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Time change representation of stochastic integrals


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TIME CHANGE REPRESENTATION OF STOCHASTIC INTEGRALS

By the Dambis and Dubins–Schwarz theorem, any stochastic integral $M = (\int_0^T H_s \, dW_s)_{t \in \mathbb{R}_+}$ with respect to a Brownian motion may be written as a Brownian motion with random time change, i.e.,

$M = (\tilde{W}_t)_{t \in \mathbb{R}_+}$

for some Brownian motion $(\tilde{W}_t)_{t \in \mathbb{R}_+}$ and some random time change $(\tilde{T}_t)_{t \in \mathbb{R}_+}$. In [7] and [5] it was shown that this statement is valid for Brownian motion with symmetric $\alpha$-stable noise. We use the cumulant process to give short new proofs. In addition, we show that this statement cannot be extended to other processes of Lévy.

Keywords and phrases: stable Lévy processes, cumulant process, stochastic integral, time change.

1. Time change representations. We generally use the notation of [2]–[4]. The transposed of a vector or matrix $X$ is denoted as $X^T$ and its components are denoted by superscripts. Stochastic and Stieltjes integrals are written as $\int H_s \, dX_s$.

Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ be a filtered probability space as in [4, Definition 1.1.2]. By a Lévy process we refer to a process with stationary independent increments in the sense of [4, II.4.1]. Similar to [8, Definition 7.5.1], we make the following

Definition 1. Let $\alpha \in (0, 2]$. An $\alpha$-stable Lévy motion is a Lévy process $X$ such that $X_1$ (or equivalently any $X_t$) has a strictly $\alpha$-stable distribution (i.e., $X_1 \sim S_\alpha(\sigma, \beta, \mu)$ for some $\alpha \in (0, 2] \setminus \{1\}$, $\sigma \in \mathbb{R}_+$, $\beta \in [-1, 1]$, $\mu = 0$ or $\alpha = 1$, $\sigma \in \mathbb{R}_+$, $\beta = 0$, $\mu \in \mathbb{R}$). We call $X$ a symmetric $\alpha$-stable Lévy motion if the distribution of $X_1$ (or equivalently any $X_t$) is even symmetric $\alpha$-stable (i.e., $X_1 \sim S_\alpha(\sigma, 0, 0)$ for some $\alpha \in (0, 2]$, $\sigma \in \mathbb{R}_+$).

Definition 2. 1. A time change is a right-continuous increasing $[0, \infty)$-valued process $(T_\theta)_{\theta \in \mathbb{R}_+}$ such that $T_\theta$ is a stopping time for any $\theta \in \mathbb{R}_+$. It is called finite if $T_\theta < \infty$ almost surely for any $\theta \in \mathbb{R}_+$.

2. By $\tilde{\mathcal{F}}_\theta := \mathcal{F}_{T_\theta}$ we define the time-changed filtration $(\tilde{\mathcal{F}}_\theta)_{\theta \in \mathbb{R}_+}$.

3. The inverse time change $(\tilde{T}_t)_{t \in \mathbb{R}_+}$ is defined as $\tilde{T}_t := \inf\{\theta \in \mathbb{R}_+: T_\theta > t\}$.

4. A process $X$ is called $(T_\theta)_{\theta \in \mathbb{R}_+}$-adapted if $X$ is constant on $[T_\theta, T_\theta^+]$ for any $\theta \in \mathbb{R}_+$.

Remark. If $X$ is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ and $(T_\theta)_{\theta \in \mathbb{R}_+}$ is a finite time change, then the process $(\tilde{X}_\theta)_{\theta \in \mathbb{R}_+}$ defined by $\tilde{X}_\theta := X_{T_\theta}$ is a semimartingale.

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on \( \Omega, \mathcal{F}, (\bar{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P} \) (cf., e.g., [2, (10.12)]). Many other properties are generally only preserved if \( X \) is \((\bar{T}_t)_{t \in \mathbb{R}_+}\)-adapted. For details cf. [2, Chapter X].

We are concerned with the following representation property which follows from the Dambis and Dubins-Schwarz theorem (cf. [6, Theorem V.1.6]).

