the solution \( u_B = Q_B(u) \) is moderate. Let \( \nu_B \) stand for the trace of \( u_B \). By 3.D, union of two moderate sets is moderate. Suppose that \( B \) is moderate and let \( \tilde{B} \subset B \). By 3.G, \( \tilde{B} \) is moderate and \( \nu_{\tilde{B}} \) is the restriction of \( \nu_B \) to \( \tilde{B} \).

A relatively open subset \( A \) of \( E' \) is called \textit{moderate} if all compact subsets of \( A \) are moderate. The union \( O \) of all moderate open sets is moderate. Clearly, there exists a unique measure \( \nu \) on \( O \) such that its restriction to an arbitrary compact subset \( B \) coincides with \( \nu_B \).

The measure \( \nu \) has the property: for every compact \( B \subset O \) the restriction of \( \nu \) to \( B \) belongs to the class \( \mathcal{K}_1 \). We denote by \( \mathcal{K}_1(O) \) the class of measures with this property. We call closed set \( \Gamma = \partial E \setminus O \) the \textit{special set of the solution} \( u \) and we call pair \((\Gamma, \nu)\) the \textit{rough trace} of \( u \) on \( \partial E \).

5. Solutions \( u_\nu \) and \( w_B \). The class \( \mathcal{U} \) is closed under pointwise convergence. Moreover, the class of solutions vanishing on a relatively open subset \( O \) of \( \partial E \) is also closed under pointwise convergence (see, for instance [2, Theorem 1.2]). Suppose that \( B_n \) are closed and \( \bigcup B_n \subset O \).

If \( u \in \mathcal{K}_1(O) \) and if \( \nu_n \) is the restriction of \( \nu \) to \( B_n \), then \( \nu_n \in \mathcal{K}_1 \) and \( u_{\nu_n} \) is an increasing sequence. We denote its limit by \( u_\nu \). It is easy to see that it does not depend on the choice of \( B_n \) and that \( u_\nu \leq u_\nu' \) if and only if \( \mu \leq \mu' \).

For every closed \( B \subset \partial E \), we put

\[
\nu_B = \sup \{ u \in \mathcal{U} : u \equiv 0 \text{ on } \partial E \setminus B \}.
\]

Note that, for every \( x \in E \), \( \nu_B(x) \) is equal to the supremum of \( u(x) \) over all \( u \in \mathcal{U} \). We have:

5.A. \( \nu_{B_1} \leq \nu_{B_2} \) if \( B_1 \subset B_2 \).
5.B. \( Q_B(u) \leq \nu_B \) for all \( u \in \mathcal{U} \).
5.C. \( Q_B(\nu_B) = \nu_B \).
5.D. \( \nu_{B_1 \cup B_2} \leq \nu_{B_1} + \nu_{B_2} \).
5.E. If \( B_n \subset B \), then \( \nu_{B_n} \leq \nu_B \).

5.A and 5.B follow from (3.1). By 5.B, \( Q_B(\nu_B) \leq \nu_B \). On the other hand, \( v = \nu_B \) satisfies conditions \( v \leq \nu_B, v \equiv 0 \) on \( \partial E \setminus B \) and \( Q_B(\nu_B) \) is a maximal solution with these properties. Hence, \( \nu_B \leq Q_B(\nu_B) \) which proves 5.C. To prove 5.D, we put \( B = B_1 \cup B_2 \) and we note that, by 5.C, 3.D and 5.B,

\[
\nu_B = Q_B(\nu_B) \leq Q_B(\nu_{B_1} + \nu_{B_2}) \leq \nu_{B_1} + \nu_{B_2}.
\]

Let us prove 5.E. Function \( \nu_n = \nu_{B_n} \) is a maximal element of \( \mathcal{U} \) vanishing on \( O_n = \partial E \setminus B_n \). By 5.A, \( \nu_n \downarrow \nu \geq \nu_B \). The limit \( \nu \) is a solution equal to 0 on \( O = \partial E \setminus B \). Hence, \( \nu \leq \nu_B \).

