GROUPS GENERATED BY INVOLUTIONS,
GELFAND-TSETLIN PATTERNS, AND
COMBINATORICS OF YOUNG TABLEAUX

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Abstract. We construct certain families of piecewise linear representations (cpI-representations) of the symmetric group $S_n$ and of the affine Weyl group $\tilde{S}_n$ of type $A_n^{(1)}$ acting on the space of triangles $X_n$. We find a nontrivial family of local cpI-invariants for the action of the symmetric group $S_n$ on the space $X_n$ and construct a global invariant with respect to the action of the affine Weyl group $\tilde{S}_n$ (the so-called cocharge). We find continuous analogs for the Kostka-Foulkes polynomials and for the crystal graph. We give an algebraic version of some combinatorial transformations on the set of standard Young tableaux.

Introduction

In this paper we define and study a new class of representations of the symmetric group $S_n$, namely, the class of continuous piecewise linear representations (cpI-representations) of $S_n$ in the space of triangles $X_n$. By definition, a triangle $x \in X_n$ is a triangular array of real numbers $x = (x_{ij})$, $x_{ij} \in \mathbb{R}$, $1 \leq i \leq j \leq n$. More precisely, as in [GZ1, GZ2, BZ1], we consider the Gelfand-Tsetlin cone $K_n$ consisting of all triangles $x \in X_n$ such that

\[
x_{ij} \geq x_{i+1,j+1}, \quad 1 \leq i \leq j \leq n - 1,
\]
\[
x_{ij} \geq x_{i,j-1}, \quad 1 \leq i \leq j \leq n,
\]
\[
x_{ij} \geq 0, \quad 1 \leq i \leq j \leq n.
\]

This is a nondegenerate convex polyhedral cone in the space $X_n \cong \mathbb{R}^{n(n+1)/2}$ having $2^n$ generators (see Proposition 2.3). It is well known (see, e.g., [GZ1]) that the set of integral points, $(K_n)_\mathbb{Z}$, of the cone $K_n$ is in one-to-one correspondence with the set $STY(n)$.

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of standard* Young tableaux having all entries not exceeding $n$. The $cpl$-action of the symmetric group $S_n$ on the space $X_n$ given in our paper is such that it preserves the Gelfand–Tsetlin cone $K_n$ and, on the set $STY(n)$, it coincides with the action of the symmetric group on the set of standard Young tableaux given by A. Lascoux and M.-P. Schützenberger [LS2, LS3] (see Theorem 2.3).

Our main observation is that a great many of combinatorial constructions on the set of standard Young tableaux, e.g., the Schützenberger involution [Sch1, EG, Ki1], the dual Schützenberger involution [Sch1], the promotion transformation [Sch1, EG], the action of the symmetric group [LS2, LS3], the crystal graph structure on the set $STY(\lambda, \leq n)$, [Ka1, Ka2], the construction of the cocharge [LS1, Ki1], can be transferred to the Gelfand–Tsetlin cone $K_n$, and even to the entire space of triangles $X_n$. Our constructions are based on the study of "elementary transformations" $t_j, 1 \leq j \leq n - 1$, and $T_e, e \in \mathbb{R}$.

**Definition 0.1.** If $x \in X_n$, then we set $t_j(x) = \tilde{x}$, where

$$
\begin{align*}
  x_{ik} &= x_{ik} \quad \text{if } k \neq j, \\
  \tilde{x}_{ij} &= \min(x_{i,j+1}, x_{i-1,j-1}) + \max(x_{i,j-1}, x_{i+1,j+1}) - x_{ij},
\end{align*}
$$

and we assume that $x_{0j} := +\infty, x_{j,j-1} := -\infty, 1 \leq j \leq n - 1$; also, $T_e(x) = \tilde{x}$, where

$$
\begin{align*}
  \tilde{x}_{ik} &= x_{ik} \quad \text{if } (i, k) \neq (1, 1), \\
  \tilde{x}_{11} &= x_{11} - e.
\end{align*}
$$

We denote by $G_n = \langle t_1, \ldots, t_{n-1} \rangle$ the group generated by $t_i, 1 \leq i \leq n - 1$. The transformations $t_i, 1 \leq i \leq n - 1$, satisfy the following relations (see Corollary 1.1):

$$
\begin{align*}
  &\text{(i) } t_i^2 = 1, \ t_it_j = t_jt_i \quad \text{if } |j - i| \geq 2, \\
  &\text{(ii) } (t_1t_2)^6 = 1, \\
  &\text{(iii) } (t_1q_i)^4 = 1 \quad \text{if } 3 \leq i \leq n - 1,
\end{align*}
$$

where $q_i := t_1t_2t_1t_3t_2t_1 \cdots t_{i-1}t_i \cdots t_1$.

The restriction of the involutions $t_i$ to the set of standard Young tableaux $STY(\lambda, \beta)$ of a given shape $\lambda$ and content $\beta$ admits a simple combinatorial interpretation. Precisely these restrictions are commonly used in order to prove that Schur functions are symmetric, see e.g., [BK, SW, Sa], and Section 2A. We conjecture that relations (0.2) are the defining relations for the group $G_n$. Anyway, the group $G_n$ seems to be very interesting. It is easy to see that the order of the group $G_3$ is equal to 12. But if $n \geq 4$, then $G_n$ is infinite and for any $N$ there exists an epimorphism of the group $G_4$ onto the symmetric group $S_N$ (see comment v) after Corollary 1.3). The group $G_n$ admits an extension $\tilde{G}_n$ by means of $\mathbb{R}^1$:

$$
\tilde{G}_n := \langle t_1, \ldots, t_{n-1}, T_e, e \in \mathbb{R} \rangle.
$$

*The term "semistandard" is often used for this type of tableaux (ed.).
We have the following relations between the generators in the group $G_n$:

(i) $T_e \cdot T_{\delta} = T_{e+\delta}$,

(ii) $(t_1 T_e)^2 = 1$, $t_j T_e t_j = T_e$, $3 \leq j \leq n - 1$, $e \in \mathbb{R}$,

(iii) $t_i t_j = t_j t_i$ if $|j - i| \geq 2$,

(iv) $(T_i t_1 T_\delta t_2 t_1)^3 = 1$ for any $\epsilon, \delta \in \mathbb{R}$,

(v) $T_\epsilon t_2 T_{-\delta} t_2 T_{-\delta} t_2 = t_2 T_\delta t_2 T_{-\delta} t_2 T_{-\delta}$,

(vi) $(t_i T_\delta q \delta t_j T_\delta q \delta)^2 = 1$, $3 \leq j \leq n - 1$,

(vii) $T_\epsilon q_j T_\delta q_j = q_j T_\delta q_j T_\epsilon$, $3 \leq j \leq n - 1$, $\epsilon, \delta \in \mathbb{R}$,

(viii) $T_\epsilon t_2 T_\epsilon t_1 T_\delta t_2 T_\delta t_1 t_2 = t_2 T_\epsilon t_1 t_2 T_\delta t_2 T_\epsilon t_1 t_2 T_\theta$ for any $\epsilon, \delta, \theta \in \mathbb{R}$,

where the transformation $q_j$ is defined in (0.2).

The proofs of relations (i)-(v) are based on direct computations. The main difficulties arise in the proof of (vi). In order to better understand relations (0.2) and (0.3), we consider the following elements in the group $G_n$:

\begin{align*}
s_i &:= q_i t_1 q_i^{-1}, \quad 1 \leq i \leq n - 1, \\
s_i^e &:= q_i T_{-\epsilon} q_i^{-1}, \quad 1 \leq i \leq n - 1.
\end{align*}

The main result of Section 1 is Theorem 1.1, which is equivalent to relations (i)-(vi) and asserts that

1*. The involutions $s_i$, $1 \leq i \leq n - 1$, satisfy the relations of the symmetric group $S_n$, i.e.,

a) $s_i^2 = 1$, $1 \leq i \leq n - 1$,

b) $(s_i s_{i+1})^3 = 1$, $1 \leq i \leq n - 2$, (0.6)

c) $s_i s_j = s_j s_i$ if $|i - j| \geq 2$.

2*. The transformations $s_i^{(\epsilon)}$, $1 \leq i \leq n - 1$, $\epsilon \in \mathbb{R}$, satisfy the colored braid relations, i.e., for any $\epsilon, \delta \in \mathbb{R}$ we have

\begin{align*}
a) & \quad s_i^{(\epsilon)} \cdot s_i^{(\delta)} = s_i^{(\epsilon+\delta)}, \quad 1 \leq i \leq n - 1, \\
b) & \quad s_i^{(\epsilon)} \cdot s_{i+1}^{(\epsilon+\delta)} \cdot s_i^{(\delta)} = s_{i+1}^{(\epsilon+\delta)} \cdot s_i^{(\epsilon)} \cdot s_{i+1}^{(\delta)}, \quad 1 \leq i \leq n - 2, \\
c) & \quad s_i^{(\epsilon)} \cdot s_j^{(\delta)} = s_j^{(\delta)} \cdot s_i^{(\epsilon)} \quad \text{if} \quad |i - j| \geq 2.
\end{align*}

Relation (0.3), (vi) is equivalent to the statement that the transformations $s_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute if $|i - j| \geq 2$. In order to prove this latter statement, we use another expression for $s_i^{(\epsilon)}$, $1 \leq i \leq n - 1$, namely, a product of Lusztig involutions [Lu2]. We recall the corresponding definitions, since we use not exactly the involutions contained in [Lu2] but rather their analogs for the space of triangles $X_n$. 

Definition 0.2. For each triple of integers \((ijk)\), \(1 \leq i < j < k \leq n\), we define a transformations \(R_{ijk} : X_n \to X_n\) in the following way:

\[
R_{ijk}(x) = \tilde{x}, \quad x \in X_n,
\]

where

\[
\tilde{x}_{i,j} = x_{i,j} + x_{i,k} - x_{j,k-1} - \min(x_{j,k} - x_{j,k-1}, x_{i,j} - x_{i,j-1}),
\]

\[
\tilde{x}_{j,j} = x_{j,j} + x_{i,k} - x_{j,k-1} + \min(x_{j,k} - x_{j,k-1}, x_{i,j} - x_{i,j-1}),
\]

\[
\tilde{e}_{a} = x_{a} \quad \text{if} \quad (a, \beta) \neq (i, j) \quad \text{or} \quad (j, j).
\]

We denote by \(L_n\) the group generated by all \(R_{ijk}\) with \(1 \leq i < j < k \leq n\). We have the following relations between the generators of the group \(L_n\):

(i) \((R_{ijk})^2 = 1\),

(ii) \(R_{ijk} \cdot R_{i'j'k'} = R_{i'j'k'} \cdot R_{ijk}\) if \(|(ijk) \cap (i'j'k')| \neq 2\),

(iii) \((R_{ijk}R_{i'j'i})^2 = 1\),

(iv) \((R_{ij}R_{i'j'})^2 = 1\) if \(1 \leq i < j < k < l \leq n\).

We conjecture that relations (0.9) are defining relations for the group \(L_n\).

To go further, we define some transformations \(T_{i}^{(i)}\) acting on the space \(X_n\):

\[
T_{i}^{(i)} = \tilde{x}, \quad x \in X_n,
\]

where

\[
\tilde{x}_{ii} = x_{ii} - \epsilon, \quad \tilde{e}_{a} = x_{a} \quad \text{if} \quad (a, \beta) \neq (i, i).
\]

The statement that \(s_{i}^{(i)}\) and \(s_{j}^{(i)}\) commute if \(|i - j| \geq 2\) can be derived from the following expression for \(s_{i}^{(i)}\), \(1 \leq i \leq n - 1\) (Theorem 1.2):

\[
s_{i}^{(i)} = R_{i-1,i}R_{i-2,i} \cdots R_{1,i}T_{i}^{(i)}R_{1,i} \cdots R_{i-2,i}R_{i-1,i},
\]

where \(R_{jk} := R_{jk+k+1}\), \(1 \leq j < k \leq n - 1\). The proof of (0.10) is based on induction, and on the recurrence formula for \(s_{i}^{(i)}\) (see (1.29)).

In order to obtain the corresponding results for the involutions \(s_{i}\) (see (1.28a)), we use

Crucial Lemma. Assume that \(x \in X_n\), \(\beta(x) = (\beta_1, \cdots, \beta_n)\). Then

\[
s_{i}(x) = s_{i}^{(\beta_i-\beta_{i+1})}(x),
\]

where \(\beta(x)\) is the weight of a triangle \(x \in X_n\), i.e., \(\beta_i(x) := |x^{(i)}| - |x^{(i-1)}|, 1 \leq i \leq n\), \(x^{(0)} := \phi\).

Relations (0.3) between the generators of the group \(\bar{G}_n\) allow us to construct many interesting subgroups in \(\bar{G}_n\). For example:

*The notation \(|x^{(i)}|\) is explained in the line after formula (1.1) below (ed.)
1°. Let $\epsilon \in \mathbb{R}$ be fixed. Then the elements $s_1 \epsilon s_1^{(\epsilon)}$, $s_2 \epsilon s_2^{(\epsilon)}$, $\ldots$, $s_{n-1} \epsilon s_{n-1}^{(\epsilon)}$ are the standard generators of the symmetric group $S_n$.