**Theorem 1.** Let \( X \) be a standard Brownian motion. Moreover, let \( M := H \cdot X \) for some \( H \in L(X) \) such that \( \int_0^t H_s^2 \, ds \to \infty \) for \( t \to \infty \). Then there exist a filtration \((\bar{F}_t)_{t \in \mathbb{R}_+}\) on \( (\Omega, \mathcal{F}) \), a process \((\bar{W}_t)_{t \in \mathbb{R}_+}\), and a finite time change \((\bar{t}_t)_{t \in \mathbb{R}_+}\) such that

1. \( \bar{W} \) is a \((\bar{t}_t)_{t \in \mathbb{R}_+}\)-adapted Brownian motion on \((\Omega, \mathcal{F}, (\bar{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\),
2. \( M = (\bar{W}_{\bar{t}_t})_{t \in \mathbb{R}_+} \).

If we define the \((\Omega, \mathcal{F}, (\bar{t}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\)-time change \((\bar{T}_t)_{t \in \mathbb{R}_+}\) by \( T_\theta := \inf\{t \in \mathbb{R}_+: \int_0^t H_s^2 \, ds > \theta\} \), then we may choose \( \bar{F}_\theta = \mathcal{F}_{T_\theta} \), \( \bar{W}_\theta = M_{T_\theta} \) and \( \bar{T}_t \) as the inverse time change of \( T \). In particular, \( \bar{t}_t = \int_0^t H_s^2 \, ds \) for \( t \in \mathbb{R}_+ \).

The previous theorem can be generalized to symmetric \( \alpha \)-stable Lévy motions (cf. [7] and [5]).

**Theorem 2.** Let \( X \) be a symmetric \( \alpha \)-stable Lévy motion. Moreover, let \( M := H \cdot X \) for some \( H \in L(X) \) such that \( \int_0^t H_s^\alpha \, ds \to \infty \) for \( t \to \infty \). Then there exist a filtration \((\bar{F}_t)_{t \in \mathbb{R}_+}\) on \((\Omega, \mathcal{F})\), a process \((\bar{W}_t)_{t \in \mathbb{R}_+}\), and a finite time change \((\bar{t}_t)_{t \in \mathbb{R}_+}\) such that

1. \( \bar{W} \) is a \((\bar{t}_t)_{t \in \mathbb{R}_+}\)-adapted Lévy process on \((\Omega, \mathcal{F}, (\bar{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\) with \( \text{Law}(\bar{W}) = \text{Law}(X) \)
2. \( M = (\bar{W}_{\bar{t}_t})_{t \in \mathbb{R}_+} \).

If we define the \((\Omega, \mathcal{F}, (\bar{t}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\)-time change \((\bar{T}_t)_{t \in \mathbb{R}_+}\) by \( T_\theta := \inf\{t \in \mathbb{R}_+: \int_0^t H_s^\alpha \, ds > \theta\} \), then we may choose \( \bar{F}_\theta = \mathcal{F}_{T_\theta} \), \( \bar{W}_\theta = M_{T_\theta} \) and \( \bar{T}_t \) as the inverse time change of \( T \). In particular, \( \bar{t}_t = \int_0^t H_s^\alpha \, ds \) for \( t \in \mathbb{R}_+ \).

**Proof.** First step. Obviously, \((\bar{T}_t)_{t \in \mathbb{R}_+}\) is a finite time change (cf. [4, 1.1.28]). Choose \( A_t = t \) for any \( t \in \mathbb{R}_+ \). Let \( q \in \mathbb{Q}_+ \) and \( \tau_q := \inf\{t \geq q: \int_0^t H_s^\alpha \, ds > 0\} \). Note that \( H = 0 \) \( \mathbb{P} \)-almost everywhere on \([q, \tau_q]\), which implies that \((\mathbb{P} \otimes A)\cdot([q, \tau_q]) = 0 \). By Lemma 3, \( 1_{[q, \tau_q]} \cdot M = (1_{[q, \tau_q]}H) \cdot X \) is a semimartingale with characteristics \((0,0,0)\) and hence it is 0 up to indistinguishability. Therefore, \( M \) is constant on \([q, \tau_q]\) outside some \( \mathbb{P} \)-null set. Since \((T_{\theta_-}, T_{\theta}) = \bigcup([q, \tau_q]: q \in \mathbb{Q}_+ \text{ with } T_{\theta_-} < q \leq T_{\theta})\), we have that \( M \) is constant on any interval \((T_{\theta_-}, T_{\theta})\), \( \theta \in \mathbb{R}_+ \). By right-continuity, it follows that \( M = (\bar{T}_t)_{t \in \mathbb{R}_+} \)-adapted.