6. Removable and thin boundary sets. We say that a compact set \( B \subset \partial E \) is \textit{removable} if 0 is the only element of \( \mathcal{U} \) vanishing on \( \partial E \setminus B \). (In the literature, such sets are called removable boundary singularities for solutions of (2.1).) Clearly, \( B \) is removable if and only if \( \nu_B = 0 \). A set \( A \) is called \textit{thin} if all its compact subsets are removable.\(^2\) It follows from 5.A and 5.D that:

6.A. All closed subsets of a removable set are removable and all subsets of a thin set are thin.
6.B. The class of all removable sets and the class of all thin sets are closed under the finite unions.

We also have:

6.C. All thin Borel sets are not charged by any \( \nu \in \mathcal{K}_1 \).

\textit{Proof.} Since \( \mathcal{K}_1 \) contains, with every \( \nu \) its restriction to any \( B \), it is sufficient to show that, if \( \nu \in \mathcal{K}_1 \) is concentrated on a removable compact set \( B \), then \( \nu = 0 \). The Poisson kernel \( k(x,y) \) is bounded on every set \( \{|x-y| > \epsilon\}, \epsilon > 0 \), and it tends to 0 as

\(^2\) If \( X \) is a \((L, \psi)\)-superdiffusion, then, by Theorem 1.2 in [5], a compact set \( B \) is removable if and only if \( \mathbb{P}_x\{A \cap B \neq \emptyset\} = 0 \), where \( A \) is the range of \( X \).
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$y \rightarrow \tilde{y} \in \partial E, \tilde{y} \neq y$. Therefore $h = K\nu = 0$ on $\partial E \setminus B$. The solution $i(h)$ satisfies the same condition because $i(h) \leq h$. Therefore $i(h) \leq \nu_B$. If $\nu_B = 0$, then $i(h) = 0$. Thus $h = 0$ and $\nu = 0$.

6.D. If compact $B \subset \Gamma$ and if $Q_B(w_T)$ is moderate, then $B$ is removable.

Proof. By 5.A, $\nu_B \leq w_T$ and, by 5.C and 3.A, $\nu_B = Q_B(w_B) \leq Q_B(w_T)$. If $Q_B(w_T)$ is moderate, then so is $\nu_B$. Hence, $\nu_B = i(h)$ for some $h \in \mathcal{H}$. By Theorem 4.3 in [6], $\nu_B = 2h \leq 2i(h)$, we have $v \leq 2w_B$. Hence, $v = 0$ on $\partial E \setminus B$ which implies that $v \leq \nu_B$. By the monotonicity of $j$, $2h = j(v) \leq j(\nu_B) = h$. Therefore $h = 0$ and $w_T = i(h) = 0$.

6.E. Suppose $\Gamma$ is removable and $B_n = \{x \in \partial E: d(x, \Gamma) \geq \varepsilon_n\}$. If $\varepsilon_n \downarrow 0$, then $Q_B(u) \uparrow u$ for every $u \in \mathcal{U}$.

Proof. Put $\Gamma_n = \{y \in \partial E: d(y, \Gamma) \leq 2\varepsilon_n\}$. Note that $\Gamma_n \cup B_n = \partial E$ and $\Gamma_n \downarrow \Gamma$. By 3.E, 3.D and 5.D, $u = Q_{\partial E}(u) \leq Q_{\Gamma_n}(u) + Q_B(u)$. By 5.B and 5.E, $Q_{\Gamma_n}(u) \leq \nu_T \downarrow \nu_T = 0$.

7. Principal results. We say that $x$ is an explosion point of a measure $\nu$ and we write $x \in \text{Ex}(\nu)$ if $\nu(U) = \infty$ for every neighborhood $U$ of $x$. If $\nu(B) < \infty$ and if $B$ is compact, then $\nu(B) < \infty$. Note that $O \cap \text{Ex}(\nu) = \emptyset$ for every measure $\nu \in \mathcal{H}(O)$.