2°. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$ be the real numbers, $\epsilon_0 \neq 0$, and let $\tilde{s}_i := \tilde{s}_i^{(\epsilon_i)} = s_i \epsilon s_i^{(\epsilon_i)}$ for $i = 1, \ldots, n - 1$. We put

$$\tilde{s}_0 := \tilde{s}_0^{(\epsilon_0)} = \tilde{s}_{n-1} \tilde{s}_{n-2} \ldots \tilde{s}_1 \epsilon \tilde{s}_1^{(\epsilon_1)} \tilde{s}_2 \ldots \tilde{s}_{n-1}.$$  

Then $\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{n-1}$ are the standard generators of the affine Weyl group of type $A_n^{(1)}$. It is important that the cocharge $\tilde{c}(x)$ of a triangle $x \in X_n$ (Section 3) is invariant with respect to the action of all involutions $\tilde{s}_i^{(\epsilon_i)}$, $0 \leq i \leq n - 1$.

We summarize the content of Section 1 of our paper. We construct a family of the $\epsilon$-representations of the symmetric group, find a family of $\epsilon$-representations of the affine Weyl group of type $A_n^{(1)}$ (see the paragraph after the proof of Corollary 1.3) and construct a $\epsilon$-representation of the colored braid relations. We develop geometric techniques for proving some (nontrivial) identities between piecewise linear maps (see Theorem 1.3). Our next step is to give a combinatorial interpretation of transformations under consideration and to study continuous piecewise linear invariants with respect to the action of the symmetric group generated by $s_1, \ldots, s_{n-1}$, on the space of triangles $X_n$. In Section 3 we prove (see Theorem 3.2) that the following $\epsilon$-functions on the space of triangles $X_n$ are $s_j$-invariants:

$$\psi_{1j}(x) = \min(x_{1,j+1} - x_{1,j}, x_{j,j} - x_{j+1,j+1}),$$

$$\psi_{ij}(x) = \min(x_{i-1,j} - x_{i-1,j-1}, x_{i,j+1} - x_{i,j}),$$

where $2 \leq i \leq j \leq n - 1$,

$$\psi_{1j}(x) = (\min(x_{j-1,j} + x_{j,j} - x_{j-1,j-1} - x_{j,j+1}, x_{1,j+1} + x_{j+1,j+1} - x_{1,j} - x_{j,j}))^+,$$

$$\psi_{2j}(x) = (\min(x_{1,j} + x_{j,j} - x_{1,j-1} - x_{j+1,j+1}, x_{1,j-1} + x_{2,j+1} - x_{1,j} - x_{2,j}))^+,$$

$$\psi_{ij}(x) = (\min(x_{i-2,j} + x_{i-1,j} - x_{i-2,j-1} - x_{i-1,j+1}, x_{i-1,j-1} + x_{i,j+1} - x_{i-1,j} - x_{i,j}))^+,$$

where $3 \leq i \leq j \leq n - 1$, and for any $a \in \mathbb{R}$, $(a)^+ := \max(a, 0)$. We recall that a $\epsilon$-function $\varphi$ defined on the space $X_n$ is said to be an $s_j$-invariant if $\varphi(s_j(x)) = \varphi(x)$ for all $x \in X_n$.

It seems very interesting to find a fundamental system $I_j$ of $\epsilon$-invariants for $s_j$, that is, a system such that any $\epsilon$-invariant under the action of $s_j$ is a min–max linear combination of those from $I_j$. It is not clear whether or not such a set can be chosen finite, because we have trivial examples of the following kind:

$$x_{ik} \text{ if } k \neq j, \text{ or } \min(\varphi(x), \varphi(s_j(x))),$$
where \( \varphi(x) \) is any cpl-function on \( X_n \). However, the number of fundamental cpl-\( S_n \)-invariants having complexity bounded by some integer \( k \) is finite. Here we define the complexity of a cpl-function as the smallest number of min's and max'es needed for its representation. (It should be noted that in this definition of complexity we regard min and max as functions of two variables, e.g., \( \min: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). Thus, the complexity of the function \( \min(x_1, x_2, \ldots, x_m) \) is equal to \( m - 1 \).) Anyway, we can construct (see the proof of Theorem 3.2) other nontrivial examples of cpl-\( S_n \)-invariants, and it is not clear whether or not they are independent (in the cpl-sense) of the invariants (0.12) and (0.13).

The next interesting problem is to describe the fundamental system of continuous piecewise polynomial functions (cpp-functions) on the space \( X_n \) that are invariant with respect to the action of the subgroup \( S_n \subseteq G_n \). Of course, we have such trivial examples as \( x_i, 1 \leq i \leq n, \) or \( \min\{ \varphi(\sigma(x)) \mid \sigma \in S_n \} \), or \( \sum_{\sigma \in S_n} \varphi(\sigma(x)) \), where \( \varphi(x) \) is cpp-function on \( X_n \). But the question is as follows: Do there exist nontrivial cpl-\( S_n \)-invariants under the action of the symmetric group \( S_n \) on the space \( X_n \)? In Section 3 we give an affirmative answer to this question (see Theorem 3.1). In fact, we define a stable (ibid.) cpl-invariant with respect to the action of a family of affine Weyl groups of type \( A_n^{(1)} \) (namely, \( \tilde{S}_n := < s_0, s_1, \ldots, s_{n-1} > \), see Corollary 1.3) that takes nonnegative values on the Gelfand–Tsetlin cone \( K_n \). Note that the complexity of the invariant under consideration is equal to \( (n - 1)! - 1 \), if \( n \geq 3 \). The origin of such invariant is very clear. We have a distinguished \( S_n \)-invariant function on the set \( STY(n) \), namely, the cocharge \( \overline{c}_n(T) \) of a tableau \( T \in STY(n) \), as defined by A. Lascoux and M.-P. Schützenberger [LS1]. Note that it is possible (at least for \( n \leq 5 \) and conjecturally for all \( n \)) to reformulate the definition of the cocharge of a tableau \( T \in STY(n) \) in terms of the corresponding Gelfand–Tsetlin pattern \( x(T) \) and to obtain some function \( \overline{c}_n \) on the set \( (K_n)_\mathbb{Z} \). We consider a natural extension of the function \( \overline{c}_n \) to the whole space \( X_n \), by changing its domain of definition from \( (K_n)_\mathbb{Z} \) to \( X_n \). This yields the desired cpl-\( \tilde{S}_n \)-invariant, \( \overline{c}_n(x), x \in X_n \), which we still call cocharge. Details are contained in Section 3. We define another cpl-\( S_n \)-invariant \( \psi(x) \) (which is not \( \tilde{S}_n \)-invariant!) by setting \( \psi(x) = \overline{c}_n(q_{n-1}(x)) \), \( x \in X_n \), where the involution \( q_{n-1} \) is defined in (0.2). As an example, we give an expression for the cocharge \( \overline{c}_4 \):

\[
\overline{c}_4(x) = \min(x_{13} - x_{12}, x_{22} - x_{33}) + x_{14} - x_{13} + x_{33} - x_{44} \\
+ \min(x_{23} - x_{34}, x_{24} - x_{23}, x_{13} - x_{12}, x_{22} - x_{33}, \beta_4(x) - x_{34}, x_{24} - \beta_4(x)).
\]

It is known (see, e.g., [H]) that the volume of the convex polytope \( K^\lambda(\beta) \) consisting of all Gelfand–Tsetlin patterns with highest weight \( \lambda \) and weight \( \beta \) (see Section 1) may be regarded as a continuous analog of the weight multiplicity \( K_{\lambda,\beta} := \dim V_{\lambda}(\beta) \). We define a continuous analog of the Kostka–Foulkes polynomial \( (q\text{-analog of the weight multiplicity}) \) [LS1, Ma, Ki1], by means of following integral:

\[
K_{\lambda,\beta}(q) = \int_{K^\lambda(\beta)} \exp(h\overline{c}_n(x))dx, \quad q = \exp(h). \tag{0.14}
\]

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We expect to study the properties of the integral (0.14), and the toric variety corresponding to the Gelfand–Tsetlin cone \( K_n \), in a separate publication. In Remark 3.2 we give a generalization of the Lascoux–Schützenberger algorithm for computing the charge of a dominant weight standard Young tableau to the case of a standard Young tableau with an arbitrary weight.

In Section 2 we study the restrictions of the transformations considered in Section 1 to the set of integral points \((K_n)_\mathbb{Z}\) of the Gelfand–Tsetlin cone \( K_n \). We show that the involution \( q_i : STY(n) \to STY(n), \ 1 \leq i \leq n - 1 \), (see (0.2)) coincides with the partial Schützenberger involution \( S_i : STY(n) \to STY(n) \). For a given tableau \( T \in STY(n) \), the involution \( S_i \) acts nontrivially only on the part \( T_{\leq i+1} \) of the tableau \( T \) filled with the numbers \( 1, \ldots, i + 1 \), and on this part coincides with the ordinary Schützenberger’s involution, see [Sch1] or Section 2C. We note that the group generated by the involutions \( q_i, \ 1 \leq i \leq n - 1 \), coincides with the group \( G_n = \langle t_1, \ldots, t_{n-1} \rangle \), and, thus, we may describe the relations between the partial Schützenberger involutions (see Remark 1.3).

Further on, we have the following relation between the partial Schützenberger involution \( q_i \) and the action of the symmetric group \( S_n \) on the set \( STY(n) \), as defined by A. Lascoux and M.-P. Schützenberger [Sch2, Sch3]:

\[
s_i = q_i q_1 q_i, \quad 1 \leq i \leq n - 1,
\]

(0.15)

where \( s_i := (i, i+1) \in S_n \) is a simple transposition.

The Schützenberger involution \( S \) possesses many interesting properties in connection with the Robinson–Schensted correspondence [Sch2], with the rigged configurations [Ki1], and so on. In addition, we explain in Remark 2.4 that the involution \( S \) allows one to give a simple purely combinatorial proof of the following symmetry property of the Littlewood–Richardson numbers (see Proposition 2.8):

\[
e_{\lambda\mu}^\nu = e_{\lambda\nu^*}^\mu^*.
\]

(0.16)

All other symmetries of the LR-numbers follow from (0.16) and the symmetries of the Berenstein–Zelevinsky triangles (see [BZ3] and Remark 2.4).

Finally, it is interesting to point out the following relationship between our transformations \( s_i^{(\pm 1)}, \ 1 \leq i \leq n - 1 \), restricted to the set of integral points of the convex polytope \( K^\lambda \) (see Section 1), and the crystal graph corresponding to the irreducible representation \( V_\lambda \) of the Lie algebra \( \mathfrak{gl}_n \) with highest weight \( \lambda \); see [Ka1, Ka2, KN].

Denote by \( f_i \) (respectively, \( e_i \)) the restriction of the map \( s_i^{-1} \) (respectively, \( s_i^{+1} \)) to the GT-polytope \( K^\lambda \) (the transformations \( s_i^{(\pm 1)} \) do not preserve the GT-cone \( K_n \), so that for \( x \in K_n \) we define \( f_i(x) = s_i^{-1}(x) \), if \( s_i^{-1}(x) \in K_n \) and \( f_i(x) = 0 \), if \( s_i^{-1}(x) \notin K_n \)). Let \( (L(\lambda), B(\lambda)) \) be the crystal base of an irreducible \( \mathfrak{gl}_n \)-module \( V_\lambda \) with highest weight \( \lambda \), and let \( \overline{F}_i, \overline{E}_i, \ 1 \leq i \leq n - 1 \), be the deformed generators of the Hopf algebra \( U_q(\mathfrak{gl}_n) \), as defined by M. Kashiwara [Ka1, Ka2]. Then there exist a bijection

\[
N : B(\lambda) \to K^\lambda \cap (X_n)_\mathbb{Z}
\]
such that

(i) \( b \in B(\lambda)_\beta := B(\lambda) \cap V_\lambda(\beta) \) if and only if \( N(b) \in K^\lambda(\beta) \cap (X_n)_\mathbb{Z}, \)

(ii) if \( \tilde{F}_i(b) \in B (\iff \tilde{F}_i(b) \neq 0), \) then \( N(\tilde{F}_i(b)) = s_i^{-1}(N(b)), \)

(iii) if \( \tilde{E}_i(b) \in B (\iff \tilde{E}_i(b) \neq 0), \) then \( N(\tilde{E}_i(b)) = s_i^{+1}(N(b)). \)

The classical representation theory of the symmetric and general linear groups is based on various combinatorial constructions among which those involving Young tableaux play an essential role. The main reason is that Young tableaux parametrize, in a natural way, a basis of an the irreducible representation of the symmetric or the general linear group [Ru, JK, St]. In a sense, the choice of a specific realization of an irreducible representation predetermines the corresponding combinatorial structures. If we consider the realization of irreducible representations of the general linear group in the space of Gelfand–Tsetlin patterns, see [GZ1], then we deal with combinatorics of convex polytopes of a special kind. If we consider the realization of representations of the symmetric (or the general linear) group by means of the Specht (or Weyl) module, see [JK], then we deal with combinatorics of Young tableaux. While the combinatorics of Young tableaux is extensively developed (see, e.g., [Sa]), the study of the combinatorics of Gelfand–Tsetlin patterns seems only to have been initiated, see [GZ1, GZ2, BZ1].

The main goal of our paper is to show that the majority of combinatorial constructions over Young tableaux admit an “algebraization” and may be defined on the space of triangles. Of course, we have not exhausted the subject and we believe that other combinatorial constructions, e.g., the Robinson–Schensted correspondence (see [Sch2]), also admit natural continuous analogs.