Second step. Let \( U \in L(X) \). The cumulant process of \( X \) is of the form \( \mathcal{X}^U(X) = \kappa(U) \cdot A \), where \( \kappa(\cdot) \) is given by \( \kappa(u) = \gamma f(e^{iu} - 1 - iux1_{[-1,1]}(x))|x|^{-(1+\alpha)} \) for \( \alpha \neq 2 \) and \( \kappa(u) = -\gamma u^2 \) for \( \alpha = 2 \) (with some constant \( \gamma \in \mathbb{R}_+ \), cf., e.g., [8, p. 6] and Lemma 4). Straightforward calculations show that \( \kappa(|u|^{\alpha}) = |u|^\alpha \kappa(u) \) for any \( u, v \in \mathbb{R} \).

Third step. Let \( u \in \mathbb{R} \). If \( \mathcal{X}^M, \mathcal{X}^\bar{L} \) denote the cumulant processes of \( M \) and \( \bar{L} \), respectively, Lemmas 2 and 5 yield that \( \mathcal{X}^M(u) = \mathcal{X}^X(uH) \) and \( \mathcal{X}^\bar{L}(u) = \mathcal{X}^X(uH) \). Together, it follows that

\[
\mathcal{X}^\bar{L}(u) = \mathcal{X}^M(u)T_\theta = \mathcal{X}^X(uH)T_\theta = \kappa(uH) \cdot A T_\theta = \kappa(u)|H|^\alpha \cdot A T_\theta
\]

for any \( \theta \in \mathbb{R}_+ \). From Lemma 4 and [4, II.2.42, II.4.19], it follows that \( \bar{L} \) is a Lévy process with the same distribution as \( X \).

Fourth step. By [2, (10.2)] and Lemma 6, \( \bar{T} \) is a finite time change and \( \bar{t}_t = \int_0^t |H_s|^\alpha \, ds \) for \( t \in \mathbb{R}_+ \). The continuity of \( \bar{T} \) implies that \( \bar{L} \) is \((\bar{t}_t)_{t \in \mathbb{R}_+}\)-adapted. Since \( t \in [T_{\theta_-}, T_{\theta}] \) and \( M = (\bar{T}_t)_{t \in \mathbb{R}_+} \)-adapted, it follows that \( \bar{L}_{\bar{t}_t} = M_{T_{\theta}T_{\bar{t}_t}} = M_t \) for \( t \in \mathbb{R}_+ \).
If we consider only nonnegative integrands \( H \), we can extend the statement to asymmetric \( \alpha \)-stable Lévy motions (including deterministic linear functions).

**Theorem 3.** Let \( X \) be an \( \alpha \)-stable Lévy motion. Moreover, let \( M := H \cdot X \) for some nonnegative \( H \in L(X) \) such that \( \int_0^\infty H_d \) is finite for \( t \to \infty \). Then there exist a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) on \((\Omega, \mathcal{F})\), a process \((\bar{L}_t)_{t \in \mathbb{R}_+}\), and a finite time change \((\bar{T}_t)_{t \in \mathbb{R}_+}\) such that

1. \( \bar{L} \) is a \((\bar{T}_t)_{t \in \mathbb{R}_+}\)-adapted Lévy process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\) with \( \text{Law}(\bar{L}) = \text{Law}(X) \) and
2. \( M = (\int_0^t H_s \, ds)_{t \in \mathbb{R}_+} \).

If we define the \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\)-time change \((T_t)_{t \in \mathbb{R}_+}\) by \( T_t := \inf\{s \in \mathbb{R}_+ : \int_0^s H_s \, ds \geq t\} \), then we may choose \( \mathcal{F}_t = \mathcal{F}_{T_t} \), \( L_t = M_{T_t} \) and \( \bar{T} \) as the inverse time change of \( T \). In particular, \( \bar{T}_t = \int_0^t H_s \, ds \) for \( t \in \mathbb{R}_+ \).

