A Survey of some Developments in Loop Spaces: 
Associated Stochastic Processes, Statistical Mechanics, 
Infinite Dimensional Lie Groups, 
Topological Quantum Fields

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1. INTRODUCTION

We shall try to present a somewhat unified picture of several developments involving in some way loop variables and loop spaces. The choice of topics is largely influenced by the author's own directions of research, no attempt of being exhaustive has been made. As for references, we tried to find a compromise between original references and newer references which give further insight into the literature. As a general rule “see [X]” should be understood as “see [X] and references therein”.

2. SPACES OF MAPPINGS, LOOP SPACES, RANDOM FIELDS AND QUANTIZED (NON LINEAR) σ-MODELS

Consider the spaces of mappings $N^M$, where $N$ and $M$ are (Riemannian) manifolds. Important particular cases are

1. $M = S^1$, then we have the space of loops in $N$.
2. $M = [0, T]$ for some $0 < T < \infty$, then we have the space of paths (in time $[0, T]$) running in $N$.
3. $N = \text{Lie group}$, then we have a space of group-valued mappings (resp. loops, resp. paths).

A covariant (or "homogeneous") random field on $M$ with values in $N$ is by definition a family of random variables $X(x)(\omega)$, $x \in M$, $X(x) \in N$, $\omega \in (\Omega, A, P)$ is probability space. Thus, for fixed $\omega$, $X(\cdot)(\omega) \in N^M$. Covariance is meant with respect to some group $\Gamma$ acting both in $M$ by transformations $T^M_\gamma$, $\gamma \in \Gamma$, and in $N$ by transformations $T^N_\gamma$, $\gamma \in \Gamma$, and such that the transformed random field $(T^M_\gamma X)(x)$ has the same law as $X \left( (T^M_\gamma)^{-1}(x) \right)$ for all $\gamma \in \Gamma$, $x \in M$.

Such covariant random fields have been studied in relation with several problems (see e.g. [3,282,40,42,301]). In connection with quantum fields they are particularly interesting when they possess a suitable "strict Markov property" (a natural extension to our setting of the usual strict Markov property of real-valued stochastic processes) (see e.g. [262,46,280]). Recently Iwata and

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Schäfer have discussed in general Gaussian Markov fields, and achieved a complete characterization (at least in the case of vector bundles) (see [287, 288, 216]). For this lecture a few (very) particular cases are of importance:

1. \( M = \mathbb{R}^d, N = \mathbb{R}, X \) (and \( P \)) Gaussian. In this case the requirement that \( X \) be Markov and covariant with respect to the Euclidean group in \( M = \mathbb{R}^d \) and its identity representation in \( N \) are in conflict. However they are not if \( X \) is understood in the sense of generalized functions, then we get Nelson’s Markov free field [262] characterized as the generalized random field \( X \) (with values in \( S'(\mathbb{R}^d) \)), such that

\[
\langle f, X \rangle = \int f(x) X(x) \, dx, \quad f \in S(\mathbb{R}^d)
\]

(in the sense of generalized functions) and

\[
C(f) = \mathbb{E} \left( e^{i(f, X)} \right) = \int e^{i(f, X)} \, dP(X) = e^{-\frac{1}{2}(f, (-\Delta + m^2)^{-1} f)}.
\]

\( P \) is then the Gaussian measure on \( S'(\mathbb{R}^d) \) with mean zero and covariance \( \langle f, (-\Delta + m^2)^{-1} \rangle \) (\( P \) is uniquely given from its characteristic functional \( C \) by Minlos theorem). Heuristically

\[
dP(X) = Z e^{-\frac{1}{2} \int_{\mathbb{R}^d} V(X(x))^2 \, dx} e^{-m^2 \int_{\mathbb{R}^d} \mathbf{1}^2 \, dx} \prod_{x \in \mathbb{R}^d} dX(x)^n.
\]

(2.1)

This formula can actually be given rigorous meaning using nonstandard analysis (see [35, 109]).

**Remark 2.1.** For \( d = 2 \) several other examples have been constructed, heuristically described by

\[
dP_V(X) = Z e^{-\frac{1}{2} \int_{\mathbb{R}^d} V(X(x))^2 \, dx} \prod_{x \in \mathbb{R}^d} dX(x)^n.
\]

(2.2)

with \( P \) as in (2.1) and \( V \) a suitable continuous real-valued function (see e.g. [40, 111, 47]). These are called Euclidean Markov fields (measures) with interaction \( V \).

The requirement of covariance with respect to the full Euclidean group, together with the Markov property (or some substitute of it) leads, by “analytic continuation”, to models of relativistic quantum fields (see [47, 111]).

For \( d \geq 3 \) only partial constructions are known, however if one requires only covariance with respect to the proper Euclidean group, many models have been constructed [62, 72–74, 108, 39, 135].

2. \( M = \mathbb{R}^+ \), \( N = \mathbb{R}^d \). We consider a corresponding model as in 1 (with \( m = 0 \)) but with \( P \) replaced by the probability measure on \( C(\mathbb{R}^+, \mathbb{R}^d) \subset S'(\mathbb{R}^{d+1}) \) heuristically described by

\[
dP(X) = Z e^{-\frac{1}{2} \int_{\mathbb{R}^+} \mathbf{1}^2 \, dt} \prod_{t \in \mathbb{R}^+} dX(t)^n.
\]

The mathematical expression of this is just Wiener measure for a \( \mathbb{R}^d \)-valued Wiener process \( X(t), \ t \in \mathbb{R}^+ \) on \( \mathbb{R}^d \) (see e.g. [251, 9]).

3. \( M = S^1 \), \( N = \mathbb{R}^d \). \( P \equiv \) Wiener measure on the \( N \)-loop space \( C(S^1, N) \) (see e.g. [7, 8]).