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§1. Groups acting on the space of triangles

Let \( n \) be a positive integer. In this section we define the group \( G_n \) generated by involutions, and its extension \( \widetilde{G}_n \) by means of \( \mathbb{R}^1 \), which act on a space of triangles \( X_n \). By definition, the space \( X_n \) consists of all sequences

\[
\mathbf{x} = (x^{(n)}, x^{(n-1)}, \ldots, x^{(1)}).
\]
where $x^{(j)} = (x_{1j}, \cdots, x_{jj}) \in \mathbb{R}^j$. As a vector space, $X_n \cong \mathbb{R}^{n(n+1)/2}$. We shall call the vector $x^{(n)} \in \mathbb{R}^n$ the highest weight of a triangle $x \in X_n$ and denote it by $\lambda(x) := x^{(n)}$. We define the weight $\beta := \beta(x)$ of a triangle $x \in X_n$ as the vector $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$ such that

$$\beta_j = |x^{(j)}| - |x^{(j-1)}|, \quad 1 \leq j \leq n,$$

(1.1)

where $|x^{(j)}| := \sum_{i=1}^j x_{ij}$, $|x^{(0)}| = 0$. For given vectors $\lambda \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$, we define the following subspaces of the space $X_n$:

$$X^{\lambda} = \{ x \in X_n \mid \lambda(x) = \lambda \},$$

$$X^{\lambda}(\beta) = \{ x \in X^{\lambda} \mid \beta(x) = \beta \}.$$

(1.2)

Now we define the Gelfand–Tsetlin patterns, see [GZ1, GZ2, BZ1, BZ2]. By definition, a triangle $x \in X_n$ is called a Gelfand–Tsetlin pattern (GT-pattern) if the following inequalities are satisfied

$$x_{ij} \geq x_{i+1,j+1}, \quad 1 \leq i \leq j \leq n - 1,$$

$$x_{ij} \geq x_{i,j-1}, \quad 1 \leq i < j \leq n,$$

$$x_{ij} \geq 0, \quad 1 \leq i \leq j \leq n$$

(1.3)

We denote by $K = K_n \subset X_n$ the set of all GT-patterns, and by $K^{\lambda}$ and $K^{\lambda}(\beta)$ the subset of those which lie in the subspaces (1.2). It is clear that $K^{\lambda}$ and $K^{\lambda}(\beta)$ are convex, compact polytopes in $\mathbb{R}^{n(n+1)/2}$. It is well known (see §2, or [GZ2]) that, if $\lambda$ is a partition and $\beta$ is a composition, then the number of integral points in the convex polytope $K^{\lambda}$ (respectively, in $K^{\lambda}(\beta)$) is equal to the dimension of the irreducible representation $V_{\lambda}$ of the Lie algebra $gl_n$ with highest weight $\lambda$ (respectively, to the dimension of the subspace $V_\lambda(\beta) \subset V_{\lambda}$ of weight $\beta$). On the other hand, as is also well known, the dimension of the weight subspace $V_\lambda(\beta) \subset V_{\lambda}$ admits a pure combinatorial description as the number of standard* Young tableaux of shape $\lambda$ and content $\beta$:

$$|K^{\lambda}(\beta) \cap \mathbb{Z}^{n(n+1)/2}| = \dim V_{\lambda}(\beta) = |\text{STY}(\lambda, \beta)|.$$  

(1.4)

Equality (1.4) is a starting point of our investigations. We shall try to translate the combinatorial information about the set of standard Young tableaux $\text{STY}(\lambda, \beta)$ into the language of GT-patterns, and vice versa.

Now we begin the construction of the main objects of this paper: the group $G_n$ and its extension $\hat{G}_n$. First, we define the "elementary" transformations $t_j$ of the space of triangles $X_n$. For this purpose, we fix a positive integer $j$, $1 \leq j < n$, and introduce two sequences of numbers $a_1, \cdots, a_j$ and $b_1, \cdots, b_j$. Given a triangle $x = (x^{(n)}, \cdots, x^{(1)}) \in X_n$, we define

$$a_i = x_{i,j+1}, \quad a_i = \min(x_{i,j+1}, x_{i-1,j-1}), \quad 2 \leq i \leq j,$$

$$b_i = x_{j+1,j+1}, \quad b_i = \max(x_{i,j-1}, x_{i+1,j+1}), \quad 1 \leq i \leq j - 1.$$

(1.5)

*See the footnote on page 92 (ed.).
Definition 1.1. The transformation $t_j : X_n \rightarrow X_n$ is given by the following formulas:

$$
t_j(x) = \tilde{x}, \quad \text{where} \quad \begin{cases} 
\tilde{x}_{ij} = a_i + b_i - x_{ij}, & 1 \leq i \leq j \\
\tilde{x}_{kl} = x_{kl} & \text{if } l \neq j.
\end{cases}
$$

(1.6)

Proposition 1.1. We have

1) $t_j^2 = 1$, $1 \leq j \leq n - 1$, (1.7)
2) $t_i t_j = t_j t_i$ if $|i - j| \geq 2$, (1.8)
3) $\beta(t_j(x)) = (j, j + 1) \cdot \beta(x)$, $1 \leq j \leq n - 1$, (1.9)

where the action of the transposition $(j, j + 1)$ on the weight space $\mathbb{R}^n$ is given by $(j, j + 1)(\beta_1, \ldots, \beta_j, \beta_{j+1}, \ldots, \beta_n) = (\beta_1, \ldots, \beta_{j+1}, \beta_j, \ldots, \beta_n)$.

Proof. Assertions a) and b) follow directly from (1.6). As for c), we observe that

$$
b_i + a_{i+1} = x_{i, j-1} + x_{i+1, j+1}, \quad 1 \leq i \leq j - 1.
$$

So

$$
|\tilde{x}(j)| = \sum_i (a_i + b_i) - |x(j)| = |x(j+1)| - |x(j)| + |x(j-1)|.
$$

Consequently,

$$
\beta_{j+1}(\tilde{x}) = |x(j+1)| - |\tilde{x}(j)| = |x(j)| - |x(j-1)| = \beta_j(x)
$$

and

$$
\beta_j(\tilde{x}) = |\tilde{x}(j)| - |x(j-1)| = |x(j+1)| - |x(j)| = \beta_{j+1}(x). \quad \blacksquare
$$

Proposition 1.2. For $1 \leq j \leq n - 1$, we have

$$
a) \quad t_j : K_n \rightarrow K_n, \\
b) \quad t_j : K^\lambda \rightarrow K^\lambda, \quad t_j : K^\lambda(\beta) \rightarrow K^\lambda((j, j + 1) \cdot \beta). \quad (1.10) \quad (1.11)
$$

Proof. It is sufficient to show that the involutions $t_j$ preserve inequalities (1.3). Consider an "elementary neighborhood" of $x := x_{ij} \in K_n$:

\[
\begin{array}{cccc}
a & \geq & c & \geq \\
\leq & t_j & \geq & \min(a, b) + \max(c, d) - x \\
\leq & d \\
b & \leq & b & \leq \\
\end{array}
\]
It is clear that $\min(a, b) \geq x \geq \max(c, d)$. So

$$a \geq \min(a, b) \geq \min(a, b) - (x - \max(c, d)), \quad (\min(a, b) - x) + \max(c, d) \geq c.$$ 

Thus, all necessary inequalities are satisfied.

We denote by $G_n$ the group generated by the involutions $t_1, \ldots, t_{n-1}$:

$$G_n = \langle t_1, \ldots, t_{n-1} \rangle.$$ (1.12)

This group acts on the space of triangles $X_n$ and for any fixed vector $\lambda \in \mathbb{R}^n_+$ transforms the convex polytope $K^\lambda$ into itself and interchanges the polytopes $K^\lambda(\beta)$.

The group $G_n$ may be embedded in a bigger group $\overline{G}_n$. In order to construct such an extension, we define a transformation $T_e$ of the space $X_n$ in the following way: assume $e \in \mathbb{R}^1$ and $x \in X_n$, then put $T_e(x) = \tilde{x}$, where $\tilde{x}^{(j)} = x^{(j)}$ if $2 \leq j \leq n$, and $\tilde{x}^{(1)} = x_{11} - e$.

We denote by $\overline{G}_n$ the group generated by $G_n$ and $T_e, e \in \mathbb{R}^1$:

$$\overline{G}_n = \langle t_1, \ldots, t_{n-1}, T_e, e \in \mathbb{R}^1 \rangle.$$ (1.13)

The next proposition gives the description of some relations between the generators $t_i$, $1 \leq i \leq n - 1$, and $T_e, e \in \mathbb{R}^1$.

**Proposition 1.3.** We have

a) $t_1 T_e = T_{-e} t_1$, or, equivalently, $(t_1 T_e)^2 = 1$,
b) $t_i T_e = T_e t_i$, $3 \leq i \leq n - 1$,
c) $(t_1 t_2)^6 = 1$,
d) $[t_i, t_j] = 0$ if $|i - j| \geq 2$,\n
e) $t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e t_2 T_e$.

f) $(T_e t_2 t_1 T_e t_2 t_1)^3 = 1, e, \delta \in \mathbb{R}^1$,
g) $T_\theta t_2 T_\epsilon t_1 t_2 T_\epsilon t_2 T_\epsilon t_1 t_2 = t_2 T_\epsilon t_1 t_2 T_\epsilon t_2 T_\epsilon t_1 t_2 T_\epsilon$ for any $\epsilon, \delta, \theta \in \mathbb{R}^1$.

**Proof.** Assertions a), b), and d) are evident.

c) It is sufficient to show that

$$(t_1 t_2)^3 = (t_2 t_1)^3 : X_3 \rightarrow X_3.$$ 

By direct computation we find $t_2 t_1(x) = \tilde{x}$, where

$$\tilde{x}_{12} = x_{13} - x_{12} + \max(x_{23}, \beta_2(x)), \quad \tilde{x}_{22} = x_{33} - x_{22} + \min(x_{23}, \beta_2(x)),$$
$$\beta_2(\tilde{x}) = \beta_2(x), \quad \tilde{x}_{11} = x_{12},$$
$$\tilde{x}_{13} = x_{13}, \quad \tilde{x}_{23} = x_{23}, \quad \tilde{x}_{33} = x_{33}.$$
Consequently, \((t_2 t_1)^3(x) = \tilde{x}\), where \(\tilde{x}_{\alpha\beta} = x_{\alpha\beta}\) if \((\alpha, \beta) \neq (1, 2)\) or \((2, 2)\), and 
\[
\tilde{x}_{12} = x_{13} - x_{12} - \max(x_{23}, \beta_3(x)) + \max(x_{23}, \beta_2(x)) + \max(x_{23}, \beta_1(x)), \\
\tilde{x}_{22} = x_{33} - x_{22} - \min(x_{23}, \beta_3(x)) + \min(x_{23}, \beta_2(x)) + \min(x_{23}, \beta_1(x)).
\]

(1.15)

Similarly, \(t_1 t_2(x) = \bar{x}\), where 
\[
\bar{x}_{12} = x_{13} - x_{12} + \max(x_{23}, \beta_1(x)), \\
\bar{x}_{22} = x_{33} - x_{22} + \min(x_{23}, \beta_1(x)), \\
\bar{x}_{11} = \beta_3(x), \\
\bar{x}_{21} = \beta_2(x), \\
\bar{x}_{\alpha 3} = x_{\alpha 3}, \quad \alpha = 1, 2, 3.
\]

Using these formulas, it is easy to see that for \((t_1 t_2)^3(x)\) we obtain the same expression (1.15).

e) Put define \(t_2^\epsilon = T_{t_2} t_{-\epsilon} t_{-\epsilon} t_{-\epsilon}\), \(\epsilon \in \mathbb{R}^1\). It is sufficient to show that 
\[t_2^\epsilon t_2^\delta = t_2^\epsilon t_2^\delta : X_3 \to X_3\]

for any \(\epsilon, \delta \in \mathbb{R}^1\). Given \(x \in X_3\), the following relations hold:
\[
x_{\alpha, \beta}(t_2^\epsilon(x)) = x_{\alpha, \beta} \quad \text{if} \quad (\alpha, \beta) \neq (1, 2) \quad \text{or} \quad (2, 2),
\]
\[
x_{12}(t_2^\epsilon(x)) = x_{12} + \max(x_{23}, x_{11} + \epsilon) - \max(x_{23}, x_{11}),
\]
\[
x_{22}(t_2^\epsilon(x)) = x_{22} + \max(x_{23}, x_{11} + \epsilon) - \max(x_{23}, x_{11}).
\]

Consequently,
\[
x_{12}(t_2^\epsilon t_2^\delta(x)) = x_{12} + \max(x_{23}, x_{11} + \epsilon) \\
+ \max(x_{23}, x_{11} + \delta) - 2\max(x_{23}, x_{11}) \\
= x_{12}(t_2^\epsilon t_2^\delta(x))
\]

By the same reasoning,
\[
x_{22}(t_2^\epsilon t_2^\delta(x)) = x_{22}(t_2^\epsilon t_2^\delta).
\]
f) It is sufficient to prove (1.14), f), for \(n = 3\). We consider a correspondence \(\psi_{\epsilon, \delta} : X_3 \to X_3\) given by
\[
\psi_{\epsilon, \delta} : \begin{array}{cccc}
\psi_{\epsilon, \delta} & x_{13} & x_{23} & x_{33} & x_{13} + \epsilon + \delta, & x_{23} + \delta, & x_{33} \\
x_{12} & x_{22} & \rightarrow & x_{12} + \epsilon + \delta, & x_{22} + \delta & x_{11} + \epsilon + \delta
\end{array}
\]

It is clear that \(\psi_{\epsilon, \delta}\) is an automorphism of \(X_3\). The following identities, which may be verified by direct computation, reduce the proof of (1.14), f) to that of item c):
\[
\psi_{\epsilon, \delta}^{-1} t_1 T_\epsilon \psi_{\epsilon, \delta} = t_1,
\]
\[
\psi_{\epsilon, \delta}^{-1} T_{-\epsilon} t_2 t_2 T_{-\epsilon} t_2 t_2 t_1 \psi_{\epsilon, \delta} = (t_2 t_1)^2.
\]
Note that c) is a particular case of f).
g) It is sufficient to prove (1.14,g) for \( n = 3 \). Put

\[
K = t_2T_1t_2T_0t_2T_1t_2.
\]

We must prove that \( \theta K = K\theta \) for any \( \theta \in R^1 \). For this purpose we compute \( \tilde{x} := K(x) \), \( x \in X_3 \). The answer is as follows: \( \tilde{x}_{ij} = x_{ij} \) if \( i \neq 2 \), and

\[
\tilde{x}_{21} = x_{21} + \max(x_{32}, \beta_3(x) - \epsilon) - \max(x_{32}, \beta_3(x) - \epsilon - \delta),
\]

\[
\tilde{x}_{21} + \tilde{x}_{22} = x_{21} + x_{22}.
\]

Consequently, \( \theta K(x) = K(x)\theta \).