We say that $(\Gamma, \nu)$ is a normal pair if:

(a) $\Gamma$ is a closed subset of $\partial E$;

(b) $\nu \in \mathcal{H}(O)$, where $O = \partial E \setminus \Gamma$;

(c) the conditions

\[ \Lambda \subset \Gamma \text{ is thin and contains no explosion points of } \nu, \]

\[ \Gamma \setminus \Lambda \text{ is closed imply that } \Lambda = \emptyset. \tag{7.1} \]

We demonstrate that these conditions hold for the trace of an arbitrary solution $u$.

First, we prove a few auxiliary propositions.

7.A. If $u \in \mathcal{U}$ vanishes on a compact set $B \in \partial E$, then $Q_B(u) = 0$.

Indeed, by (3.1), $v = Q_B(u) = 0$ on $\partial E \setminus B$ and $v \leq u$. Hence, $v = 0$ on $\partial E$, and $v = 0$ by the comparison principle (see, e.g., [1, p. 113]).

7.B. Suppose that $\text{tr}(u) = (\Gamma, \nu)$. If $u = 0$ on an open subset $O_1$, then $O_1 \subset \Gamma = \emptyset$ and $\nu(O_1) = 0$.

Indeed, by 7.A, $Q_B(u) = 0$ for all compact subsets of $O_1$.

7.C. Let $\text{tr}(u) = (\Gamma, \nu)$. If $\Gamma$ is removable and $\nu$ is finite, then $u$ is moderate.

To prove this, we apply 6.E. Note that $h = K\nu \in \mathcal{E}$ and $u_n = Q_B(u) \leq K\nu \leq h$.

By 6.E, $u_n \uparrow u$. Hence, $u \leq h$.

Theorem 7.1. The trace of an arbitrary solution $u$ is a normal pair.

Proof. Properties (a) and (b) follow immediately from the definition of the trace. Let us prove (c). Suppose that $\Lambda$ satisfies the conditions (7.1) and let $\Gamma_0 = \Gamma \setminus \Lambda$. Theorem will be proved if we show that $v = Q_B(u)$ is moderate for every closed subset $B_1$ of $\Gamma_0 = \partial E \setminus \Gamma_0$. Indeed, this implies $O_0 \subset O$ and therefore $\Gamma_0 \subset \Gamma$, $\Lambda = \emptyset$.

Let $(\Gamma_1, \nu_1)$ be the trace of $u$. By 7.C, it is sufficient to prove that $\Gamma_1$ is removable and $\nu_1$ is finite.

By 3.B, $v \leq u$ and, by 3.A, all moderate sets for $u$ are also moderate for $v$. Hence $\Gamma_1 \subset \Gamma$. By 7.B, $\partial E \setminus B_1 \subset \partial E \setminus \Gamma_1$ because $v = 0$ on $\partial E \setminus B_1$. Hence, $\partial E \setminus B_1$ is moderate for $v$ and it is contained in $O_1 = \partial E \setminus \Gamma_1$. We conclude that $\Gamma_1 \subset B_1 \cap \Gamma \subset \Lambda$. Hence, $\Gamma_1$ is removable.

Note that $B_1 \subset O \cup \Lambda$ does not contain explosion points of $\nu$ and therefore $\nu(B_1) < \infty$. Since $Q_B(v) = 0$ for $B \cap B_1 = \emptyset$, measure $\nu_1$ vanishes on $\partial E \setminus B_1$. Since $v \leq u$, $v_1 \leq \nu$ on $O \subset O_1$. We have

\[ \nu_1(O_1) = \nu_1(O_1 \cap B_1) = \nu_1((O_1 \setminus O) \cap B_1) + \nu_1(O \cap B_1) < \nu(O \cap B_1) < \infty \]

because $\Gamma \cap B_1$ is removable and therefore $\nu_1(\Gamma \cap B_1) = 0$ by 6.C. Theorem 7.1 is proved.