4. \( M = \mathbb{R}^+, \mathbb{R}, [0, T] \) resp. \( S^1 \), \( N = \) compact Lie group, \( P \) distribution of a (pinned in the case of \( S^1 \)) Brownian motion on \( N \). \( X \) is constructed e.g. as solution of a (Stratonovich) stochastic
equation \( dX(t) = X(t) \circ dw(t) \), with \( w \) a (pinned) Brownian motion with values in the Lie algebra \( g \) of \( N \) (see e.g. [257,261]). This can be looked upon as a (one-dimensional) "non-linear \( \sigma \)-model or chiral model" (with base manifold \( M \) and target space \( N \)).

**Remark 2.2.** In all these examples \( P \) is given by an action-functional \( W(X) \) such that

\[
dP(X) = \frac{1}{Z} \exp \left[ -W(X) \right] \prod_{x \in \mathbb{R}^d} dX(x) .
\]

(2.3)

For the case 4, e.g., we have

\[
W(X) = \frac{1}{2} \int_0^t |X(t)^{-1}dX(t)|^2 dt
\]

(well defined for \( X \) taking values in the natural \( H^{1,2} \)-Sobolev space for \( N \) (see [49]); as well known, however, \( P \) is supported on a larger space (see e.g. [235])).

An algebraic substitute for the knowledge of \( X \) (and its law) is provided by the representation of the groups of mapping \( H^{1,2}(M, N) \) given by

\[
(T^L_\gamma f)(X) = \sqrt{\frac{dP(\gamma X)}{dP(X)}} f(\gamma X)
\]

with \( \gamma \in H^{1,2}(M, N) \), \( f \in L^2(P) \) (left translation), resp.

\[
(T^R_\gamma f)(X) = \sqrt{\frac{dP(X\gamma^{-1})}{dP(X)}} f(X\gamma^{-1})
\]

(right translation).

The Radon–Nikodym derivatives can be shown to exist [41,45] for 1, 2; [49,252] for 3, 4.

\( T^R \), \( T^L \) can be shown to be unitary representations in the Hilbert space \( L^2(P) \), unitarily equivalent. From the representations one can recover \( P \) and \( X \).

In [65,67] also the equivalence of the representations \( T^L \), \( T^R \) with the (vacuum) component of the "energy representation" was given.

In fact, let \( M \) be a Riemannian manifold with \( \rho \) a \( C^\infty \) function, let \( N \) be a compact semisimple Lie group. Let \( \Omega(TM; g) \) be the smooth 1-forms. Giving \( g \) the Killing structure, \( \Omega \) becomes a Hilbert space \( H \) with scalar product

\[
(\omega_1, \omega_2) \equiv \int_M \text{Tr} (\omega_1(x) \omega_2(x)^*) \rho(x) dx
\]

(\( dx \) being the volume measure). We have an unitary representation \( V \) of the groups of mappings \( G^M \) from \( M \) into \( G \), in \( H \) given by:

\[
(V(\psi)\omega)(x) \equiv A\psi(x)\omega(x) , \ \psi \in G^M , \ \omega \in H , \ x \in M .
\]

Let \( \beta(\psi)(x) \equiv d\psi(x)\psi(x)^{-1} \), then \( \beta(\psi) \) is a Maurer–Cartan 1-cocycle for \( V \). The corresponding exponential representation \( U_{V,\beta} \) of \( G^M \) constructed from \( V, \beta \) (obtained by the general methods of Streater, Araki, Parthasarathy–Schmidt) is

\[
(U_{V,\beta}(\psi) f)(\omega) \equiv e^{i\beta(\psi) f} f(V^{-1}(\psi) \omega).
\]
This is called "energy representation" (see [49,64]). The restriction of $V$ to the cyclic component of the vacuum (the function 1 in $L^2(P)$) is unitary equivalent $T^R$, $T^L$ (by a theorem of Albeverio and Høegh-Krohn [49]).

**Remark 2.3.**
- It is conjectured (by Gelfand, Graev and Vershik) that for $M = \mathbb{R}$ the vacuum is cyclic for $U_{(V,\beta)}$. The cyclicity of $T^L$, $T^R$ has been recently shown in [167].
- A detailed study of the case $\dim M = 1$ (in particular for the case of loop groups) has been made in [65,67], i.e., the irreducible components (of the reducible $U_{(V,\beta)}$, $T^L$, $T^R$) have been found:

$$U(\psi) = \int \mathbb{E}^{U(\psi)} d\eta(\chi),$$

with $\eta$ a Gaussian white noise (belonging to the space $\mathcal{M}(M; t)$ of measures on $M$ with values in the Lie algebra $t$ to a maximal Abelian torus in $G$). The commutants $W^R$, resp. $W^L$, of $T^R$, resp. $T^L$, are factors, $(W^R)' = W^L$ and the above is a spectral representation of $W^R$, resp. $W^L$. For $M = \mathbb{R}, W^L$ and $W^R$ are asymptotic Abelian with respect to translations in $\mathbb{R}$, hence of type III. For further results see also [303,304,259].

**Remark 2.4.** A detailed study for $N = SU(n)$ has been done, and an extension to the equivariant case has been given [298,64].

Irreducibility of the energy representation has been proven (see [215,194] for $d \geq 5$; [65] for $d \geq 3$, see also [302]). The proof of irreducibility is actually based on a connection of the representation $U_{(V,\beta)}$ with the Høegh-Krohn model of quantum fields (which has been shown to be Gaussian for $d \geq 3$ [50,38]).

For $d \geq 2$ one has irreducibility for "large roots" [65], reducibility for "small roots" (the former is again connected with properties of Høegh-Krohn's model for $d = 2$ (see [65]).

**Remark 2.5.** The general case for $d = 2$ is still open. Most recently irreducibility on a suitable subspaces has been shown [167]. Looking at the case $d = 1$ as an Euclidean representation of the quantized nonlinear $\sigma$-model, reducibility in the case $d = 2$ should be connected with the existence of a similar $\sigma$-model. For other work connected with these topics see [310,96].