One of our main results in this paper, namely, Theorem 1.1, gives a description of additional relations between the generators in the group \( G_n \). We do not know whether or not the set of relations given by (1.24) and (1.25), is complete, but conjecture it is. In any case, relations (1.24) allow us to construct a section of the following exact sequence

\[
1 \rightarrow \operatorname{Ker} \pi \rightarrow G_n \xrightarrow{\pi} S_n \rightarrow 1, \quad \pi(t_i) = (i, i + 1),
\]

and to find a subgroup (not a normal subgroup!) isomorphic to the symmetric group \( S_n \).

Now we pass to the construction of the involutions \( s_i \), which will generate the symmetric group and the transformations \( s_i^{(e)} \). The transformations \( s_i^{(e)} \) will satisfy the colored braid relations (Theorem 1.1).

For this purpose, we consider the following elements in the groups \( G_n \) and \( \widehat{G}_n \):

\[
p_i = t_it_{i-1}\cdots t_1,
\]

\[
v_i = t_{i+1}\cdots t_{n-1},
\]

\[
q_i = p_1p_2\cdots p_i,
\]

\[
u_i = v_{n-1}v_{n-2}\cdots v_{n-i},
\]

\[
\sigma = t_1t_2\cdots t_{n-1} := p_{n-1}^1 = v_1,
\]

\[
s_i = \sigma^i_1t_1\sigma^{1-i}, \quad s_i^{(e)} = \sigma^i_1T_e\sigma^{1-i},
\]

(1.16)

where \( 2 \leq i \leq n - 1 \). We put \( p_1 = q_1 = s_1 = t_1, \ v_n = 1 \).

**Proposition 1.4.** We have

a) \( q_i = q_{i-1}p_i = p_i^{-1}q_{i-1}, \ u_i = u_{i-1}v_{n-i} = v_{n-i}u_{i-1} \), consequently, \( q_i^2 = 1, \ u_i^2 = 1, \ 1 \leq i \leq n - 1 \).

b) \( s_i = q_it_1q_i, \ s_i^{(e)} = q_iT_{-\epsilon}q_i, \ 1 \leq i \leq n - 1 \).

(1.17)

c) \( q_{n-1}s_is_{n-1} = s_{n-i}, \ q_{n-1}s_i^{(e)}q_{n-1} = s_i^{(e)}, \ 1 \leq i \leq n - 1 \).

d) \( (s_{i-1}s_i)^3 = q_{i-1}t_1p_it_1(t_2t_1)^6t_1p_i^{-1}t_1q_{i-1}, \ 2 \leq i \leq n - 1 \).
e) **Crucial Lemma.** If \( x \in X_n \), \( \beta(x) = (\beta_1, \ldots, \beta_n) \), then
\[
s_i(x) = s_i(\beta_i - \beta_{i+1})(x).
\]
(1.18)

f) \( s_i \cdot s_i^{(-e)} = s_i^{(-e)} \cdot s_i \), \( 1 \leq i \leq n - 1 \).

g) Assume that \( s_i t_j = t_j s_i \) for all \( j < i - 1 \), then
\[
s_i = t_{i-1} t_i s_{i-1} t_i t_{i-1}.
\]
(1.19)

h) **Formulas for the weights:**
\[
\begin{align*}
\beta(s_i(x)) &= (i, i + 1)\beta(x), \\
\beta(s_i^{(-e)}(x)) &= (\beta_1, \ldots, \beta_{i-1}, \beta_i - \epsilon, \beta_{i+1} + \epsilon, \beta_{i+2}, \ldots, \beta_n), \\
\beta(\sigma(x)) &= (n, n - 1, \ldots, 2, 1)\beta(x), \\
\beta(q_{n-1}(x)) &= \tilde{\beta}(x) := w_0 \beta(x),
\end{align*}
\]
where \( w_0 \in S_n \) is the element of maximal length in the symmetric group \( S_n \).

**Proof.** a) We must prove that \( p_{i+1} q_i p_{i+1} = q_i \). We proceed by induction:
\[
p_{i+1} q_i p_{i+1} = t_{i+1} p_i q_i -1 p_i t_{i+1} p_i = t_{i+1} q_i -1 t_i t_{i+1} p_i = q_i -1 p_i = q_i,
\]
because \([t_{i+1}, q_i -1] = 0\). Consequently,
\[
q_i^2 = q_i q_i = q_i -1 p_i^{-1} p_i q_i -1 = q_i -1 = \cdots = q_1^2 = t_1^2 = 1.
\]
Similarly, we can prove that
\[
v_{n-i} u_{i-1} v_{n-i} = u_{i-1} \quad \text{and} \quad u_i^2 = 1.
\]

b) By induction, it is easy to see that
\[
\begin{align*}
\sigma^{i-1} &= q_i u_{i+1} v_1^{-1} v_2^{-1} t_1, \quad 2 \leq i \leq n - 1, \\
\sigma^{n-1} &= q_{n-1} u_{n-1} v_1^{-1}, \\
\sigma^n &= q_{n-1} u_{n-1}.
\end{align*}
\]
(1.20)

So, we have
\[
s_i = \sigma^{i-1} t_1 \sigma^{-(i-1)} = q_i (u_{i+1} v_1^{-1} v_2^{-1} t_1 v_2 v_1 u_{i+1}^{-1}) q_i = q_i t_1 q_i,
\]
because \([u_i v_1^{-1} v_2^{-1}, t_1] = 0 \) if \( i \geq 2 \).

c) It is sufficient to show that
\[
[q_{n-i} q_{n-1} q_i, t_1] = 0, \quad [q_{n-i} q_{n-1} q_i, T_c] = 0, \quad 1 \leq i \leq n - 1.
\]
(1.21)
We assume that $2 \leq i \leq n - 2$. (For $i = 1$ or $i = n - 1$ equalities (1.21) are clear.) By induction, we find from a) that

$$q_i = q_j p_{j+1} \cdots p_i = p_i^{-1} \cdots p_j^{-1} q_j \quad \text{if } i \geq j. \tag{1.22}$$

Consequently,

$$q_{n-i} q_{n-1} = p_{n-i+1} \cdots p_{n-1}.$$ 

Now it is easy to see by induction that

$$p_{n-i+k-1} = v_{n-i+k} \cdot p_k, \quad 2 \leq k \leq i.$$ 

So

$$p_{n-i+1} \cdots p_{n-1} = \left( \prod_{k=2}^{i} v_{n-i+k} v_{k+1}^{-1} \right) \cdot t_1 q_i.$$ 

Thus, we have

$$q_{n-i} q_{n-1} q_i = \left( \prod_{k=2}^{i} v_{n-i+k} v_{k+1}^{-1} \right) \cdot t_1.$$ 

But the product $\prod_{k=2}^{i} v_{n-i+k} v_{k+1}^{-1}$ does not contain the involutions $t_1$ and $t_2$ and, therefore, commutes with $t_1$.

d) We use identity (1.22). Let $i - j \geq 1$, then

$$s_i s_j = q_i t_1 q_i t_1 q_j = q_j (p_{j+1} \cdots p_i t_i t_i^{-1} \cdots p_{j+1} t_1) q_j.$$ 

Now we take $j = i - 1$. Then

$$(s_{j+1} s_j)^3 = q_j (p_{j+1} t_1 p_{j+1}^{-1} t_1)^3 q_j = q_j t_1 p_{j+1} t_1 (t_1 p_{j+1}^{-1} t_1 p_{j+1} t_1)^3 t_1 p_{j+1}^{-1} t_1 q_j$$

since

$$t_1 p_{j+1}^{-1} t_1 p_{j+1} = t_1 t_1 \cdots t_{j+1} t_1 t_{j+1} \cdots t_1 = t_2 t_1 t_2 t_1.$$ 

e) It is sufficient to show (see (1.17) of Proposition 1.4) that if $\beta(x) = (\beta_1, \cdots, \beta_n)$, then

$$t_1 (q_i(x)) = T_{\beta_{i+1}} - \beta_i (q_i(x)).$$ 

We note that

$$\beta(q_i(x)) = (\beta_{i+1}, \beta_i, \cdots, \beta_1, \beta_{i+2}, \cdots, \beta_n), \quad 1 \leq i \leq n - 1.$$
Consequently,
\[ x_{11}(t_1(q_i(x))) = \beta_i, \]
\[ x_{11}(T_{\beta_{i+1} - \beta_i}(q_i(x))) = \beta_{i+1} - (\beta_{i+1} - \beta_i) = \beta_i. \]

Crucial Lemma (see e) in Proposition 1.4 allows us to reduce any statement about \( s_i \) to a statement about \( s_i^2 \). For example, relation (1.24), b, follows from (1.25), b, if \( \epsilon = \beta_{i+1} - \beta_{i+2} \) and \( \delta = \beta_i - \beta_{i+1} \).

\[ \text{g) By (1.17) and part a), we have} \]
\[ s_i = q_i t_i q_i = p_i^{-1} q_{i-1} t_i q_{i-1} p_i = p_i^{-1} s_i^{-1} p_i \]
\[ = t_1 \cdots t_{i-2} (t_{i-1} t_i s_{i-1} t_i t_{i-1}) t_{i-2} \cdots t_1. \]

However, by assumption we have \([\bar{s}_i, t_1] = \cdots = [\bar{s}_i, t_{i-2}] = 0\), where \( \bar{s}_i \) is given by (1.19). So \( s_i = \bar{s}_i = t_{i-1} t_i s_{i-1} t_i t_{i-1} \).

\[ \text{f) This follows from Proposition 1.3, item a).} \]

Note that from (1.17) we obtain a representation of \( s_i \) as a product of \((i^2 + i - 1)\) simple involutions \( t_j \), whereas formula (1.19) gives an expression for \( s_i \) as a products of \( 4i - 3 \) involutions. For example,
\[ s_1 = t_1, \quad q_1 = t_1, \quad s_2 = t_1 t_2 t_1 t_2 t_1, \quad q_2 = t_1 t_2 t_1, \quad q_3 = t_1 t_2 t_1 t_3 t_2 t_1, \]
\[ s_3 = t_1 t_2 t_1 t_3 t_2 t_1 t_2 t_1 t_3 t_2 t_1 = t_2 t_3 t_1 t_2 t_1 t_2 t_1 t_3 t_2. \] (1.23)

Note that equality (1.23) for \( s_3 \) is equivalent to the following relation in the group \( G_n, n \geq 4 \) (see Corollary 1.1): \( (t_1 g_3)^4 = 1 \).

Now we formulate the main result of this section.

**Theorem 1.1.** 1°. The involutions \( s_i, 1 \leq i \leq n - 1 \), satisfy the relations of the symmetric group \( S_n \), i.e.,
\[ a) \quad s_i^2 = 1, \quad i = 1, \ldots, n - 1, \]
\[ b) \quad (s_i s_{i+1})^3 = 1, \quad i = 1, \ldots, n - 2, \]
\[ c) \quad s_i s_j = s_j s_i \quad \text{if} \ 1 \leq i, j \leq n - 1, \ |i - j| \geq 2. \] (1.24)

2°. The transformations \( s_i^{(\epsilon)} \) satisfy the colored braid relations, i.e., for any \( \epsilon, \delta \in R^1 \) we have
\[ a) \quad s_i^{(\epsilon)} \cdot s_i^{(\delta)} = s_i^{(\epsilon + \delta)}, \quad \text{in particular,} \quad s_i^{(-\delta)} = (s_i^{(\epsilon)})^{-1}, \]
\[ b) \quad s_i^{(\epsilon)} \cdot s_{i+1}^{(\epsilon + \delta)} \cdot s_i^{(\delta)} = s_{i+1}^{(\delta)} \cdot s_i^{(\epsilon - \delta)} \cdot s_{i+1}^{(\epsilon)}, \ 1 \leq i \leq n - 2, \] (1.25)
\[ c) \quad s_i^{(\epsilon)} \cdot s_j^{(\delta)} = s_j^{(\delta)} \cdot s_i^{(\epsilon)} \quad \text{if} \ |i - j| \geq 2. \]

Theorem 1.1 follows from a sharper result, namely, Theorem 1.2.
Corollary 1.1 (to Theorem 1.1). 1°. We have the following relations between the generators $t_i$ in the group $G_n$:

1) $t_i^2 = 1$, $t_it_j = t_jt_i$ if $|i - j| \geq 2$,

2) $(t_1t_2)^6 = 1$, 

3) $(t_1q_i)^4 = 1$ if $3 \leq i \leq n - 1$,

where $q_i = p_1p_2 \cdots p_i = t_1 \cdot \underbrace{t_2t_1 \cdot t_3t_2 \cdots}_{t_{i-1} \cdots t_2t_1}$.

