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Differentiable manifolds in Euclidean space

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We present here a summary of some theorems on the imbedding of abstract differentiable manifolds in Euclidean space $E^n$ and on the approximation to such manifolds by analytic manifolds. As a corollary it is noted that on any differentiable manifold may be given an analytic Riemannian metric.

I. The imbedding of a differentiable manifold in Euclidean space

Let $M$ be a topological space with neighborhoods $U_1, U_2, \ldots$. Let each $U_i$ be homeomorphic with the interior of the unit $m$-sphere $S^m$. If $U_i$ and $U_j$ have common points $U_{ij}$, then the homeomorphisms of $U_i$ and $U_j$ with $S^m$ induce a mapping of one part of $S^m$ on another part. If all such maps are of class $C^r$ (i.e., have continuous partial derivatives through the $r$th order), $r \geq 1$, with non-vanishing Jacobian, we say $M$ is differentiable, and of class $C^r$.

If $M$ is in $E^n$ and each point of $M$ is in a neighborhood which may be defined by expressing $n - m$ of the coordinates in terms of the remaining $m$, the functions being of class $C^r$, then $M$ is of class $C^r$ in the above sense; we say $M$ is of class $C^r$ in $E^n$. Suppose $M$ is of class $C^r$, and is mapped into $E^n$. The $n$ coordinates at points of $M$ are $n$ functions defined over $M$. If these functions are of class $C^s$ (with the obvious definition for $s \leq r$), and are independent (so that $m$ independent directions at any point of $M$ go into $m$ independent directions in $E^n$), we call the map of $M$ in $E^n$ a regular $C^r$-map. Such a map is locally one-one: a neighborhood of any point of $M$ is mapped in a one-one manner in $E^n$.

Theorem I. Any $m$-manifold of class $C^r$ ($r \geq 1$ finite or infinite) may be imbedded by a regular $C^r$-map in $E^{2m}$, and by such a map in a one-one manner in $E^{2m+1}$.

The proof runs as follows:

If $M$ is closed, a finite number of neighborhoods $U_1, \ldots, U_s$ cover $M$. Corresponding to these neighborhoods we define functions $f_1, \ldots, f'_\mu$, $\mu = (m + 1)v$, of class $C^r$ over $M$, which used as coordinates, map $M$ in a regular $C^r$-manner in $E^v$. If $r = 1$, we next approximate to $M$ by a manifold of class $C^2$. We now project $M$ or the new manifold along straight lines into spaces of lower dimension, till we have it in $E^{2m+1}$ or in $E^{2m}$. If $M$ is not closed, we define such a map successively over larger and larger parts of $M$.

II. Approximations to differentiable manifolds by analytic manifolds

A manifold of class $C^r$ in $E^n$ was defined above; it is a analytic if the functions defining its neighborhoods are analytic. If $M$ and $M^*$ are homeomorphic manifolds in $E^n$ and $\eta(p)$ is a positive continuous function defined on $M$, we say $M^*$ approximates to $M$ through the $r$th order with an error $<\eta(p)$ if the distance from any point $p$ of $M$ to the corresponding point $p^*$ of $M^*$ is $<\eta(p)$, and corresponding partial derivatives of order $\leq r$ (in a suitable coordinate system) differ by $<\eta(p)$.

Theorem II. Let $M$ be of class $C^r$ in $E^n (r \geq 1$ finite), and let $\eta(p)$ be a positive continuous function defined on $M$. Then there is an analytic manifold $M^*$ in $E^n$ which approximates to $M$ through the $r$th order with an error $<\eta(p)$.

To prove the theorem, we first define a positive function $f$, analytic in $E^n - M$, and approaching 0 as we approach $M$. The points $f = c > 0$ define a "tube" about $M$; the $(n - m)$-plane orthogonal to $M$ at $p$ cuts $f = c$ in an $(n - m - 1)$-sphere. We define in an analytic fashion a "centre" to the tube; the set of centre points form $M^*$.

From Theorems I and II follows

Theorem III. Any $m$-manifold of class $C^r (r \geq 1$ finite) is homeomorphic with an analytic manifold in $E^{2m+1}$, the homeomorphism being of class $C^r$.

We may define $ds^2$ on the manifold as the $ds^2$ in $E^{2m+1}$; hence

Theorem IV. To any manifold $M$ of class $C^r (r \geq 1$ finite) may be given an analytic Riemannian metric, the $g_{ij}$ being of class $C^r$ in terms of the original neighborhoods in $M$.

III. Imbedding of manifolds in families of analytic manifolds

We state here a generalization of Theorem II for certain classes of manifolds in $E^n$.

We say $M$ is in regular position in $E^n$ if there exist $n - m$ independent continuous unit vector functions $v_1(p), \ldots, v_{n-m}(p)$ defined over $M$ with the following property: each point $p_0$ of $M$ is in a neighborhood $U$ of $p_0$ in $M$ which is an $m$-cell, and such that any vector through two points of $U$ makes an angle $> \rho(p_0)$ with the $(n - m)$-plane determined by the $v_1(p_0), \ldots, v_{n-m}(p_0)$; $\rho(p)$ is a positive continuous function defined on $M$.

If $M$ is differentiable, the condition reduces to: the normal $(n - m)$-planes to points of $M$ may be determined by $n - m$ vector functions on $M$. The class of such (differentiable) manifolds is the same as the class of manifolds which may be determined by the simultaneous vanishing of $n - m$ (independent) differentiable functions.

Theorem V. Let $M$ be an $m$-manifold of class $C^r (r \geq 1$ finite) in regular position in $E^n$, and let $\eta(p)$ be a positive continuous function on $M$. Then $M$ can be imbedded in an $(n - m)$-parameter family of manifolds $M(c_1, \ldots, c_{n-m})$, each $|c_i| < 1$, such that

1) $M(0, \ldots, 0) = M$,

2) $M(c_1, \ldots, c_{n-m})$ is analytic if $(c_1, \ldots, c_{n-m}) \neq (0, \ldots, 0)$.

2 If $M$ is differentiable, it is in regular position if and only if the normal sphere-space is a product space. See the following paper, especially 3, C) and 8, d).
3) each \( M(c_1, \ldots, c_{n-m}) \) approximates to \( M \) through the \( r \)th order with an error \( \leq \eta(p) \),

4) the manifolds fill out a neighborhood of \( M \) in a one-one way.
Дифференцируемые многообразия в евклидовом пространстве

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Изложение некоторых результатов о включении абстрактных дифференцируемых многообразий в евклидовы пространства и об аппроксимации этих многообразий аналитическими многообразиями. В частности: каждое $r$ раз дифференцируемое $n$-мерное многообразие топологически отображается с непрерывностью первых $r$ производных на некоторое аналитическое многообразие, лежащее в $(2n+1)$-мерном евклидовом пространстве. В качестве следствия получается, что в каждом дифференцируемом многообразии может быть введена аналитическая риманова метрика.