**Remark 2.6.** As well known, there are other types of representations of $G^M$, the “highest weight representation”, discussed especially for $d = 1$ [275,64,258], but also partly for $d \geq 2$ [211,299,258]. The relations between these types of representation and the energy representation are not fully clarified. Also the relation between representations of current algebras, gauge groups, Sugawara currents and (quantized) Hamiltonian systems is not clear, especially not in connection with relativistic models.

**Remark 2.7.** For various extensions to flag manifolds and equivariant representations of affine Lie algebras see e.g. [255]. For relations with non-commutative geometry see [254].

3. **STOCHASTIC QUANTIZATION AND PROCESSES WITH VALUES IN LOOP SPACES**

Let us first briefly expose the idea of "stochastic quantization". Let $\mu$ be a given measure on $\mathbb{R}^d$. One looks for a diffusion $X_\mu^t$, $t \geq 0$ on $\mathbb{R}^d$ such that $X_\mu^t$ has $\mu$ as invariant measure. If $\mu$ has the density $|\psi|^2$ with respect to Lebesgue measure, with $\psi \in L^2_{\text{loc}}(\mathbb{R}^d)$, then $X_\mu^t$ can be defined as
the solution (in the weak sense, whenever it exists) of the stochastic differential equation

$$dX_t^\mu = \beta^\mu(X_t^\mu)dt + dw_t,$$

(3.1)

with start measure $\mu$.

Here $\beta^\mu$ is by definition the logarithmic derivative of $\mu$, i.e.,

$$\beta^\mu = \nabla \ln \psi .$$

$w_t$ is by definition a standard Brownian motion (with respect to the filtration generated by $X_t^\mu$).

**Remark 3.1.** $\beta^\mu$ can also be written as $\nabla^{*}1$, with $*$ the adjoint in $L^2(\mathbb{R}^d; \mu)$.

One says then $X_t^\mu$ is the stochastic quantization of $\mu$. This is a physical terminology, related to the fact that in the case where $\psi$ is the “ground state” of a Schrödinger operator $-\frac{1}{2}\Delta + V$ for a certain potential $V : \mathbb{R}^d \to \mathbb{R}$, in the sense that $\psi \in L^2(\mathbb{R}^d)$ and $(-\frac{1}{2}\Delta + V)\psi = E_0\psi$, with $E_0 > -\infty$ the eigenvalue at the bottom of the spectrum of $-\frac{1}{2}\Delta + V$, then using the ergodic theorem, for any $f \in C_b(\mathbb{R}^d),

$$\int f d\mu = L^2(\mu) - \lim_{t \to \infty} \frac{1}{T} \int_{-T}^{T} f(X_t^\mu) dt .$$

(3.2)

Under regularity assumptions on $\psi$ the r.h.s. is actually also equal to the limit in sup-norm

$$\lim_{t \to \infty} \frac{1}{T} \int_{-T}^{T} f(X_t^\mu) dt ,$$

(3.3)

with $X_t^\mu$ defined as the solution of (3.1) with initial condition $x$.

In this sense $\mu$ can be “reconstructed from (3.1)” (cfr. [82,86,98,223,97] concerning the general procedure of stochastic quantization).

We note that $X_t^\mu$ can be defined (even more generally) as the (unique) diffusion process associated with the Dirichlet form $\varepsilon(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u \cdot \nabla v d\mu$, defined for a dense domain $D(\varepsilon)$ of $u, v$ in $L^2(\mathbb{R}^d)$. $\varepsilon$ is called classical Dirichlet form given by $\mu$ [98]. In the above situation of (3.2), (3.3) we have also $\int f d\mu = P_{t}f(x)$ for $\mu$ a.e.x, $P_{t}$ being the symmetric semigroup in $L^2(\mu)$ associated with $(\varepsilon, D(\varepsilon))$.

The theory of Dirichlet forms is the presently most general setting for relations between potential theory and Markov process theory, at least in the symmetric case. In particular, starting from a Dirichlet form is the most general way to define a symmetric Markov process (like $X_t^\mu$ e.g.).

In fact, the theory of Dirichlet forms of the above type works also in infinite dimensions, i.e., when $\mathbb{R}^d$ is replaced e.g. by a Banach space (or a conuclear space) and the Hilbert space structure intrinsic in the definition of $\nabla u \cdot \nabla v$ is provided by a separable Hilbert space $\mathcal{H}$, densely embedded in $E$ (“tangent” space to $E$). In this case however it is better to assume that $\mu$ is a probability measure. Dirichlet forms of the form

$$\varepsilon(u, v) = \frac{1}{2} \int_{E} \nabla u \cdot \nabla v d\mu$$
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(with · the scalar product in $\mathcal{H}$) with a certain natural minimal domain on which they are closed, are known to be “properly associated” with symmetric Markov diffusion processes $(X_t; t \geq 0)$ in the sense that

$$(P_t u)(z) = E^u(X_t),$$

the r.h.s. being defined for quasi-every $z \in E$ and being quasi-continuous, and where $P_t$ is the Markov semigroup associated with $E$. For these concepts and results see e.g. [249].

In this way the problem of the construction of a (symmetric) diffusion (or more generally a Hunt process) on a state space $E$ (finite or infinite dimensional, linear or not) is reduced to the one of constructing a Dirichlet form (in fact, this also extends to non-commutative state spaces, see e.g. [9-13,162,208,156] for recent developments).

We shall now mention two recent results on the construction of diffusion processes on loop spaces.

A) Free loop spaces. In [93] a process on the free loop space $\mathcal{L} \equiv \mathcal{L}(\mathbb{R}^d) \equiv (\mathbb{R}^d)^{S^1}$ with invariant measure $\mu(\cdot) \equiv \int e^{\Delta g}(x,x)P_1^x(\cdot)dx$ is constructed (with $\Delta g$ the Laplace operator given by a uniformly elliptic metric $g$ on $\mathbb{R}^d$), $P_1^x$ the probability measure for a Brownian loop $w(0) = w(1) = x$, $dx$ Lebesgue measure (following Léandre, this is called the BHK measure on $\mathcal{L}$).