2°. Besides the above relations 1)-3), we have the following relations between the generators $t_i$ and $T_\epsilon$ in the group $\widetilde{G}_n$:

1) $T_\epsilon \cdot T_\delta = T_{\epsilon+\delta}$,

2) $t_1T_\epsilon = T_{-t_1}, t_iT_\epsilon = T_{t_i} t_\epsilon$ if $i \geq 3$,

3) $(T_\epsilon t_2t_1T_\epsilon t_2t_1)^3 = 1$ for any $\epsilon, \delta \in \mathbb{R}^1$,

4) $T_\epsilon t_2T_{-\epsilon}t_2T_{-\delta}t_2 = t_2T_{\epsilon}t_2T_{-\delta}t_2T_{-\epsilon}$,

5) $T_\epsilon q_jT_\delta q_jT_\epsilon = T_{q_j}$ if $3 \leq j \leq n - 1$,

6) $(t_1T_\epsilon q_jt_1T_\delta q_j)^2 = 1$, $3 \leq j \leq n - 1$.

Proof. 1°. Assertions 1) and 2) were proved in Proposition 1.3. On the other hand, if $i - j \geq 2$, then $s_is_i = \sigma^{i-1}(t_is_{i-j+1})\sigma^{-(j-1)}$ and $(s_is_i)^2 = 1$ if and only if $(t_is_{i-j+1})^2 = 1$. But if $i \geq 3$, then $(t_is_i)^2 = (t_1q_i)^4$.

2°. Properties 1)-4) follow from Proposition 1.3. Identity 5) follows from the fact that the transformations $s_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute if $|i - j| \geq 2$ (see Theorem 1.1). Finally, identity 6) follows from the observation that the elements $s_is_i^{(\epsilon)}$ and $s_j^{(\delta)}$ commute if $|i - j| \geq 2$, which is a consequence of Corollary 1.4.

Corollary 1.2 (to Theorem 1.1). Define

$$s_0 := s_0^{(\epsilon)} = s_{n-1}s_{n-2} \cdots s_2s_1T_\epsilon s_2 \cdots s_{n-1}.$$ 

Then $s_0, s_1, \ldots, s_{n-1}$ are the standard generators of the affine Weyl group of type $A_{n-1}^{(1)}$, and $\beta(s_0^{(\epsilon)}(x)) = s_0^{(\epsilon)}(\beta(x)) = (\beta_n + \epsilon, \beta_2, \ldots, \beta_{n-1}, \beta_1 - \epsilon)$.

Proof. It is sufficient to show that $(s_0s_1)^3 = 1$ and $(s_0s_{n-1})^3 = 1$. We have the following chains of the equivalent statements

i) $(s_0s_1)^3 = 1 \iff (s_2s_1T_\epsilon s_2s_1)^3 = 1 \iff (T_\epsilon s_1s_2)^3 = 1$. The latter relation is equivalent to the following one: $(T_\epsilon t_2t_1t_2t_1)^3 = 1$, which is a particular case of (1.14), f, for $\delta = 0$.

ii) $(s_0s_{n-1})^3 = 1 \iff [(s_2s_3 \cdots s_{n-2}s_{n-1}s_{n-2} \cdots s_3s_2)T_\epsilon]^3 = 1$. The latter relation is equivalent to the relation $(s_2s_1T_\epsilon)^3 = 1$, which is also a particular case of (1.14), f. It is also clear that $s_0^2 = 1$ and $s_0s_j = s_j$ if $2 \leq j \leq n - 2$. 

•
Corollary 1.3. The elements $s_i s_i^{(e)}$, $1 \leq i \leq n$, $e \in \mathbb{R}^1$, satisfy the following relations:

1) $(s_i s_i^{(e)})^2 = 1$, $1 \leq i \leq n - 1$;
2) $(s_i s_i^{(e)} s_{i+1} s_{i+1}^{(e)})^3 = 1$, $1 \leq i \leq n - 2$, $e, \delta \in \mathbb{R}^1$,
3) $s_i s_i^{(e)} s_j s_j^{(e)} = s_j s_j^{(e)} s_i s_i^{(e)}$ if $|i - j| \geq 2$, $e, \delta \in \mathbb{R}^1$.

Proof. We must prove 2) only for $i = 1$, i.e., the relation to be checked is $(s_1 T_e s_2 s_2^{(e)})^3 = 1$. But this is exactly (1.14), f. Statement 3) follows from Corollary 1.4.

Using the same arguments as in the proof of Corollary 1.2, we can prove that, if

$$\tilde{s}_0 := s_0^{(e, \delta)} = s_{n-1}^{(e, \delta)} s_{n-2}^{(e, \delta)} \cdots s_2^{(e, \delta)} s_1^{(e, \delta)} T_e s_2^{(e, \delta)} \cdots s_{n-1}^{(e, \delta)},$$

then $\tilde{s}_0$, $s_1 s_1^{(e, \delta)}$, \ldots, $s_{n-1} s_{n-1}^{(e, \delta)}$ are the standard generators of the affine Weyl group of type $A_{n-1}^{(1)}$ and

$$\beta(s_0^{(e, \delta)}(x)) = (\beta_n + n\delta + e, \beta_2, \ldots, \beta_{n-1}, \beta_1 - n\delta - e).$$

Some comments are in order.

i) Part 1o of Theorem 1.1 follows from part 2o and equality (1.18).

ii) Assertion b) of part 2o follows from Proposition 1.3, part e). In fact, formula (1.25), b, is equivalent to the following identity:

$$T_e \cdot s_2^{(e, \delta)} \cdot s_1^{(e, \delta)} \cdot T_e \cdot s_2^{(e, \delta)} = s_2^{(e, \delta)} \cdot T_e \cdot s_2^{(e, \delta)}.$$
The group $G_3$ is finite of order 12 and contains a normal subgroup $N := \langle 1, (t_1 t_2)^2 \rangle$ of order 3 with factor group $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The group $G_4$ seems to be very interesting. For example, for arbitrary $n$ the symmetric group $S_n$ is a factor of $G_4$. Namely, define an epimorphism $\rho := \rho_n : G_4 \to S_n$ by setting

$$
\rho(t_1) = s_1, \quad \rho(t_2) = s_2 s_4 s_6 \cdots, \quad \rho(t_3) = s_3 s_5 s_7 \cdots,
$$

where $s_i = (i, i + 1) \in S_n$, $1 \leq i \leq n - 1$, is a simple transposition.

One can show that $\rho(t_1)$, $\rho(t_2)$, $\rho(t_3)$ satisfy the relations in the group $G_4$ and generate the symmetric group $S_n$. Consequently, the group $G_4$ is infinite if $n \geq 4$.

The main difficulties in the proof of Theorem 1.1 seem to lie in the verification of the fact that the generators $s_i^{(e)}$ and $s_i^{(e)}$ commute if $|i - j| \geq 2$, or, equivalently, in the verification of identity (1.26). Our strategy is to prove a sharper result about the action of the transformations $s_i^{(e)}$ on triangles. Before stating the theorems about the structure of the mapping $s_i^{(e)}$, we define some additional operators acting on the space of triangles $X_n$.

**Definition 1.2.** For each triple of integers $(ijk)$, $1 \leq i < j < k \leq n$, the transformation $R_{ijk} : X_n \to X_n$ is given by the following formulas:

$$
R_{ijk}(x) = \tilde{z}, \ x \in X_n,
$$

where

$$
\begin{align*}
\tilde{z}_{ij} &= x_{ij} + x_{ik} - x_{i,k-1} - \min(x_{jk} - x_{j,k-1}, x_{ij} - x_{i,j-1}), \\
\tilde{z}_{jj} &= x_{jj} - x_{ik} + x_{i,k-1} + \min(x_{jk} - x_{j,k-1}, x_{ij} - x_{i,j-1}), \\
\tilde{z}_{\alpha\beta} &= x_{\alpha\beta} \quad \text{if } (\alpha, \beta) \neq (i, j) \text{ or } (j, j).
\end{align*}
$$  

(1.27)

**Proposition 1.5.** The operators $R_{ijk}$ satisfy the following relations:

a) $(R_{ijk})^2 = 1$,

b) $R_{ijk} \cdot R_{i'j'k'} = R_{i'j'k'} \cdot R_{ijk}$ if $|(ijk) \cap (i'j'k')| \neq 2$,

c) $(R_{ijk} R_{ijl} R_{ikl} R_{jkl})^3 = 1$,

d) $(R_{ijl} R_{ikl})^3 = 1$,

e) $\beta(R_{ijk}(x)) = \beta(x), \ x \in X_n$.

**Proof.** Assertions a), b), and e) are almost evident, and c), d) may be checked by direct computation. •

**Remark 1.1.**

i) Assertions a)–c) are essentially due to G. Lusztig [Lu2].

ii) It seems plausible that relations a)–d) are defining relations for the group $L_n$ generated by all $R_{ijk}$ with $1 \leq i < j < k \leq n$. 

To go further, we define certain transformations $T_e^{(i)}$ and $\varphi_i$ acting on the space $X_n$ in accordance with the following formulas:

$$T_e^{(i)}(x) = \tilde{x}, \quad \varphi_i(x) = \tilde{x}, \quad x \in X_n,$$

where

$$\tilde{x}_{ii} = x_{ii} - \epsilon, \quad \tilde{x}_{ii} = x_{ii} + \beta_{i+1}(x) - \beta_i(x);$$

$\tilde{x}_{\alpha\beta}$ and $\tilde{x}_{\alpha\beta}$ are equal to $x_{\alpha\beta}$ if $(\alpha, \beta) \neq (i, i)$.

It is clear from the definitions that if $x \in X_n$, $\beta = \beta(x) = (\beta_1, \ldots, \beta_n)$, then $\varphi_i(x) = T_{\beta_i - \beta_{i+1}}(x)$, and

$$\beta(T_e^{(i)}(x)) = (\beta_1, \ldots, \beta_i - \epsilon, \beta_{i+1} + \epsilon, \ldots, \beta_n),$$
$$\beta(\varphi_i(x)) = (i, i + 1) \cdot \beta(x).$$

Now we are ready to state our result about the structure of the transformations $s_i$ and $s_i^{(e)}$.

**Theorem 1.2.** The following relations hold:

1. $s_i = R_{i-1,i} R_{i-2,i} \cdots R_{1,i} \cdot \varphi_i \cdot R_{i,i} \cdots R_{i-2,i} R_{i-1,i}$, \quad (1.28a)

2. $s_i^{(e)} = R_{i-1,i} R_{i-2,i} \cdots R_{1,i} \cdot T_e^{(i)}(R_{i,i} \cdots R_{i-2,i} R_{i-1,i})$, \quad (1.28b)

where $1 \leq i \leq n - 1$ and $R_{jk} := R_{jkk+1}$, $1 \leq j < k \leq n - 1$.

The proof of Theorem 1.2 will be given in the Appendix.

**Corollary 1.4.** Let $x \in X_n$, $x = (x^{(n)}, \ldots, x^{(1)})$. Then

1. $s_i(x) = (x^{(n)}, \ldots, x^{(i+1)}, \tilde{x}^{(i)}, x^{(i-1)}, \ldots, x^{(1)}), 1 \leq i \leq n-1$, and the vector $\tilde{x}^{(i)} \in \mathbf{R}^i$ depends only on the components of the vectors $x^{(i-1)}, x^{(i)}$, and $x^{(i+1)}$.

2. $s_i^{(e)}(x) = (x^{(n)}, \ldots, x^{(i+1)}, \tilde{x}^{(i)}, x^{(i-1)}, \ldots, x^{(1)}$, and the vector $\tilde{x}^{(i)}$ depends only on $\epsilon$ and the components of the vectors $x^{(i-1)}, x^{(i)}$, and $x^{(i+1)}$.

In particular, from Corollary 1.2, part 1o, it follows that $s_i t_j = t_j s_i$ if $1 \leq j < i - 1$ and, consequently, $s_i s_j = s_j s_i$ if $i - j \geq 2$. This proves part 1o of Theorem 1.1, and also the recurrence relation (1.19) for $s_i$.

Similarly, from Corollary 1.2, part 2o, it follows that $T_e \cdot s_i^{(e)} = s_i^{(e)} \cdot T_e$ for any $\epsilon, \delta \in \mathbf{R}$ and $3 \leq i \leq n - 1$. This proves part 2o of Theorem 1.1 (see (1.26)).

As another application of Theorem 1.2, we deduce a recurrence relation for the transformations $s_i^{(e)}$, but before doing that it is necessary to introduce some additional notation.

**Definition 1.3.** Define a mapping $[i, i+1]: X_n \rightarrow X_n$, $1 \leq i \leq n - 1$, by

$$[i, i+1](x) = \tilde{x}, \quad x \in X_n,$$
where

\[ \tilde{x}_{ki} = x_{k,i+1} + x_{k,i-1} - x_{ki} \quad \text{if} \quad k < i, \]
\[ \tilde{x}_{ii} = x_{i,i+1} + x_{i+1,i+1} - x_{ii}, \]
\[ \tilde{x}_{ik} = x_{i+1,k} \quad \text{and} \quad \tilde{x}_{i+1,k} = x_{i,k} \quad \text{if} \quad k > i; \]

and for all remaining elements \( \tilde{x}_{\alpha\beta} = x_{\alpha\beta} \).