**Proof.** This is shown along the same lines as the previous theorem. Only the second step has to be modified slightly. Here,

\[
\kappa(u) = \gamma_1 \int_{(0, \infty)} (e^{iuX} - 1)x^{-(1+\alpha)} \, dx \\
+ \gamma_2 \int_{(-\infty, 0)} (e^{iuX} - 1)|x|^{-(1+\alpha)} \, dx, \quad \alpha < 1,
\]

\[
\kappa(u) = iu \mu + \gamma_1 \int (e^{iuX} - 1 - iux 1_{[-1, 1]}(x)) |x|^{-(1+\alpha)} \, dx, \quad \alpha = 1,
\]

\[
\kappa(u) = \gamma_1 \int_{(0, \infty)} (e^{iuX} - 1 - iux) x^{-(1+\alpha)} \, dx + \gamma_2 \int_{(-\infty, 0)} (e^{iuX} - 1 - iux) |x|^{-(1+\alpha)} \, dx, \quad 1 < \alpha < 2,
\]

\[
\kappa(u) = -\gamma_1 u^2, \quad \alpha = 2
\]

(with some constants \( \gamma_1, \gamma_2 \in \mathbb{R}_+, \mu \in \mathbb{R} \), cf. [9, Theorem 14.7] or [10, III.1.c, Theorem 3]). Again, straightforward calculations show that \( \kappa(uv) = v^\alpha \kappa(u) \) for any \( u, v \in \mathbb{R}_+ \).

The following result shows that there are no other Lévy processes such that an analogous statement holds.

**Theorem 4.** Suppose that \( X \) is a real-valued Lévy process with the following property: For any \( H \in \mathbb{R} \setminus \{0\} \) there exist a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) on \((\Omega, \mathcal{F})\), a process \((\bar{L}_t)_{t \in \mathbb{R}_+}\), and a finite time change \((\bar{T}_t)_{t \in \mathbb{R}_+}\) such that

1. \( \bar{L} \) is a \((\bar{T}_t)_{t \in \mathbb{R}_+}\)-adapted Lévy process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})\) with \( \text{Law}(\bar{L}) = \text{Law}(X) \) and
2. \( HX = (\int_0^t H_s \, ds)_{t \in \mathbb{R}_+} \).

Then \( X \) is a symmetric \( \alpha \)-stable Lévy motion.

If we claim the above property only for any \( H \in (0, \infty) \), it follows that \( X \) is an \( \alpha \)-stable Lévy motion.

**Proof.** Let \( H \in (0, \infty) \) and \( u \in \mathbb{R} \). Note that the cumulant processes of Lévy processes are deterministic, linear in time, and characterized by the Lévy–Khinchine triplet. By Lemmas 2 and 5, the cumulant process of \( HX \) is of the form

\[
\kappa(u) A_t = \mathcal{X}_X(uH)_t = \mathcal{X}_H(u) \bar{T}_t = \kappa(u) A_{\bar{T}_t}
\]

if we set \( A_t = t \) and \( \mathcal{X}_X(u) = \kappa(u) \cdot A \) denotes the cumulant process of \( X \) and \( \bar{L} \). The process \((A_{\bar{T}_t})_{t \in \mathbb{R}_+}\) can be written as \( A_{\bar{T}_t} = a \cdot A_t + N_t \) for some \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)-predictable, nonnegative process \( a \) and some increasing process \( N \) which is orthogonal to Lebesgue measure (cf. [4, 1.3.13]). Since the left-hand side of equation (1) is absolutely continuous with respect to Lebesgue measure, it follows that \( \kappa(uH) \cdot A = (\kappa(u)a) \cdot A \), which implies that \( \kappa(uH) \) is a multiple of \( \kappa(u) \), say \( \kappa(uH) = f(H)\kappa(u) \) with some function \( f : (0, \infty) \to (0, \infty) \). Note that \( f(H H') = f(H) f(H') \) and \( f \) is continuous because of the continuity.
of \( \kappa \). Therefore, \( \ln(f(\exp(-))) \) is a linear function, hence \( f(H) = H^\alpha \) for some \( \alpha \in \mathbb{R} \). Note that \( \alpha = 0 \) is only possible for \( \kappa = 0 \), in which case \( X = 0 \). Suppose that \( \alpha 
eq 0 \).

**Second step.** Let \( k_1, k_2 \in (0, \infty) \). Since \( u \mapsto \exp(\kappa(u)) \) is the characteristic function of \( X_1 \), we have that

\[
\left( p^{k_1}X_1 + p^{k_2}X_1 \right)^\wedge(u) = \exp\left( \kappa(uk_1) + \kappa(uk_2) \right) = \exp\left( (k_1^\alpha + k_2^\alpha)\kappa(u) \right) = \exp\left( \kappa(u(k_1^\alpha + k_2^\alpha)^{-\alpha}X_1) \right)^\wedge(u),
\]

where \( (p^Y)^\wedge \) denotes the characteristic function of \( Y \). It follows that \( X_1 \) is strictly stable.