There is a natural definition of a classical Dirichlet form $\varepsilon(u,v) = \int_{\mathcal{L}} (DU_DV)_{\mathcal{H}_0}d\mu$ on $\mathcal{L}(\mathbb{R}^d)$, where $(\cdot,\cdot)_{\mathcal{H}_0}$ is the scalar product in $\mathcal{H}_0$, a dense Hilbert subspace of $\mathcal{L}$. In fact, $\varepsilon(u,v) = \frac{1}{2} \sum_n \int \partial_{h_n}u\partial_{h_n}vd\mu$, where $\partial_h$ is the Gâteaux derivative in direction $h$, $\{h_n\}$ being an orthonormal base in $\mathcal{H}_0$.

By showing that integration by parts is possible, one proves that $\varepsilon$ is indeed closed and in fact a Dirichlet form.

By using tightness and the general theory of Dirichlet forms [94,249] one gets the existence of a diffusion process properly associated with the Dirichlet form $\varepsilon$. By construction, the diffusion is invariant (homogeneous) under “reparametrizations” of $S^1$.

This construction actually extends (at least) to the loop space $\mathcal{L}(M)$ of a compact Riemannian manifold $M$.

B) Pinned loop space of a compact Riemannian manifold $M$. Here, instead of $\mathcal{L}(M)$, one considers the subset $\mathcal{L}_0(M)$ of $(M^{S^1})$ consisting of all loops starting and ending in a chosen point $0$ of $M$.

Otherwise the construction is similar, using as $\mu$ the probability measure for Brownian loops starting and ending at $0$. The closability of the form, first defined on a minimal dense domain in $L^2(\mu)$, is proven using Driver’s Cameron–Martin formula on manifolds (see [174]).

**Remark 3.2.** 1. It is well known that the loop space $\mathcal{L}(M)$ over a manifold $M$ contains interesting topological information about $M$ (based on the natural action of $S^1$). Processes over manifolds usually also contain interesting geometrical/topological information about $M$ (e.g. recurrence/transience as related to dimensions or global structure of the manifold). Can one from properties of the diffusion on $\mathcal{L}(M)$, resp. $\mathcal{L}_0(M)$, deduce information about the structure of $\mathcal{L}(M)$, resp. $\mathcal{L}_0(M)$, and from this about $M$ itself? This problem seems to be open, and is somewhat related to the one discussed below in relation with Log-Sobolev inequalities.

2. The process $X$ in A) resp. B) has an infinitesimal generator $L$ (the self-adjoint operator in $L^2(\mu)$ uniquely associated with the Dirichlet form $\varepsilon$). $-L$ is positive, is it essentially self-adjoint.
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(i.e., "already determined") on smooth cylinder functions? There is a result like this in [2–6], for the simpler case of path spaces instead of loop spaces, and where the manifold is replaced by a group. The problem seems to be open in the general case.

3. What are the ergodic properties of the processes $X$ in A), resp. B)? By known relations between ergodic properties of processes and those of the associated generators of transition semigroups (see e.g. [41]), this problem is related to the one of the ergodic properties of the semigroups associated with the corresponding Dirichlet forms, hence to spectral properties at the lower end of the spectrum (non-degenerate zero eigenvalues, resp. "spectral gap", i.e., 0 as simple, resp. isolated, point in the spectrum of $L$, corresponding to ergodicity, resp. exponential mixing).

These properties have recently received a lot of interest in work by Gross [203–205], Aida–Elworthy [6], Fang [182] (see also references therein).

In particular Gross studied Log-Sobolev inequalities for the case B) having as a long range goal the construction of an infinite dimensional version of the Hodge–de Rham theory, a key step being precisely the establishment of the above mentioned spectral gap. The holding of Log-Sobolev inequalities (LS) would be a tool to get a spectral gap (a sufficient, but not necessary condition), however the problem seems to be still open at the moment of writing, although Gross has proven LS in the case $L$ is modified by the addition of a suitable potential $V$ (but the result is not sufficient to prove a spectral gap for $L$).

**Remark 3.3.** The relation of the dimension of the kernel of the Laplace–Beltrami operator on $d$–forms on loop space with the topology of $M$ has been discussed by Kusuoka for $d = 0, 1$ [236].

**Remark 3.4.** 1. Sobolev spaces as well as $L^p$-Chen forms over loop spaces have been studied in [225].

2. An equivariant Ornstein–Uhlenbeck operator over loop space (with a suitable drift) has been discussed in [238].

3. For the relation between exterior stochastic derivative on loop spaces and the Hochschild boundary see [238].

4. For a discussion of a conjecture on an equivariant Riemann–Roch formula over free loop spaces see [238].

5. For a supersymmetric extension of Dirichlet forms and a corresponding Hodge–de Rham theory see [83].

6. For the relations between (the geometry of) loop spaces and index theory see e.g. [224]; this relation works in the case of finite dimensional index theory [142] as well as for infinite dimensional index theory (e.g. elliptic cohomology, see e.g. [224]).

7. In principle, the index of suitable infinite dimensional operators in free loop space should give topological invariants of the underlying manifold (see e.g. [290,306]).

4. QUANTIZATION OF LATTICE STRINGS

Let us consider, following [90], the following infinite product $E$ of free loop spaces

$$E \equiv \mathcal{L}(\mathbb{R}^N)^{Z^d}, \text{ with } \mathcal{L}(\mathbb{R}^N) = C(S^1; \mathbb{R}^N).$$

$Z^d$ is a $d$-dimensional lattice. To each point $k \in Z^d$ there is associated a loop $x_k(s), s \in S^1$, with values in $\mathbb{R}^N$. 