**Lemma 1.1.** The mappings \([i, i + 1], 1 \leq i \leq n - 1,\) satisfy the relations of the symmetric group \( S_n, \) i.e.,

a) \([i, i + 1]^2 = 1, \]
b) \([i, i + 1] \cdot [i + 1, i + 2] \cdot [i, i + 1] = [i + 1, i + 2] \cdot [i, i + 1] \cdot [i + 1, i + 2], \]
c) \([i, i + 1] \cdot [j, j + 1] = [j, j + 1] \cdot [i, i + 1] \quad \text{if} \quad |i - j| \geq 2, \]
d) \(\beta([i, i + 1](x)) = (i, i + 1)\beta(x), \quad x \in X_n.\)

It is not difficult to show that the group generated by all \([i, i + 1], 1 \leq i \leq n - 1,\) is in fact isomorphic to the symmetric group \( S_n. \) •

**Proposition 1.6.** For \( i \geq 2, \) we have

\[ s_i = R_{i-1,i}[i-1,i][i,i+1]s_{i-1}[i-1,i][i,i+1]R_{i-1,i}, \]
\[ s_i^{(e)} = R_{i-1,i}[i-1,i][i,i+1]s_{i-1}^{(e)}[i-1,i][i,i+1]R_{i-1,i}. \]  \hfill (1.29)

**Proof.** It is easy to see that

\[ R_{k,i} = [i-1,i][i,i+1]R_{k,i-1}[i-1,i][i,i+1], \]
\[ T_{e}^{(i)} = [i-1,i][i,i+1]T_{e}^{(i)}[i-1,i][i,i+1], \]  \hfill (1.30)

where \( 1 \leq k < i \leq n - 1. \) Identity (1.29) follows by induction from (1.19), (1.30), and Theorem 1.2, parts 1\(^\circ\) and 2\(^\circ\).

The recurrence relations (1.19) and (1.29) are used as an induction base in the first proof of Theorem 1.2. However, it is possible to solve these recurrence relations in an explicit form and, consequently, to obtain another proof of Theorem 1.2. Before describing the solutions of (1.19) and (1.29), we give appropriate definitions. It is convenient to use the notation

\[ (a)_+ = \max(a, 0), \quad (a)_- = \min(a, 0), \quad a \in \mathbb{R}. \]

First, we define the sequence of piecewise linear functions \( Q_k^n(a_1, \ldots, a_n), 1 \leq k \leq n, \) inductively, in the following way:

\[ Q_1^1(a_1) := -a_1, \]
\[ Q_1^n(a_1, \cdots, a_n) := (Q_{n-1}^1(a_1, \cdots, a_n))_- + (Q_{n-1}^1(a_2, \cdots, a_n))_+, \]
\[ Q_k^n(a_1, \cdots, a_n) := Q_k^1(a_k, \cdots, a_n, a_1, a_2, \cdots, a_{k-1}), \quad 1 \leq k \leq n. \]  \hfill (1.31)
GROUPS GENERATED BY INVOLUTIONS ...

Second, we define some linear functionals

$$\varphi_{ij} := \varphi_{ij}^{(n)} : X_n \to \mathbb{R}, \quad 1 \leq i < j \leq n - 1,$$

don the space of triangles $X_n$ by the formulas:

$$\varphi_{ij}(x) := x_{i-1,j} + x_{i,j} - x_{i-1,j-1} - x_{i,j+1} \quad \text{if } 2 \leq i < j \leq n - 1,$$

$$\varphi_{11}(x) := 0, \quad \varphi_{1,j}(x) := x_{1,j} + x_{j,j} - x_{1,j+1} - x_{j+1,j+1} \quad \text{if } 1 < j \leq n - 1. \tag{1.32}$$

**Theorem 1.3.** Fix a positive integer $k$, $2 \leq k \leq n - 1$, and a triangle $x \in X_n$. Assume that

$$s_k(x) = \tilde{x}, \quad s_k^{(e)}(x) = \tilde{x}.$$

Then

$$\tilde{x}_{ik} = x_{ik} + Q_k(\varphi_{2k}(x), \ldots, \varphi_{kk}(x), \varphi_{1k}(x)); \tag{1.33}$$

$$\tilde{x}_{ik} = x_{ik} + Q_k(\varphi_{2k}(x), \ldots, \varphi_{kk}(x), \varphi_{1k}(x) + \beta_{k+1}(x) - \beta_k(x) + e). \tag{1.34}$$

The proof of Theorem 1.3 will be given elsewhere. We shall give precise formulas for $s_k$, $k \leq 3$, in Section 3.

**Remark 1.2.** We know (see (1.20)) that $\sigma^n = q_{n-1}u_{n-1}$. Assume additionally that $(q_{n-1}u_{n-1})^h = 1$, $h$ may be equal to $\infty$. Consider the subgroup $\Sigma_n \subset G_n$ generated by the elements $s_i = \sigma^{i-1}t_1\sigma^{1-i}$, $1 \leq i < nh$. Then from Corollary 1.1 it follows that

a) $s_i^2 = 1$, $1 \leq i < nh$,

b) $(s_is_{i+1})^3 = 1$, $1 \leq i < nh - 2$,

c) $s_is_j = s_js_i$ if $2 \leq |i - j| \leq n - 2$.

**Proof.** From the definition, it is easy to see that $s_is_j = \sigma^{-i}(t_1s_{j-i+1})\sigma^i$ if $j \geq i$. So $(s_is_{i+1})^3 = 1$ if and only if $(t_1s_2)^3 = 1$. But $t_1s_2 = (t_2t_1)^2$, and by Proposition 1.3 part c) we know that $(t_1t_2)^6 = 1$. Similarly, $(s_is_j)^2 = 1$ if and only if $(t_1s_{j+i+1})^2 = 1$, and under the assumption $2 \leq |i - j| \leq n - 2$ assertion c) follows from Theorem 1.1, part 10). 

In particular, for any $1 \leq a \leq (n - 1)h$ the involutions $s_a, \ldots, s_{a+n-2}$ generate a subgroup of $G_n$ isomorphic to the symmetric group $S_n$.

**Remark 1.3.** The involutions $q_i$, $1 \leq i \leq n - 1$, also give a system of generators for the group $G_n$, because we have

$$t_1 = q_1, \quad t_i = q_{i-1}q_iq_{i-1}q_{i-2} \quad \text{if } i \geq 2 \quad (g_0 := 1). \tag{1.35}$$

The proof of (1.35) follows from Proposition 1.4, item a). In fact, we have

$$q_{i-1}q_{i-2}q_{i-3}q_{i-4} = q_{i-1}q_{i-1}p_{i-1}p_{i-1}q_{i-2}q_{i-3}q_{i-4}p_{i-1}p_{i-1} = t_i \quad \text{if } i \geq 2. \quad \blacksquare$$
The relations between the generators $q_i$, $1 \leq i \leq n - 1$, follow from Corollary 1.1 and have the following form:

1) $q_i^2 = 1$, $1 < i < n - 1$,
2) $(q_1 q_2)^6 = 1$, $(q_1 q_i)^4 = 1$ if $3 \leq i < n - 1$,
3) $[q_{i+1}, q_i q_{i-1} q_i] = 0$ if $3 \leq i \leq n - 2$,
4) $[q_i q_{i+1} q_i q_{i-1}, q_j q_{j+1} q_j q_{j-1}] = 0$ if $|i - j| \geq 2$.

In Section 2 we shall show that, in the case where $\lambda \in \mathbb{Z}_+^n$ is a partition, the restriction of the involution $q_{n-1}$ to the set $K_2^2 \simeq STY(\lambda, \leq n)$ coincides with the Schützenberger involution $S$ (see e.g., [Sch1, Sch2, EG, Ki1]). The involutions $q_i$, $1 \leq i \leq n - 1$, admit a similar combinatorial interpretation. So, our construction gives the extension of the Schützenberger involutions $q_i$ to the space of triangles $X_n$ and describes the group generated by $q_1, \ldots, q_{n-1}$, i.e., the relations between $q_1, \ldots, q_{n-1}$. The next steps are Theorems 1.1 and 2.3 (see Section 2), which describe a relationship between the natural action of the symmetric group $S_n = < s_1, \ldots, s_{n-1} >$ on the set $STY(\lambda, \leq n)$, which was introduced and studied by A. Lascoux and M.-P. Schützenberger [LS2, LS3], and the Schützenberger involutions

$$s_i = q_i q_1 q_i.$$ (1.37)

Equality (1.37), restricted to the set $STY(\lambda, \leq n)$, is a pure combinatorial assertion and may be deduced directly from the properties of the plactic monoid (see [LS2]) and the Robinson–Schensted correspondence. Details will appear elsewhere.

Remark 1.4. In a particular case of a weight $\beta \in \mathbb{R}_+^n$ of the form $\beta = (a^n)$, i.e., $\beta_1 = \beta_2 = \cdots = \beta_n = a$, the group $G_n$ preserves the convex polytope $K^\lambda(\beta)$ for any highest weight $\lambda \in \mathbb{R}_+^n$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. It is easy to see that the restriction of $t_i$ to the polytope $K^\lambda(\beta)$ is the identity map and, consequently, for all $1 \leq i \leq n - 1$ we have

$$s_i|_{K^\lambda(\beta)} = Id_{K^\lambda(\beta)}.$$

The example

\[
\begin{array}{cccc}
7 & 6 & 3 & 0 \\
6 & 5 & 1 \\
5 & 3 \\
4
\end{array}
\]

shows that $(t_3 t_2)^6 x \neq x$ (in fact, in this example $(t_3 t_2)^{24} x = x$), and it is not clear whether or not it is possible to find a subgroup in $G_n$ that is isomorphic to the symmetric group $S_N$ (for some $N$) after the restriction of the action of $G_n$ to the polytope $K^\lambda(\beta)$.

We finish this section with introducing some additional transformations of the space $X_n$ and considering the stable behavior of the involutions $s_i$. First, consider a map $I: X_n \rightarrow X_n$ defined in the following way: let $x \in X_n$, then $I(x) = \tilde{x}$, where

$$\tilde{x}_{ij} = x_{1n} - x_{j-i+1,j}, \quad 1 \leq i \leq j \leq n.$$ (1.38)
Proposition 1.7. We have
1) $\beta_i(I(x)) = x_{1:n} - \beta_i(x)$.
2) The map $I$ commutes with the actions of $s_i$ and $q_i$, i.e., $I s_i = s_i I$ and $q_i I = I q_i$, $1 \leq i \leq n - 1$.
3) $I \cdot T_e = T_{-e} \cdot I$, $I \cdot s_i^{(e)} = s_i^{(-e)} \cdot I$, $1 \leq i \leq n - 1$.

Proof. Using Definition 1.1 for the involution $t_j$, we find that, for $x \in X_n$,

$$ x_{ij}(t_j(Ix)) = x_{1:n} - x_{j-i+1,j}(t_j(x)). $$

Consequently, $It_j = t_j I$, i.e., the actions of $t_j$ and $I$ commute. •

Second, for a partition $\lambda$ we consider the restrictions of the maps $s_i^{(\pm 1)}$ to the Gelfand-Tsetlin polytope $K^{\lambda}$:

$$ e_i := pr \cdot s_i^{(1)}, f_i := pr \cdot s_i^{(-1)}, 1 \leq i \leq n - 1, $$

(1.39)

where $pr: X_n \to K^{\lambda}$ is the projection map.

Proposition 1.8. We have

$$ s_i f_i s_i = e_i, \ 1 \leq i \leq n. $$

Proof. The assertion follows from the definitions (1.16), Proposition 1.4, item f), and (1.39). •

Now we consider the stable behavior of the involutions $s_i$ and $q_i$. Given the space of triangles $X_n-1$, we define the following embeddings $\varphi_\alpha: X_{n-1} \hookrightarrow X_n$, $\alpha = 1, 2$: if $x = (x^{(n-1)}, \cdots, x^{(1)}) \in X_{n-1}$, then

$$ \varphi_1(x) = (x^{(n)}, x^{(n-1)}, \cdots, x^{(1)}), $$

(1.40a)

where $x^{(n)} = (x_{1:n-1}, \cdots, x_{n-1:n-1}, 0)$; and

$$ \varphi_2(x) = (\tilde{x}^{(n)}, \cdots, \tilde{x}^{(1)}), $$

(1.40b)

where $\tilde{x}^{(1)} = 0$, $\tilde{x}^{(i+1)} = (x_{1,i}, \cdots, x_{ii}, 0)$, $1 \leq i \leq n - 1$.

Proposition 1.9. We have

1) $s_i(\varphi_1(x)) = s_i(x)$ \ if \ $1 \leq i \leq n - 2$,

$$ s_{n-1}(\varphi_1(x)) = (x_{1:n-1}, \cdots, x_{n-2:n-2}, 0). $$

(1.41)

2) $t_i(\varphi_2(x)) = \varphi_2(t_{i-1}(x))$,

$q_i(\varphi_2(x)) = \varphi_2(q_{i-1}(x))$,

$s_i(\varphi_2(x)) = \varphi_2(s_{i-1}(x))$, \ where \ $1 \leq i \leq n - 1$.