**Third step.** If the property in the theorem holds for negative \( H \), the same reasoning as above yields that \( \kappa(uf(-1)) = f(-1)\kappa(-u) \) for some \( f(-1) \in \mathbb{R} \) that does not depend on \( u \). From \( \kappa(uf(-1)) = f(-1)\kappa(-u) = (f(-1))^2\kappa(u) \) it follows that \( f(-1) = 1 \). This implies that \( (p^{X_1})^\wedge(u) = (p^{X_1})^\wedge(-u) \), which in turn yields that \( \operatorname{Law}(X_1) = \operatorname{Law}(-X_1) \). Hence, \( X_1 \) is symmetric stable.

It is an open question whether the previous theorem still holds if we remove the constraint that \( \tilde{L} \) is \( (\tilde{T}_t)_{t \in \mathbb{R}_+} \)-adapted.

### 2. Tools from stochastic calculus.

**Lemma 1.** Let \( Z \) be a semimartingale such that \( Z_-, Z_- \in (C \setminus \{0\}) \)-valued. Then there exists an up to indistinguishability unique \( C \)-valued semimartingale \( X \) such that \( X_0 = 0 \) and \( Z = Z_0 \mathcal{G}(X) \). It is given by \( X = (1/Z_-) Z \).

**Proof.** The existence and explicit form of \( X \) follows from \( 1 + (Z_-/Z_0) \cdot ((1/Z_-) \cdot Z) = 1 + (Z_-/(1/Z_-)) \cdot Z/Z_0 = Z/Z_0 \). Now let \( X \) be any semimartingale such that \( Z_0 = 0 \) and \( Z = Z_0 \mathcal{G}(X) \). Then \( X = ((1/Z_-) Z_-) \cdot X = (1/Z_-) \cdot (Z_- \cdot X) = (1/Z_-) \cdot Z \), which yields the uniqueness.

**Definition 3.** We call the process \( X \) in the previous lemma **stochastic logarithm** of \( Z \) and write \( \mathcal{L}(Z) := X \).

Let \( X \) be a \( \mathbb{R}^d \)-valued semimartingale whose characteristics \( (B, C, \nu) \) relative to some truncation function \( h: \mathbb{R}^d \to \mathbb{R}^d \) are given in the form \( B = b \cdot A, C = c \cdot A, \nu = A \otimes F \), where \( A \in \mathfrak{M}_{uc}^+ \) is a predictable process, \( b \) is a predictable \( \mathbb{R}^d \)-valued process, \( c \) is a predictable \( \mathbb{R}^d \times \mathbb{R}^d \)-valued process whose values are nonnegative, symmetric matrices, and \( F \) is a transition kernel from \((\Omega \times \mathbb{R}_+, \mathcal{F})\) into \((\mathbb{R}^d, \mathcal{B}^d)\). By [4, Proposition II.2.9] such a representation always exists.

**Definition 4.** Suppose \( \mathbb{R}^d \)-valued process \( U \in L(X) \). The **cumulant process** \( \mathcal{X}(U) \) of \( U \) in \( U \) is defined as the predictable part of finite variation in the canonical decomposition of the special semimartingale \( \mathcal{L}(\exp(iU \cdot X)) \) (in the sense of [4, 1.4.22]).

**Remark.** Note that \( \exp(iU \cdot \mathcal{L}(X)) \) is bounded and hence is a complex special semimartingale. Therefore, \( \mathcal{L}(\exp(iU \cdot X)) = \exp(-iU \cdot X) \cdot \exp(iU \cdot X) \) is a special semimartingale as well (cf. [2, (2.51)])

**Lemma 2.** Let \( H \in L(X) \) and \( U \in L(H^T \cdot X) \). Then \( \mathcal{X}^H \cdot \mathcal{X}(U) = \mathcal{X}^X(UH) \).

**Proof.** From \( H \in L(X), U \in L(H^T \cdot X) \), it follows that \( UH \in L(X) \) (cf. [1, Proposition 5.1]). The assertion is now readily obtained from \( \exp(iU \cdot (H^T \cdot X)) = \exp(i(UH)^T \cdot X) \).