For every $u, v \in \mathcal{U}$ we put $u \odot v = \Sup\{u \in \mathcal{U}: u \leq u + v\}$. Note that $u_\mu \odot u_\nu = u_{\mu + \nu}$.

3) Here is the only place we use the property 1.D of $\psi$. 
Theorem 7.2. If $(\Gamma, \nu)$ is a normal pair, then $u = w_\Gamma \oplus u_\nu$ is a solution with the trace $(\Gamma, \nu)$. Moreover, $u$ is maximal among such solutions.

Proof. 1°. Let $B \subset O = \partial E \setminus \Gamma$. We claim that $Q_B(u) = u_{\nu_B}$, where $\nu_B$ is the restriction of $\nu$ to $B$. Indeed, $B$ is contained in an open subset $O_1$ of $\partial E$ such that $\overline{O}_1 \subset O$. Let $\nu_1$ and $\nu_2$ be the restrictions of $\nu$ to $O_1$ and to $O \setminus O_1$. Note that $u_\nu = u_{\nu_1} \oplus u_{\nu_2}$. By 3.D, $Q_B(u) = Q_B(u_{\nu_1}) + Q_B(u_{\nu_2})$. Since $\nu_2$ does not charge a neighborhood of $B$, $u_{\nu_2}(B) = 0$. The same is true for $w_B$. By 7.A, $Q_B(w_B) = Q_B(u_{\nu_2}) = 0$. Hence, $Q_B(u) = Q_B(u_{\nu_1})$ and, by 3.A and 3.G, $Q_B(u) = Q_B(u_{\nu_1}) = u_{\nu_B}$.

2°. Denote the trace of $u$ by $(\Gamma_0, \nu_0)$. It follows from 1° that $O \subset O_0$ and $\nu = \nu_0$ on $O_0$. Since $\nu$ is concentrated on $O$, we have $\nu \leq \nu_0$ and therefore $\operatorname{Ex}(\nu) \subset \operatorname{Ex}(\nu_0) \subset \Gamma_0 \subset \Gamma$. Every compact $B \subset \Lambda = \Gamma \setminus \Gamma_0$ is moderate for $u$ (because $B \subset O_0$) and it is removable by 6.D. Thus, $\Lambda$ is thin. Since $\Lambda \cap \operatorname{Ex}(\nu) = \emptyset$ and $\Gamma \setminus \Lambda = \Gamma_0$ is closed, $\Lambda = \emptyset$ by the definition of a normal pair.

3°. Let us show that an arbitrary solution $v$ with the trace $(\Gamma, \nu)$ is dominated by $u = w_\Gamma \oplus u_\nu$. Since $u$ is the maximal element of $\mathcal{U}$ dominated by $w_\Gamma + u_\nu$, it is sufficient to show that $v \leq w_\Gamma + u_\nu$. Consider compact sets $B_n$ and $\Gamma_n$ such that $B_n \uparrow O$, $\Gamma_n \downarrow \Gamma$ and $B_n \cup \Gamma_n = \partial E$ for all $n$. By 3.E and 3.D, \begin{equation}

v = Q_{\partial E}(v) = Q_{B_n}(v) + Q_{\Gamma_n}(v). \tag{7.2}

\end{equation}

Let $\nu_n$ be the restriction of $\nu$ to $B_n$. Note that $Q_{B_n}(v) = u_{\nu_n} \leq u_\nu$. By 5.B, $Q_{\Gamma_n}(v) \leq w_{\Gamma_n}$ and, by 5.E, $w_{\Gamma_n} \downarrow w_\Gamma$. Therefore (7.2) implies that $v \leq u_\nu + w_\Gamma$. Theorem 7.2 is proved.

Remark. If $X$ is a $(L, \psi)$-superdiffusion, then \begin{equation}

(w_\Gamma \oplus u_\nu)(x) = -\ln P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z_\nu}\}, \tag{7.3}

\end{equation}

where $\mathcal{R}$ is the range of $X$ and $Z_\nu$ is the stochastic boundary value of $h = K\nu$ (see [3]).

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