(1.42)
Remark 1.5. All our main results (including those dealing with a cocharge \( c \) (see Section 3)) after small modification remain valid for the space of truncated triangles \( X_{n,m} \), \( 0 \leq m \leq n \) [BZ1]. By definition, the space \( X_{n,m} \) consists of all sequences
\[
x = (x^{(n)}, \ldots, x^{(m)}),
\]
where \( x^{(j)} = (x_{1,j}, \ldots, x_{j,j}) \in \mathbb{R}^j \). In what follows, we always regard the space \( X_{n,m} \) as the subspace of \( X_n \) distinguished by the constraint
\[
x_{i,j} = x_{i,j+1}, \ 1 \leq i \leq j \leq m - 1.
\]

Given a truncated triangle \( x \in X_{n,m} \), it is convenient to use the notation \( \lambda(x) := x^{(n)} \), \( \nu(x) := x^{(m)} \) and to define a weight \( \beta := \beta(x) \) of a truncated triangle \( x \in X_{n,m} \) as a vector \( \beta = (\beta_{m+1}, \ldots, \beta_n) \in \mathbb{R}^m \) such that \( \beta_j = |x^{(j)}| - |x^{(j-1)}|, m \leq j \leq n \). For given vectors \( \lambda \in \mathbb{R}^n, \nu \in \mathbb{R}^m \), and \( \beta \in \mathbb{R}^{n-m} \) we consider the following subspaces of the space \( X_{n,m} \):
\[
\begin{align*}
X_{n,m}^\lambda & = \{ x \in X_{n,m} \mid \lambda(x) = \lambda \} ; \\
X_{n,m}^\lambda \setminus \nu & = \{ x \in X_{n,m}^\lambda \mid \nu(x) = \nu \} ; \\
X_{n,m}^\lambda \setminus \nu(\beta) & = \{ x \in X_{n,m}^\lambda \setminus \nu \mid \beta(x) = \beta \} .
\end{align*}
\]

By definition, a truncated triangle \( x \in X_{n,m} \) is called a truncated Gelfand–Tsetlin pattern if \( x \) belongs to the cone \( K_n \). We denote by \( K_{n,m} \) the cone of all truncated GT-patterns, and its intersections with subspaces (1.44) by \( K_{n,m}^\lambda \), \( K_{n,m}^\lambda \setminus \nu \), and \( K_{n,m}^\lambda \setminus \nu(\beta) \), respectively. It is clear that \( K_{n,m}^\lambda \setminus \nu \) and \( K_{n,m}^\lambda \setminus \nu(\beta) \) are convex, compact polytopes in the space \( \mathbb{R}^{c_{n,m}} \), where \( c_{n,m} = \frac{1}{2}(n-m)(n+m+1) \). It is well known (see Section 2 or [BZ1]) that if \( \lambda \) and \( \nu \) are partitions, \( \lambda \trianglerighte = \nu \), \( |\lambda \setminus \nu| = p \), and \( \beta \) is a composition of the same integer \( p \), then the number of integral points in the convex polytope \( K_{n,m}^\lambda \setminus \nu \) (respectively, \( K_{n,m}^\lambda \setminus \nu(\beta) \)) is equal to the dimension of the representation \( V_{\lambda \setminus \nu} \) of the Lie algebra \( \mathfrak{gl}_{n-m} \) (see Section 2) (respectively, to the dimension of the subspace \( V(\lambda \setminus \nu, \beta) \subset V_{\lambda \setminus \nu} \) of weight \( \beta \)). On the other hand, as is also well known, the dimension of the weight subspace \( V(\lambda \setminus \nu, \beta) \subset V_{\lambda \setminus \nu} \) admits a purely combinatorial description as the number of (skew) standard Young tableaux of shape \( \lambda \) and content \( \beta \),
\[
|K_{Z}^{\lambda \setminus \nu}(\beta)| := |K_{Z}^{\lambda \setminus \nu}(\beta) \cap \mathbb{Z}^{c_{n,m}}| = \dim V(\lambda \setminus \nu, \beta) = \dim \text{STY}(\lambda \setminus \nu, \beta).
\]

Using the constraint (1.43), it is possible to define the action of the symmetric group \( S_{n-m} \) on the space of truncated triangles \( X_{n,m} \) by setting
\[
\sigma_i = s_{m+i}, \ 1 \leq i \leq n - m - 1.
\]

The fact that the involutions \( \sigma_i \) really generate the symmetric group \( S_{n-m} \) follows from Theorem 1.1 and Corollary 1.2.
Exercise 1.1. Fix a real number $t$ and integers $i, j$ such that $1 < i < j < n$. Define a transformation $T_t^{(i,j)} : X_n \to X_n$ in the following way:

$$T_t^{(i,j)}(x) := x, \quad x \in X_n,$$

where $\bar{x}_{ij} = x_{ij} - t$ and $\bar{x}_{kl} = x_{kl}$ if $(k, l) \neq (i, j)$. Show that

(i) $(t_j T_t^{(i,j)})^2 = 1$,
(ii) $t_k T_t^{(i,j)} = T_t^{(i,j)} t_k$ if $|k - j| \geq 2$,
(iii) $t_k T_t^{(i,j)} t_k T_t^{(i,j)} t_k = T_t^{(i,j)} t_k T_{t-\epsilon}^{(i,j)} t_k T_{t-\delta}^{(i,j)} t_k$ if $|k - j| = 1$, $\epsilon, \delta \in \mathbb{R}$,

which is a generalization of (1.14e).

Exercise 1.2. Fix a real number $q$ and an integer $j$, $1 < j < n$. Define a transformation $\tilde{t}_j : X_n \to X_n$:

$$\tilde{t}_j := t_j[q](x) = \bar{x},$$

where

$$\tilde{x}_{i,k} = x_{i,k} \quad \text{if } k \neq j,$$
$$\tilde{x}_{i,j} = \min(x_{i,j+1}, qx_{i,j-1}) + \max(x_{i+1,j+1}, qx_{i,j-1}) - qx_{i,j}.$$

Here we assume that $x_{0,j} := +\infty$, $x_{j,j-1} := -\infty$, $1 \leq j \leq n - 1$. Show that

(i) $\tilde{t}_j^2 = (1 - q)\tilde{t}_j + q \cdot Id_{X_n}$, $\tilde{t}_j \tilde{t}_j = \tilde{t}_j \tilde{t}_i$ if $|i - j| \geq 2$,
(ii) $\beta(\tilde{t}_j(x)) = (\beta_1, \cdots, \beta_j - 1, \beta_j + 1, (1 - q)\beta_j, q\beta_j, \beta_{j+1}, \cdots, \beta_n),$

where $\beta(x) := (\beta_1, \cdots, \beta_n)$ is the weight of a triangle $x$. Thus, we obtain a representation of the Hecke algebra $H_n(q)$ on the space of weights $\mathbb{R}^n$.

Problems. (i) Find the defining relations between the transformations $\tilde{t}_j$.
(ii) Is it possible to extend the above representation of the Hecke algebra $H_n(q)$ to the entire space of triangles $X_n$?
(iii) Is it possible to construct a cpl-representation of the braid group $B_n$ on the space of triangles $X_n$?

§2. Combinatorial description of the basic transformations

In this section we give a combinatorial description of the restrictions of the transformations constructed in the previous section to the set of standard Young tableaux of a given shape $\lambda$ and content $\beta$. We shall denote this latter set by $STY(\lambda, \beta)$. Also, we exploit the notation $STY(\lambda \setminus \nu, \beta)$ for the set of skew standard Young tableaux of skew shape $\lambda \setminus \nu$ and content $\beta$, and $STY(\lambda \setminus \nu, \leq n)$ for the set of all skew standard Young tableaux of shape $\lambda \setminus \nu$ with all entries not exceeding $n$. We assume in the sequel that $l(\lambda) \leq n$, $l(\nu) \leq m$, $l(\beta) \leq n$ for some fixed positive integers $m \leq n$. 


First, we recall the well-known bijection (see [GZ1, GZ2, BZ1]) between the set $\text{STY}(\lambda \setminus \nu, \beta)$ and the set of integral points in the convex polytope $K^\lambda\setminus\nu(\beta)$. Take $T \in \text{STY}(\lambda \setminus \nu, \beta)$. As is well known (see [Ma]), one may consider the tableaux $T$ as a sequence of Young diagrams
\[ \nu = \lambda^{(m)} \subset \lambda^{m+1} \subset \cdots \subset \lambda^{(n)} = \lambda \] (2.1)
such that all skew diagrams $\lambda^{(i)} \setminus \lambda^{(i-1)}, \ m < i \leq n$, are horizontal strips. Define the triangle $x = x(T) = (x^{(n)}, \ldots, x^{(m)})$, where $x^{(i)}$ is the shape of the diagram $\lambda^{(i)}$. Then
\[ T \in \text{STY}(\lambda \setminus \nu, \beta) \text{ if and only if } x(T) \in K^\lambda\setminus\nu(\beta). \]

We construct the inverse map $K^\lambda\setminus\nu(\beta) \to \text{STY}(\lambda \setminus \nu, \beta)$. Given a point $x \in K^\lambda\setminus\nu(\beta)$, we are filling the shape $\lambda \setminus \nu$ by the numbers $1, \ldots, n$ in accordance with the following rule: in the $i$-th row $(\lambda \setminus \nu)_i$ of the skew diagram $\lambda \setminus \nu$ we write exactly $x_{k,i+m} - x_{k,i+m+1}$ numbers equal to $k$, starting from $k = 1$.

Here is an explanatory example. Assume
\[ T := \begin{array}{cccc}
1 & 4 & 2 & 3 \\
2 & 2 & 3 & 5 \\
1 & 3 & 4 & 4
\end{array}, \quad \lambda = (5,5,4), \quad \nu = (3,1), \quad \mu = (2,2,2,3,1). \]

Then we have the following sequence of shapes for (2.1):
\[ \nu = (3,1) \subset (4,1,1) \subset (4,2,1) \subset (4,4,2) \subset (5,4,4) \subset (5,5,4) = \lambda. \]

Consequently,
\[ x(T) = \begin{array}{cccc}
5 & 5 & 4 & 0 \\
4 & 4 & 4 & 0 \\
3 & 1 & 0 \\
1 & 1 \\
3 & 1
\end{array} \in K^\lambda\setminus\nu(\mu). \]

We briefly remind the reader of the group-theoretic interpretation of the numbers $K_{\lambda \setminus \nu, \beta} := |\text{STY}(\lambda \setminus \nu, \beta)|$ and $d_{\lambda \setminus \nu}^{(n)} := |\text{STY}(\lambda \setminus \nu, \nu \leq n)|$. Let $g_n = gl_n$. The lattice of weights $P_n$ of the Lie algebra $g_n$ is identified in a standard way with $\mathbb{Z}^n$; the set $P_n^+$ of highest weights of finite-dimensional polynomial $g_n$-modules is identified with $\{\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$. For each $\lambda \in P_n^+$, let $V_\lambda$ be an irreducible $g_n$-module with highest weight $\lambda$. For all $m, \ 0 \leq m \leq n$, we embed the Lie algebra $g_m \times g_{n-m}$ in $g_n$ in such a way that the subalgebra $g_m$ is generated by the elements $e_{ij}, 1 \leq i, j \leq m$, where $\{e_{ij} \mid 1 \leq i, j \leq n\}$ is a standard basis of $g_n$, and $g_{n-m}$ is generated by the elements $\{e_{ij} \mid m+1 \leq i, j \leq n\}$. Accordingly, we shall write the weights $\beta \in P_{n-m}$ in the form $\beta = (\beta_{m+1}, \cdots, \beta_n)$. For all $\lambda \in P_n^+$ and $\nu \in P_n^+$, we define the skew $g_{n-m}$-module $V_{\lambda \setminus \nu}$ (see [BZ1]) by putting
\[ V_{\lambda \setminus \nu} = \text{Hom}_{g_m}(V_\nu, V_\lambda | g_m). \] (2.2)

If $m = 0$, it is convenient to assume that $P_n^+$ consists of one element $\nu = \phi$ and that $V_{\lambda \setminus \phi}$ is an irreducible $g_n$-module $V_\lambda$. 
Proposition 2.1 (see [BZ1]). The multiplicity $K_{\lambda \nu, \beta}$ of the weight $\beta$ in the skew $\mathfrak{g}_{n-m}$-module $V_{\lambda \nu}$ is equal to the number of all truncated Gelfand–Tsetlin patterns of highest weight $\lambda \nu$ and of weight $\beta$.

As a corollary, we obtain:

$$d_{\lambda \nu}^{(n)} := |K_{Z}^{\lambda \nu}| = |\text{STY}(\lambda \nu, \leq n)| = \dim V_{\lambda \nu}.$$ 

Note that the numbers $K_{\lambda \nu, \beta}$ admit a completely elementary description in terms of symmetric functions (see [Ma]): they are the coefficients in the expression of the skew Schur function $S_{\lambda \nu}(x)$ as a linear combination of monomials $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

Now we are ready to give a combinatorial interpretation of the restrictions of the transformations constructed in Section 1 to the set $\text{STY}(\lambda \nu, \beta)$. We keep the notation of Remark 1.5 and that of the beginning of Section 2.

A) The action of $t_{i+m}$ on the set $\text{STY}(\lambda \nu, \beta)$.

Given a skew tableau $T \in \text{STY}(\lambda \nu, \beta)$, consider the part $T_i$ of the tableau $T$ which is filled by the letters $i$ and $i+1$. It is clear that $T_i$ is a disjoint union of the following fragments:

We replace every such fragment by the following one:

The part $T \setminus T_i$ of the tableau $T$ remains unchanged. As a result, we obtain a new tableau $\overline{T} \in \text{STY}(\lambda \nu, (i, i+1) \beta)$.

Proposition 2.2. We have

$$t_{i+m}(T) = \overline{T}.$$ 

The proof is clear from the construction of the bijection

$$\text{STY}(\lambda \nu, \beta) \leftrightarrow K_{Z}^{\lambda \nu}(\beta).$$ 

We remark that the action of the transformations $t_i$ on the set $\text{STY}(\lambda \nu, \beta)$ (as described above) is well known, see, for example, [BK, SW, Sa].
Our main observation is that the action of $t_i$ on the set of standard Young tableaux of a given shape and content admits a continuous piecewise linear extension to the set of triangles $X_n$, and that other well-known combinatorial operations on the set of Young tableaux may be represented as superpositions of transformations $t_i$ and, consequently, also admit an extension to the space $X_n$. Anyhow, using the involutions $t_i$, for any element $\sigma$ of the symmetric group $S_n$ we may construct a piecewise linear one-to-one mapping, defined over $\mathbb{Z}$, between the GT-polytopes $K^{\lambda}v(\beta)$ and $K^{\lambda}v(\sigma\beta)$. Finally, we note that to define the involution $t_i$ it is essential to have the entire tableau $T$, but not only the corresponding word $w(T)$, so we cannot define the action of $t_i$ on the words.