**Lemma 3.** Let \( H \in L(X) \). Then the characteristics \( (\tilde{B}, \tilde{C}, \tilde{\nu}) \) of \( H^T \cdot X \) relative to some truncation function \( h_1: \mathbb{R} \to \mathbb{R} \) are of the form \( \tilde{B} = \tilde{b} \cdot A, \tilde{C} = \tilde{c} \cdot A, \tilde{\nu} = A \otimes \tilde{F} \), where

\[
\tilde{b}_t = H^T_t b_t + \int (h_1(H^T_t x) - H^T_t h(x)) F_t(dx),
\]
\[
\tilde{c}_t = H^T_t c_t H_t,
\]
\[
\tilde{F}_t(G) = \int 1_G(H^T_t x) F_t(dx) \quad \text{for any} \ G \in \mathcal{B} \ \text{with} \ 0 \notin G.
\]
Proof. For locally bounded integrands the lemma is proved in [4, IX.5.3]. Since $H \in L_{10c}^1(X^c)$, it follows that $\tilde{C} = \langle H^T \cdot X^c, H^T \cdot X^c \rangle = (H^T \cdot H) \cdot A$ (cf. [1, Proposition 5.2] and [3, 1a]). Moreover, $\Delta(H^T \cdot X) = H^T \Delta X$ implies that $1_G(x) \mu_{H^T \cdot X} = 1_G(H^T x) \mu_X$ and hence $1_G(x) \tilde{v} = 1_G(H^T x) \nu$ for any $G \in \mathbb{B}$ with $0 \notin G$. By [1, Proposition 5.2], there exist a set $\Delta \in \mathcal{P} \otimes \mathfrak{B}^d$ and a predictable process $\tilde{B} \in \mathfrak{F}^d$ such that

$$X = X_0 + X^c + z_1 \Delta_c(x) \ast (\mu_X - \nu) + z_1 \Delta(x) \ast \mu_X + \tilde{B},$$

$$\nu = (H^T \cdot X) \ast \nu = (H^T \cdot z_1 \Delta_c(x) \ast \mu_X + \tilde{B}).$$

Since $X$ can also be written in its canonical semimartingale representation $X = X_0 + X^c + h(x) \ast (\mu_X - \nu) + (x - h(x)) \ast \mu_X + B$, it follows from [1, Proposition 5.3] that $\tilde{B} = B + (z_1 \Delta_c(x) - h(x)) \ast \nu = (b + f(z_1 \Delta_c(x) - h(x)) F(dx)) \ast A$. Similarly, the canonical semimartingale representation of $H^T \cdot X$ equals $H^T \cdot X = x^c + h_1(H^T x) \ast (\mu_X - \nu) + (H^T x - h_1(H^T x)) \ast \mu_X + \tilde{B}$, which yields, again using [1, Proposition 5.3],

$$\tilde{B} = H^T \cdot \tilde{B} + (h_1(H^T x) - H^T x_1 \Delta_c(x)) \ast \nu \ast \nu \ast A.$$

This proves the assertion.

Lemma 4. Let $U \in L(X)$. Then $X^X(U) = \kappa(U) \cdot A$, where

$$\kappa(U)_i = i U_i^T b_i - \frac{1}{2} U_i^T c_i U_i + \int \left( i U_i^T x - 1 - i U_i^T h(x) \right) F_i(dx).$$

Proof. First step. From Lemmas 2 and 3 it follows that $X^X(U) = X^U \cdot X(1)$ and $\kappa(U) = \kappa U \cdot X(1)$, where $\kappa U \cdot X$ is defined as $\kappa$ but for the process $U \cdot X$ instead of $X$. Therefore, it suffices to prove the statement for $\mathbb{R}^1$-valued $X$ with $X_0 = 0$ and $U = 1$.

Second step. Let $Y := i X$. Applying Ito's formula (e.g. as in [1, Lemma 5.5]) yields that exp$(iX)$ = $\exp(Y) = s(Y + i \frac{1}{2} (Y^e, Y^c) + (e^x - 1 - x) \ast \mu_Y)$ and hence $L := \mathcal{L}(\exp(iX)) = Y + i \frac{1}{2} (Y^e, Y^c) + (e^x - 1 - x) \ast \mu_Y$. Note that $Y = i (B + X^c + h(x) \ast (\mu_X - \nu) + (x - h(x)) \ast \mu_X)$. Together, we obtain that

$$L = i B + i X^c + i h(x) \ast (\mu_X - \nu) - \frac{1}{2} (X^e, X^c) + (e^{ix} - 1 - i h(x)) \ast \mu_X.$$ 

On the other hand, the canonical decomposition of $L$ is $L = V + L^e + z \ast (\mu_L - \nu_L) = V + L^e + (e^{ix} - 1) \ast (\mu_X - \nu)$, where $V \in \mathcal{V}$ is predictable. From [1, Proposition 5.3] it follows that $V = i B - \frac{1}{2} C + (e^{ix} - 1 - i h(x)) \ast \nu = \kappa(1) \cdot A$. Therefore $X^X(U) = \kappa(1) \cdot A$ as claimed.