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Let $A \equiv (A_{kj})_{k,j \in \mathbb{Z}^d}$, $A_{kj} = A_{jk}$ and $A_{kj} : \mathbb{R}^N \to \mathbb{R}^N$ is a symmetric $N \times N$ matrix. Let $\beta > 0$ and let us identify $(S^1; \mathbb{R}^N)$ with $\{ \gamma : [0, \beta] \to \mathbb{R}^N \mid \gamma(0) = \gamma(\beta), \gamma \in C^1 \}$.

Let $\mathcal{H}$ be the Hilbert space of such a lattice of loops:

$$\mathcal{H} = \{ x = (x_k(u)), \ k \in \mathbb{Z}^d, \ u \in S^1 \mid (x, x)_{\mathcal{H}} \equiv \int_0^\beta \sum_k |x_k(u)|^2 \, du + \int_0^\beta (x(u), A x(u))_{\nu(\mathbb{Z}^d)} \, ds < \infty \}.$$ 

Let $\nu_0^\beta$ be the standard normal distribution associated with $\mathcal{H}$ heuristically described by

$$Z^{-1} e^{-\frac{1}{2}(x, x)_{\mathcal{H}}} \prod_k \prod_{s \in S^1} dx_k(s),$$

$Z$ being a "normalization constant".

Let $V$ be a given continuous function from $\mathbb{R}^N$ to $\mathbb{R}$, describing a "one lattice point interaction". Under some assumptions on $V$ one can prove that the probability measure heuristically described by

$$d\nu_\beta(x(\cdot)) = Z^{-1} e^{-\int_0^\beta \sum_k A_{kj} x_k(u) \, ds} d\nu_0^\beta(x(\cdot))$$

(with $Z$ a suitable normalization constant) exists and the classical Dirichlet form $\varepsilon_{\nu_\beta}$ given by $\nu_\beta$ also exists:

$$\varepsilon_{\nu_\beta}(u, v) = \frac{1}{2} \int (Du, Dv) d\nu_\beta.$$ 

We have here

$$(Du, Dv) = \sum_k \int_0^\beta \frac{\delta u}{\delta z_k(s)} \frac{\delta v}{\delta z_k(s)} \, ds$$

(for $u, v$ belonging to a space of cylinder functions, dense in $L^2(\nu_\beta)$).

By a direct construction (but also by the general theory of Dirichlet forms) one gets then a diffusion $X$ properly associated with $\varepsilon_{\nu_\beta}$, with values in $E$, such that

$$dX_{k,t}(u) = \frac{1}{2} \frac{\partial^2}{\partial u^2} X_{k,t}(u) \, dt - \sum_{j \in \mathbb{Z}^d} A_{kj} X_{j,t}(u) \, dt - \frac{1}{2} V'(X_{k,t}(u)) \, dt + dw_{k,t}(u),$$

$k \in \mathbb{Z}^d, t \in \mathbb{R}_+, u \in S^1 \cong [0, \beta]$.

We can look at the process $X = (X_{k,t}(u), k \in \mathbb{Z}^d, u \in S^1)_{t \geq 0}$ as describing a system of interacting strings on the lattice $\mathbb{Z}^d$. Sufficient conditions (on $A, V$) for the existence of mild solutions of the equations have been found. Moreover, ergodicity in the $L^2$ as well as sup-norm sense has been proven, with estimates on the exponential decay of correlations [90,163]. Support properties of $\nu_\beta$ have also been investigated, as well as the existence of a unique invariant distribution for $X$ [90], identified with $\nu_\beta$ [231].

**Remark 4.1.** Previous related results for the case of $\mathbb{Z}^d$ replaced by a point have been given in [191,217,192].
Remark 4.2. In all of the above considerations $\mathbb{R}^N$ can be replaced by a compact Riemannian manifold (see [31]).

Remark 4.3. The above model has a close connection with the quantum statistics of continuous quantum lattice spin systems, described heuristically by a Hamiltonian $H$ of the form

$$H = \sum_k \left( -\frac{1}{2} \Delta x_k - \sum_j A_{kj} x_j \nabla x_k + V(x_k) \right)$$

acting in $L^2(\nu)$. $x_k$ can be thought as describing (in the Schrödinger picture) a degree of freedom ("spin") with values in $\mathbb{R}^N$.

Let us call $H_\Lambda$ the restriction of $H$ to the bounded open subset $\Lambda$ of $\mathbb{Z}^d$. The quantum Gibbs state at inverse temperature $\beta > 0$ in the region $\Lambda$ is by definition the state

$$\frac{\text{Tr} (e^{-\beta H_\Lambda})}{\text{Tr} (e^{-\beta H_\Lambda})} = \omega_\beta(\cdot)$$

on the algebra $B(H_\Lambda)$ of all bounded operators on $H_\Lambda = L^2(\mathbb{R}^N|\Lambda|)$.

As shown in [42] (see also e.g. references in [12]) one can express $\omega_\beta$ (evaluated on "time-evoluted" functions of the positions $x_k$) in terms of expectations with respect to $\nu_\beta$ (of corresponding functions of the above process $X$). This relation carries over in the limit $\Lambda \uparrow \mathbb{Z}^d$. One obtains in this way new results on lattice quantum fields (see [90] and [17]).

Work on Log-Sobolev inequalities for these models is in preparation [91,248].

5. RELATIVISTIC QUANTIZED STRINGS

One possible formulation of the problem of constructive relativistic quantized strings is to ask for a construction of a probability measure of the heuristic form

$$d\mu(X,g) = Z^{-1} e^{-\frac{1}{2} \int_M \sqrt{\det g} \alpha^a_p \frac{\partial X^a(x)}{\partial x^b} \partial \frac{\partial X^b(x)}{\partial x^a} dz} \prod_{\mu=1}^d \prod_{x \in M} dX_\mu(x) \prod_{x \in M} dg(x)$$

where $M$ is a compact Riemannian surface ("parameter space" of a moving string), $g$ is a Riemannian metric on $M$, $X$ is an embedding of $M$ in $\mathbb{R}^d$.

A parametrization of the space of all metrics $g$ is by conformal maps, diffeomorphisms of $M$ and Teichmüller parameters. The (heuristic) invariance of $\mu$ under diffeomorphisms heuristically reduces the integration of diffeomorphism invariant functions of $X$ and $g$ to an integration with respect to a measure on the product space $T \times C$, where $T$ is essentially a (finite dimensional) moduli space and $C$ is a space describing conformal maps. Keeping the variable in $T$ fixed and dropping it from the notation, the dependence of the measure on the variable $X \in C$ is of the form (2.2) with $\mathbb{R}^d$ replaced by $M$, $m = 0$ and with $V$ of the form

$$V(X) = \int_M e^{aX} dx,$$
with \( \alpha_d \) a real constant depending on \( d \). This corresponds to a random field with exponential interaction called mass zero Høegh-Krohn model or Liouville model.

See [81] for, at least partial, mathematical implementation of these ideas, for \( d \leq 13 \).

The above heuristic procedure for \( d = 26 \) yields a measure on \( T \) (see [81]).

It seems too difficult at present to go further in a full mathematical justification of other heuristic procedures in string theory, see however, e.g., references in [81] for further discussions.

Some of these heuristic procedures also relate aspects of string theory with conformal field theory and topological field theory.

6. QUANTIZED YANG–MILLS FIELDS

Heuristically the construction of a quantized Yang–Mills (non-Abelian) gauge field model can be related to the problem of constructing a probability measure \( \mu \) on the space of connection one-forms \( A \) on a principal fiber bundle with base a Riemannian manifold and structure group \( G \) a compact (Lie) group of the form

\[
d\mu(A) = Z^{-1} e^{-\int_M F_A \cdot F_A} dA^n,
\]

with \( F = DA = dA + [A, A] \) the curvature 2-form given by \( A \).

More precisely, we would like to give a meaning to integrals of gauge invariant functions of \( A \), like products \( \prod_{i=1}^{n} W^A(C_i) \) of holonomy operators \( W^A \) along loops \( C_i \), against \( \mu \).

This problem has been solved for \( \dim M = 2 \): [54–61, 72, 74, 206, 290, 292, 133, 134].

For partial solutions in the cases \( d = 3, 4 \) see e.g. [39, 73, 105, 108], as well as Section 7 below.

A model describing the “coupling” of a random field of above form and a scalar field \( X \) described by a measure of the form (2.2) is the Higgs model. For results on the Higgs model for \( d = 2 \) see [59, 60, 92].

**Remark 6.1.** Random fields \( A \) on manifolds with values in \( C \cong \mathbb{R}^2 \) or a Riemann surface \( N \) described by stochastic equations of the form \( \partial A = \mu \) with \( \mu \) an “\( N \)-valued noise” have been discussed in [72, 74] (in the case where \( N = \mathbb{R}^2 \) and \( \mu \) is the Gaussian white noise, \( A \) can be identified with an Euclidean Abelian gauge field). There are interesting connections with the theory of automorphic forms.

In some cases one can discuss the continuation of these fields to relativistic fields.

For extensions of such studies to the case \( d = 4 \) see [39, 73, 105, 108]. For the corresponding Higgs fields see [59, 60, 92, 105, 107]. For a general study in arbitrary dimensions see [135, 287, 288]. The Markov property of such fields has been studied particularly in [216, 218] and [288].

7. OSCILLATORY INTEGRALS, CHERN–SIMONS MODELS, KNOTS INVARIANTS

The problem of the construction of a quantized Chern–Simons (topological gauge) field model can be formulated as the problem of constructing a complex measure \( \mu \) on the space of the \( g \)-valued connection 1-forms ("classical gauge field") (\( g \) being the Lie algebra of some Lie group \( G \)), on a 3-dimensional compact Riemannian manifold \( M \), of the heuristic form

\[
\mu(dA) = Z^{-1} e^{i W(A)} \prod_{x \in M} dA(x),
\]
with \( W(A) \) the Chern–Simons functional

\[
W(A) \equiv \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad k \in \mathbb{R}
\]

Schwarz–Witten–Atiyah’s conjecture states that the integral

\[
I = \int \prod_{i=1}^m W^A(C_i) \mu(dA)
\]

whenever it exists is a topological invariant associated with \( M \) (\( W^A(C) \) denotes as before the holonomy operator given by \( A \) along the loop \( C \)). For \( M = S^3 \), \( G = SU(3) \) Witten heuristically argued that this invariant should be Jones’ knot polynomial. The mathematical deduction of such a polynomial from a certain combinatorial substitute of the Chern–Simons model was done by Turaev and Viro, a proof of the above conjecture is however still lacking, except for the Abelian case, where the conjecture was proven by Schäfer [286] (see [103]), using Albeverio–Høegh-Krohn’s theory of oscillatory integrals [43]. A sketch of this proof goes as follows.

In the Abelian case we have \( G = U(1) \), \( W^A(C) = e^{\alpha \int_C A} \), \( \alpha \in \mathbb{Z} \). \( I \) is expressed as the normalized oscillatory integral of \( f \) for a suitable function \( f \) related to \( \prod_{i=1}^n W^A(C_i) \). Normalized oscillatory integrals in the sense of [43] are defined as follows. Let \( \mathcal{H} \) be a Hilbert space (or, more generally, a Banach space). Let \( \mathcal{F}(\mathcal{H}) \) be the Banach algebra of the functions \( f \) on \( \mathcal{H} \) such that \( \exists \mu(f) \in \mathcal{M} \mathcal{H} \equiv \{ \text{complex measures on } \mathcal{H} \} \) such that \( \hat{\mu}(f) = f \) (\( \hat{\cdot} \) meaning Fourier transform).