Remark. In particular, the cumulant process $X^X(U)$ for $U \in L(X)$ can be obtained from the cumulant processes $X^X(u), u \in \mathbb{R}^d$. Also note that by [4, II.2.42] the cumulant processes $X^X(u), u \in \mathbb{R}^d$, uniquely determine the characteristics $(B, C, \nu)$ of $X$.

Lemma 5. Let $(T_\theta)_{\theta \in \mathbb{R}^d}$ be a finite time change such that $X$ is $(T_\theta)_{\theta \in \mathbb{R}^d}$-adapted. Then the process $(\tilde{X}_\theta)_{\theta \in \mathbb{R}^d}$ defined by $\tilde{X}_\theta := X_{T_\theta}$ is a semimartingale on $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_\theta)_{\theta \in \mathbb{R}^d}, \mathbb{P})$ whose characteristics ($\tilde{B}, \tilde{C}, \tilde{\nu}$) are of the form

$$\tilde{B}_\theta = B_{T_\theta}, \quad \tilde{C}_\theta = C_{T_\theta}, \quad 1_G \ast \tilde{v}_\theta = 1_G \ast \nu_{T_\theta}.$$
for $\theta \in \mathbb{R}^+$, $G \in \mathcal{B}^d$ with $0 \notin G$. Moreover, its cumulant process satisfies
\[ \mathcal{X}^X(u)_\theta = \mathcal{X}^X(u)_{T_\theta} \quad \text{for } \theta \in \mathbb{R}^+, \ u \in \mathbb{R}^d. \]

Proof. The statement on \( \mathcal{C} \) follows from \[2, (10.17)a\]. The \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adaptedness of \( X \) implies that \( 1G^* \mu^X_{T_\theta^-} = \sum_{t \leq T_\theta} 1G(\Delta X_t) = \sum_{t \leq T_\theta} 1G(\Delta X_t) = 1G^* \mu^X_{T_\theta} \), hence \( \mu^X \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted in the sense of \[2, (10.25)\]. Since \( \Delta \mathcal{X} \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted in the sense of \[2, (10.27)\], it follows from \[2, (10.27)\] that \( 1G^* \nu_\theta \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted in the sense of \[2, (10.17)b\]. Since \( B \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted,\( \nu_\theta \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted, hence \( J^* \) is \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adapted. Finally, observe that by Lemma 4, \( \mathcal{X}^X(u)_\theta = \mathcal{X}^X(u)_{T_\theta} \).

Remark. Even though the semimartingale property is preserved under arbitrary finite time changes, Lemma 5 generally ceases to hold. For example, even a continuous process may have fixed times of discontinuity after a time change. Under \((T_\theta)_{\theta \in \mathbb{R}^+}\)-adaptedness, however, many properties are preserved (cf. \[2, Chapter X\]).

We end with a characterisation of \( L(X) \) for \( \alpha \)-stable Lévy motions \( X \) which can be found in \[5\] (cf. also \[7\]). We give a more direct proof here.

**Lemma 6.** Let \( X \) be a nonconstant \( \alpha \)-stable Lévy motion and \( H \) a predictable real-valued process. Then \( H \in L(X) \) if and only if \( \int_0^t |H_s|^\alpha ds < \infty \) \( \mathbb{P} \)-almost surely for any \( t \in \mathbb{R}^+ \).

Proof. Since \( H = H' + H'' \) with \( H'_t := H_t^{1(|H_t|>1)} + 1_{|H_t| \leq 1} \) and \( H''_t := (H_t - 1)1_{|H_t| \leq 1} \), it suffices to consider the case \( |H| \geq 1 \). By \((B,C,\nu)\) we denote the characteristics of \( X \) relative to \( h: \mathbb{R} \to \mathbb{R} \), i.e., \( B_t = bt \), \( C_t = ct \), \( \nu = \lambda \otimes F \), where \((b,c,F)\) is the Lévy–Khintchine triplet of \( \text{Law}(X_1) \). By \[1, Proposition 5.2\], \( H \in L(X) \) holds if and only if \( H \in L_{\text{loc}}^{1}(X') \), \( Hx1_{|Hx| \leq 1} \in G_{\text{loc}}(\mu^X) \), \( |Hx1_{|Hx|>1}|^\alpha \mu^X \in \mathcal{Y} \), and \( H \in L_{\text{loc}}(\mathcal{B}) \), where \( \mathcal{B} = B + (x1_{|Hx| \leq 1} - h(x)) * \nu \). By \( \alpha \in (0,2) \) we denote the index of stability of \( X \).