Let

\[
I_h(f) \equiv \int_{\mathcal{H}} e^{-i \frac{\gamma}{h}(\alpha, B^{-1} \gamma)} d\mu(f)(\alpha),
\]

where \( B \) is a self-adjoint operator on \( \mathcal{H} \) such that \( B^{-1} \) is bounded.

\( I_h(f) \) is called “normalized oscillatory integral of \( f \)” (given by \( B, h > 0 \)). A suggestive notation for it is

\[
\int_{\mathcal{H}} e^{i \frac{\gamma}{h}(\alpha, B^{-1} \gamma)} f(\gamma) d\gamma.
\]

E.g. for \( B = \mathbb{1} + L \), \( L \) trace class, \( I_h(f) \) is the limit, as \( n \to \infty \), of corresponding finite dimensional oscillatory integrals

\[
(\det B_n)^{-\frac{1}{2}} (2\pi ih)^{-\frac{5}{2}} \int_{\mathbb{R}^2} e^{i \frac{\gamma}{h}(x, B_n \gamma)} f(x) dx,
\]

\( B_n \) being the restriction of \( B \) to an \( n \)-dimensional subspace (\( \cong \mathbb{R}^n \)) of \( \mathcal{H} \).

Important properties of \( I_h(f) \) (which makes it possible to look at \( I_h(f) \) as a “generalized complex integral”) are:

1. \( f \to I_h(f) \) is a linear continuous functional, normalized in the sense that \( I_h(1) = 1 \).
2. Fubini theorem about "iterated integration" (with respect to direct splittings $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$) holds.

3. $I_h(f)$ has natural transformation properties with respect to translations and rotations in $\mathcal{H}$.

4. $I_h(f)$ has natural relations with "similar" probabilistic integrals (e.g. analytic continuation or limit of such).

$I_h(f)$ is however not the integral of $f$ with respect to a $\sigma$-additive complex measure on $\mathcal{H}$. That infinite dimensional oscillatory integrals cannot be understood as (improper) integrals with respect to $\sigma$-additive complex measures was first pointed out in connection with Wiener measure by Cameron (see [152]). For the study of above normalized oscillatory integrals see e.g. [19,22–28,43,12] and references therein.

Let us now return to our claim that the Chern–Simons functional $I$ can be interpreted as $I(f)$, for a certain $f$, for $h = 1$ and a certain $B$. First we have to give $\mathcal{H}$: $\mathcal{H}$ is here by definition $\overline{\Omega^1 + \Omega^3}$, with () meaning closure, $\Omega^i$ being i-forms (in Hodge sense). The role of $A$ in above notation is played by $\gamma$, the one of $A \wedge dA$ by $(\gamma, B\gamma)$, with $B \equiv \frac{k}{2\pi}(d + \delta + P)$, $P$ being the projection into the kernel of $(d + \delta)$ ($d$ being differentiation and $\delta$ its Hodge dual). Finally $f$ is such that

$$f(A) = \prod_{i=1}^{N} \tilde{W}^\varepsilon_i(\epsilon_i),$$

with $\tilde{W}^\varepsilon_i$ a regularized version of $W^A$ defined such that

$$\tilde{W}^\varepsilon_i(C) = K(\varepsilon, \alpha)e^{i\alpha j_i(C),A},$$

$K(\varepsilon, \alpha)$ being a positive constant depending on $\varepsilon$, $\alpha$ and $<j_\varepsilon(C),A>$ being the evaluation of $A$ along a neighborhood of $C$ in the sense of De Rham currents. It is then possible to show that $f \in \mathcal{F}(\mathcal{H})$ and that for $\varepsilon$ sufficiently small $I(f)$ is independent of $\varepsilon$ and is a topological invariant, namely

$$I(f) = e^{-\frac{i\pi}{2} \sum_{i \neq j, \epsilon \neq \epsilon'} \alpha_\epsilon \alpha_{\epsilon'} LK(C_i, C_{\epsilon'})},$$

$LK$ being the linking number.

For $M = \mathbb{R}^3$, $LK$ is the classical Gaussian linking number.

Remark 7.1. In [247] a representation of the Chern–Simons oscillatory integral $I(f)$ in terms of white noise calculus has been found. Similar methods are presently being developed for the case of the non-Abelian Chern–Simons model [104].

8. SIMPLICIAL APPROXIMATION OF QUANTUM FIELDS AND OF THE
CHERN–SIMONS ACTION

Let $M$ be a smooth compact manifold of dimension $d$. Let $K$ be a smooth triangulation. Let $\sigma^q_i \equiv [p_{0,i}, \ldots, p_{q,i}], q = 1, \ldots, d$, be a $q$-simplex of $K$. Let $C_q(K)$ be the linear space of all formal linear combinations of $q$-simplexes of $K$. Let $dC: C_q(K) \to C_{q+1}(K)$ be the natural differentiation.

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Let $\tilde{W} : C_\sigma(K) \to \Omega^s M \equiv L^2 - q$-forms such that

$$\tilde{W}(\sigma^t) \equiv q! \sum_{i=0}^q (-1)^i b_{p_i} d b_{p_0} \wedge \ldots \wedge d b_{p_i} \wedge \ldots \wedge d b_{p_q},$$

$b_{p_i}$ being the barycentric coordinates. $\tilde{W}$ is called "Whitney mapping". The mapping $R : \Omega^s M \to C_q(K)$ given by $R\omega = \sum_i (\int_{\sigma_i} \omega) \sigma_i^t$ is called De Rham mapping. $R$ and $\tilde{W}$ are "inverse" to each other in the sense that

$$\tilde{W} R \omega \longrightarrow_{n \to \infty} \omega \quad \forall \omega \in \Omega^s M,$$

the convergence being in $L^2$. When $\{K_n\}$ is a convergent simplicial approximation of $M$ and $\tilde{W}_n \equiv \tilde{W}(K_n)$, then $R_n \equiv R(K_n)$. This construction has been applied to quantum fields in [110]. In that paper simplicial quantum fields $\chi_n$ on a manifold have been constructed.

We shall now discuss the simplicial approximation of the Chern-Simons action. Let us define the following modified cup-product $\tilde{U} : C_p(K) \times C_q(K) \to C_{p+q}(K)$

$$\langle C\sigma^t \tilde{U} C\sigma^t, \{v_0, \ldots, v_{p+q}\} \rangle \equiv \sum_{\sigma} (p+q+1)!^{-1} \langle \sigma^t, \{v_i, \ldots, v_q\} \rangle \langle C\sigma^t, \{v_i, \ldots, v_{p+q}\} \rangle \text{ sign } \sigma.$$

In the Abelian case, the combinatorial Chern-Simons action $W_{C,S}(R_n A)$ can be defined by

$$W_{C,S}(R_n A) \equiv \int_M \tilde{W}(R_n A \tilde{U} d R_n A), \quad R_n A \in C_q.$$

It is shown in [102] that $W_{C,S}(R_n A)$ converges in $L^2$ to the Chern-Simons action $W_{C,S}(A) = A \wedge dA$ used in the definition of the Chern-Simons model. As discussed in [102], we expect the convergence "$Z^{-1} \int e^{i W_{C,S}(R_n A)} \prod_{1 \leq i \leq n} W^{R_n A}(C_i) d(R_n A)$" as $n \to \infty$ to $I(f)$ as defined above.

9. PERIODIC ORBITS IN CLASSICAL HAMILTONIAN SYSTEMS AND ENERGY EIGENVALUES OF THE CORRESPONDING QUANTUM SYSTEMS.

THE TRACE FORMULA FOR SCHröDINGER OPERATORS

Let us consider a classical Hamiltonian system with classical Hamiltonian

$$H_{cl} = \frac{p^2}{2} + \frac{1}{2} (x, A^2 x) + V - c,$$

$c$ being a constant, $x, p \in \mathbb{R}^d$, $A^2$ a symmetric positive $d \times d$-matrix, $V \in \mathcal{F}(\mathbb{R}^d)$ (in the notation of Section 7). This describes a particle moving in $\mathbb{R}^d$ under the influence of a harmonic (linear) force $-Ax$ and a nonlinear force $-\nabla V(x)$. The corresponding quantum mechanical Hamiltonian is

$$H = -\frac{\hbar^2}{2} \Delta + \frac{1}{2} (x, A^2 x) + V - \frac{1}{2} \text{tr } A$$

as acting in $L^2(\mathbb{R}^d, dx)$ ($\hbar$ being Planck's constant) ($c$ can be chosen so that the infimum of the spectrum of $H$ when $V = 0$ is the simple, isolated eigenvalue zero). $H$ has spectrum consisting...
of eigenvalues $\lambda_n(K) \geq 0$, $n \in \mathbb{N}$. Considering the setting of Section 7, with $\mathcal{H}$ the Sobolev space $H^{1,2}([0,t];\mathbb{R}^d)$ and

$$
(\gamma, B\gamma) = \frac{1}{2} \int_0^t |\gamma(s)|^2 ds - \frac{1}{2} \int (\gamma, A^2\gamma) ds,
$$

$\gamma \in H^{1,2}([0,t];\mathbb{R}^d)$, it is possible to express $\text{Tr} \ e^{i\mathcal{H}}$ by a normalized oscillatory integral in the sense of Section 7.

Now from the general (rigorous, mathematical) method of stationary phase for such infinite dimensional integrals developed in [44,278,19,22–28], we get (under some additional assumptions on $V$ and for all $t$ outside a certain precise discrete set) that

$$
\text{Tr} \ e^{i\mathcal{H}} = \int_{\mathbb{R}^d} e^{i\mathcal{H}(x,x)} dx = \sum_{n} e^{i\lambda_n(h)}
$$

has an asymptotic expansion in powers of $h$ of the form $c(h)(1 + c_1 h + c_2 h^2 + \ldots)$, where $c_n$, $n \in \mathbb{N}$, are independent of $h$, and $c_0(h)$ is explicitly known. The $c_i$, $i \in \mathbb{N} \cup \{0\}$, are given in terms of the closed periodic orbits of $H_{cl}$. The leading term in $h$ of this formula corresponds to Selberg's trace formula, e.g. in the case where $H$ is replaced by the Laplace–Beltrami operator on a surface of constant negative curvature. In the physical literature the approximate equality of the above trace with this leading term is often called Guttwiller's trace formula (and the left hand side is set equal to $c_0(h)$ in a "semiclassical approximation"). For these results see [19,28] These results can be extended to the case of Hamiltonians containing an additional magnetic field (see [24]). For corresponding results for the trace of the heat kernel see [213,214,256]. For other trace formulae (obtained after [19]) see e.g. [148] and references therein.

10. A FINAL SHORT COMMENT

There are many other problems and areas of research where loop variables and spaces play a role. E.g. there are in the physical literature formulations of gauge fields and gravitational fields in terms of loop variables (see e.g. [120–122,127–130,283]). Most of these considerations do not yet have a mathematical form, in fact they depend on the development of more powerful mathematical methods to cope with the heuristic use of functional integration. It would be interesting to integrate proposals in the physical literature like the ones in above references into the existing (or extended) present settings for probabilistic, resp. oscillatory, path integrals (see [14] and references therein). Also the extensive use of supersymmetric tools and $\varphi$-deformations, often on a heuristic level (see e.g. [276]), should receive a more thorough mathematical treatment. Finally let us mention methods of non-commutative differential geometry [157] which have been used in loop space theory, often without much contact with other developments. Loop space theory has been developed from analytic, probabilistic, algebraic-topological, differential-topological, differential geometric and physical points of view. More often these developments have only had an explorative character. It can be hoped that in the future some stronger coordination of efforts can be achieved, probably the only possible way to really obtain major breakthrough in this exciting area of research.
A SURVEY OF SOME DEVELOPMENTS IN LOOP SPACES

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