Case 1. \( \alpha = 2 \). In this case \( \mu^X = 0 \), \( B = 0 \), and \( X \) is a continuous local martingale. Therefore, \( H \in L(X) \) if and only if \( H \in L_{\text{loc}}^{1}(X') \), i.e., if and only if \( H^2 \cdot (X,X) \in \mathcal{Y} \). Since \( (X,X)' = ct \), the claim follows.

Case 2. \( \alpha \not= 2 \). Note that \( \mathcal{X}^X = 0 \). Moreover, \( |Hx1_{|Hx|>1}|^\alpha \mu^X \in \mathcal{Y} \) holds automatically because \( |Hx1_{|Hx|>1}|^\alpha \mu^X \) is a finite sum for fixed \( t \). By \[4, II.1.33c\], \( Hx1_{|Hx| \leq 1} \in G_{\text{loc}}(\mu^X) \) if and only if \( H^2x^21_{|Hx|<1} \nu \mu^X \in \mathcal{Y} \). But note that \( \nu = \lambda \otimes F \) and \( \int H^2x^21_{|Hx|<1} F(dx) = |H|^\alpha \int y^21_{|y|<1} F(dy) \), because \( F \) is the density of \( X \), \( x \in \mathbb{R}^d \), and \( 0 < \alpha < 2 \). Since \( \int y^21_{|y|<1} F(dy) < \infty \), it follows that \( H^2x^21_{|Hx|<1} \nu \mu^X \in \mathcal{Y} \) if and only if \( F \in L^{1\alpha} \) or \( \int_0^t |H|^\alpha dt \in \mathcal{Y} \). For the drift part, we have to distinguish several cases.

Firstly, let \( \alpha = 1 \). If we choose a symmetric \( h \), then \( x1_{|Hx| \leq 1} - h(x) \) \( \nu \mu^X \in \mathcal{Y} \) if and only if \( B = 0 \) or \( \int_0^t |H|^\alpha dt \in \mathcal{Y} \).

Secondly, let \( \alpha \in (1,2) \). By \[9, Theorem 14.7\], we can use \( h(x) = x \) and have \( B = 0 \) for this choice of \( h \). Consequently, \( \mathcal{B} = -x1_{|Hx|>1} \nu \). Therefore, \( H \in L_{\text{loc}}(\mathcal{B}) \) holds if \( |Hx1_{|Hx|>1}| \nu \in \mathcal{Y} \), i.e., \( \int |y1_{|y|>1} F(dy) < \infty \), a similar reasoning as above yields that \( |Hx1_{|Hx|>1}| \nu \in \mathcal{Y} \) if and only if \( F = 0 \) or \( \int_0^t |H|^\alpha dt \in \mathcal{Y} \).

Finally, let \( \alpha \in (0,1) \). By \[9, Theorem 14.7\], we can use \( h(x) = 0 \) and have \( B = 0 \) for this choice of \( h \). Therefore, \( \mathcal{B} = x1_{|Hx| \leq 1} \nu \) in this case. Note that \( \int |y1_{|y|<1} F(dy) < \infty \). As for \( \alpha > 1 \), it follows that \( H \in L_{\text{loc}}(\mathcal{B}) \) holds if \( \int_0^t |H|^\alpha dt \in \mathcal{Y} \).
REFERENCES


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Введение. Первоначальный вариант этого сообщения был направлен в редакцию журнала «Теория вероятностей и ее применения» еще в начале 1985 года. Однако в том варианте отсутствовали указания на иллюстрации таких применений полученных результатов и предложенной комбинаторной техники, которые могли бы представлять чисто вероятностный интерес. Поэтому автор решил в конечном счете ограничиться публикацией формулировок главных результатов в тезисах [6]. Впоследствии к ним были даны комментарии в докладе на IV Всемирном конгрессе Общества им. Бернулли в Вене (август 1996 г.), а в докладе на Международном