Remark 2.1. The action of the involutions $t_i$ on the set of standard Young tableaux $STY(\lambda)$, i.e., in the case of the weight $\beta = (1^n)$, $|\lambda| = n$, has been studied in many papers. We only mention the work of A. Garsia and T. McLarnan [GM], which contains, among other things, some interesting results concerning the relationship between the Robinson–Schensted correspondence and the transformations $t_i$. Now we talk about the group $G_n$ generated by $t_i$, $1 \leq i \leq n-1$, in the case under consideration. It is not difficult to verify that $t_i = Id_{STY(\lambda)}$, and $(t_it_{i+1})^6 = 1$, $1 \leq i \leq n-2$. Since $t_1 = Id_{STY(\lambda)}$, we see that $a_i = Id$ on the set $STY(\lambda)$ for all $1 \leq i \leq n-1$. We note that, if the diagram $\lambda$ is a hook, then $(t_it_{i+1})^3 = 1$, $1 \leq i \leq n-2$, and the group $G_n$, viewed as a subgroup of $\text{Aut}(STY(\lambda))$, is isomorphic to the symmetric group $S_{n-1}$.

Remark 2.2. Below we give another combinatorial description of the action of the transformation $t_i$ on the Gelfand–Tsetlin cone $K_n$.

For this, we begin with a description of some special triangulation of the cone $K_n$. We denote by

$$B_n = K_n \cap \{0,1\}^{n^2+n}$$

the set of all $(0,1)$-patterns.

Lemma 2.1. There exist a bijection

$$B_n \rightarrow \{0,1\}^n,$$

in particular, $|B_n| = 2^n$.

Proof. The bijection under consideration is given by the correspondence

$$x \in B_n \rightarrow \beta(x) \in \{0,1\}^n.$$

It is easy to see that the GT-pattern consisting of 0's and 1's only is uniquely determined by its weight. •

Now we introduce a partial order on the set of triangles $X_n$: $x \succeq \tilde{x}$ means $x - \tilde{x} \in (X_n)_{\mathbb{R}+} = (\mathbb{R}^n)^{\pi_{n^2+n}}$.
Proposition 2.3. The set $B_n$ possesses the following properties:
1) $B_n$ is the set of all generators of the cone $(K_n)_\mathbb{Z} = K_n \cap (X_n)_\mathbb{Z}$ (consequently, also of the cone $K_n = \mathbb{R}_+ \otimes (K_n)_\mathbb{Z}$).
2) $B_n$ is a lattice with respect to the partial order "\geq".
3) The length of a maximal chain in the lattice $B_n$ is equal to $\frac{n(n+1)}{2}$.
4) Assume $x \in K_n$, then there exists a unique representation

$$x = \alpha_1 x_1 + \cdots + \alpha_m x_m,$$

(2.3)

where $x_1, \ldots, x_m \in B_n$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$, $\alpha_1 > 0, \ldots, \alpha_m > 0$, and $x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$ ($m < \frac{n^2+n}{2}$).

Proposition 2.4 (The relationship between the involutions $t_i$ and the Gelfand–Tsetlin cone $K_n$).
1) If $x \in B_n$, then $t_i(x) \in B_n$, $1 \leq i \leq n - 1$.
2) Assume that $x$ is written uniquely in the form (2.3). Then

$$t_i(x) = \alpha_1 t_i(x_1) + \cdots + \alpha_m t_i(x_m).$$

Note that if the elements $x_1, \ldots, x_m$ lie in a chain of the lattice $B_n$, then their images $t_i(x_1), \ldots, t_i(x_m)$ may belong to a different chain.

Conjecture 2.1. Assume that $\lambda, \beta \in \mathbb{Z}^n$, then all vertices of the Gelfand–Tsetlin polytope $K^\lambda(\beta)$ belong to the set $K_n \cap (X_n)_\mathbb{Z}$.

B) The action of the transformations $p_i$ and $p_i^{-1}$.
Let $\lambda$ be a partition and $\beta$ a composition, $\beta = (\beta_1, \ldots, \beta_n)$. First, we observe that it is sufficient to describe the action of the transformations $\sigma = p_n^{-1}$ and $\sigma^{-1} = p_{n-1}$ on the set

$$STY(\lambda, \beta) \simeq K^\lambda(\beta) \hookrightarrow X_n.$$ 

In fact, we have a natural embedding $X_i \hookrightarrow X_n$, $x \rightarrow \tilde{x}$, where for a triangle $x = (x^{(i)}, \ldots, x^{(1)}) \in X_i$ we define

$$\tilde{x} = (x^{(i)}, \ldots, x^{(i)}, x^{(i)}, \ldots, x^{(1)}) \in X_n.$$ 

On the level of Young tableaux this embedding corresponds to considering the part $T_{\leq i}$ of $T$, filled by the numbers $1, \ldots, i$. It is easy to verify the commutativity of the following diagram:

$$
\begin{array}{ccc}
X_i & \hookrightarrow & X_n \\
p_{i-1} & \downarrow & \downarrow p_{i-1} \\
X_i & \hookrightarrow & X_n
\end{array}
$$
This means that $p_{i-1} \in G_n$ acts nontrivially only on the tableau $T_{\leq i}$. Moreover, we observe that

$$
\beta(\sigma^{-1}(x)) = (\beta_2 \beta_3 \cdots \beta_n \beta_1) = (1, 2, \ldots, n) \beta(x),
$$

$$
\beta(\sigma(x)) = (\beta_n \beta_1 \cdots \beta_{n-1}) = (n, n-1, \ldots, 1) \beta(x).
$$

Given a tableau $T \in STY(\lambda \setminus \nu, \beta)$, we may assume that $\beta_1 > 0$. We delete the most upperleft box filled with 1's. After this, we apply a "jeu de taquin" (see [Sch1, Sch2]) in order to obtain a standard tableau $T'$ having the same left border strip, and one box less than $T$. Repeating this procedure, we eliminate all 1's from the tableau $T$. As a result, we obtain a new tableau $T''$ of content $(0, \beta_2, \ldots, \beta_n)$, and a horizontal strip $\theta_n$, $|\theta_n| = \beta_1$. We fill the strip $\theta_n$ with the number $n$. Now we subtract 1 from all numbers situated in the boxes of $T''$. We obtain a tableau $\overline{T}$ of content $\overline{\beta} = (\beta_1, \ldots, \beta_{n-1})$, where $\beta_i = \beta_{i+1}$, $1 \leq i \leq n-1$. The tableau $\overline{T}$ and the horizontal strip $\theta_n$ determine a tableau $\overline{T} \in STY(\lambda, \overline{\beta})$, where $\overline{\beta} = (\beta_2, \ldots, \beta_n, \beta_1)$.

Proposition 2.5. We have (in the case $\nu = \phi$)

$$
\sigma^{-1}(T) = p_{n-1}(T) = \overline{T}.
$$

Similarly, we may describe the action of the transformation $\sigma = p_{n-1}^{-1}$. However, in this case it is more convenient to consider the action of $\sigma$ on the set of skew Young tableaux. So, let $\nu$ be a partition, $\nu \subseteq \lambda$. Given a tableau $T \in STY(\lambda \setminus \nu, \beta)$, we may assume that $\beta_n > 0$. We delete the most right and bottom box in $T$ is filled in with the number $n$. After this, we apply a "jeu de taquin" (see [Sch2, Sa]) in order to obtain a standard tableau $T' \in STY(\lambda \setminus \nu, \beta)$ with the same right border strip, which has one box less than $T$ (in fact $\beta = (\beta_1, \ldots, \beta_{n-1}, \beta_n - 1)$, $|\beta| = |\nu| + 1$). Repeating this procedure, we eliminate all numbers equal to $n$ from the tableau $T$. As a result, we obtain a new tableau $T''$ of content $\beta = (\beta_1, \ldots, \beta_{n-1})$, where $\beta_i = 0$, $2 \leq i \leq n$. Further, we fill in the strip $\theta_1$ with the number 1. The tableau $\overline{T}$ and the horizontal strip $\theta_1$ generate a tableau $\overline{T} \in STY(\lambda \setminus \nu, \beta)$, where $\beta = (\beta, \beta_1, \beta_2, \ldots, \beta_{n-1})$.

Proposition 2.6. We have (in the case $\nu = \phi$)

$$
\sigma(T) = p_{n-1}^{-1}(T) = \widehat{T}.
$$

The proof of Propositions 2.5 and 2.6 is based on the paper of M.-P. Schützenberger [Sch2] and Proposition 2.2. Details will appear elsewhere.

C) The action of the involutions $q_i \in G_n$, $1 \leq i \leq n$.

Let a tableau $T \in STY(\lambda, \beta)$ be given. Note at the beginning that the involution $q_{i-1}$ acts nontrivially only on the tableau $T_{\leq i}$ (see Section B), so it is sufficient to describe the action of the transformation $q_{n-1}$ on the set

$$
STY(\lambda, \beta) \cong K_{2n}^{2}(\beta) \subset X_n.
$$

Remark 2.3. We use the term "involution" for the restriction of $q_i$ to the set $STY(\lambda, \beta)$ in spite of the fact that in general the transformation $q_i$ does not preserve this set, but, of course, $q_i^2 = Id_{STY(\lambda, \beta)}$. 
Theorem 2.1. The restriction of the involution $q_{n-1}$ to the set $STY(\lambda, \beta)$ coincides with the Schützenberger involution $S$.

We recall the corresponding definitions. Let $\lambda$, $\nu$, $\beta$, and $n$ be as above. Let a tableau $T \in STY(\lambda \setminus \nu, \beta)$ be given. We may define four natural operations on the set $STY(\lambda \setminus \nu, \beta)$. In this subsection we define two of them, the definition of the remaining ones being postponed till subsection D). Note that it is sufficient to consider the case where all parts of the composition $\beta$ are not equal to zero. We have already given a combinatorial description of the transformations $p_i$ and $p_i^{-1}$. It is a "jeu de taquin" that is crucial for the description of these transformations. We recall (see subsection B) that $p_{n-1}(T) = \overline{T}$, where $T \in STY(\lambda \setminus \nu, \overline{\beta})$, $\overline{\beta} = (\beta_2, \beta_3, \ldots, \beta_n, \beta_1)$, and, in fact, the tableau $\overline{T}$ is a disjoint union of the tableau $\tilde{T}$ of content $\beta = (\tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1})$, where $\tilde{\beta}_i = \beta_{i+1}$, $1 \leq i \leq n-1$, and of the filled horizontal strip $\theta_n$, $|\theta_n| = \beta_1$, with the number $n$ in all boxes. After this we may act in two ways. The first one consists in applying the same algorithm to the tableau $\overline{T}$, and so on. As a result, we obtain a sequence of horizontal strips $\theta_1, \theta_2, \ldots, \theta_n$ such that $|\theta_i| = \beta_{n-i+1}$, $1 \leq i \leq n$, which defines (see [Ma]) a standard Young tableau $T_0 \in STY(\lambda \setminus \nu, \overline{\beta})$, where $\overline{\beta} = (\beta_n, \beta_{n-1}, \ldots, \beta_1)$. Therefore, the following mappings arise:

$$S: STY(\lambda \setminus \nu, \beta) \rightarrow STY(\lambda \setminus \nu, \overline{\beta}), \ T \rightarrow T_0,$$

$$S: K^\lambda Z^\nu \rightarrow K^\lambda Z^\nu. \quad (2.4)$$

The mapping $S$ is called the Schützenberger involution. If the weight $\beta$ has equal parts, then we obtain an involution $S: STY(\lambda, \beta) \rightarrow STY(\lambda, \beta)$.

The second way is to apply the transformation $p_{n-1}$ to the tableau $T$ $n$ times, i.e., to consider the tableau $p_{n-1}^n(T)$. It should be noted that the transformation $U := p_{n-1}^n \cdot S$ is also an involution, which will be described in subsection D. From our description of the Schützenberger involution $S$, it is clear that $q_{n-1|STY(\lambda, \beta)} = S$. This completes the proof of Theorem 2.1.

Remark 2.4 (the symmetries of the Littlewood-Richardson numbers). Let $V_\lambda$, $V_\mu$, $V_\nu$ be three irreducible finite-dimensional $sl_n$-modules with highest weights $\lambda$, $\mu$, $\nu$. The multiplicity $c^{\nu}_{\lambda \mu}$ of $V_\nu$ in the tensor product $V_\lambda \otimes V_\mu$ is called the Littlewood–Richardson (LR) number and plays an important role in the representation theory of the symmetric and general linear groups. The combinatorial rule for computing the LR-numbers was given by Littlewood and Richardson (LR-rule), [Li] (see also [T, Ma]). There are many various interpretations of the LR-rule: [GZ1, GZ2, T, LS2, KR, W, Z]. We shall use an interpretation in terms of the number of standard Young tableaux of certain kind [GZ2, KR], and another one in terms of BZ-triangles [BZ3]. Recall the corresponding definitions. To begin with (see, e.g., [Ma]), we observe that $c^{\nu}_{\lambda \mu} \neq 0$ if and only if both $\lambda$ and $\mu$ are contained in $\nu$ and $|\nu| \equiv |\lambda| + |\mu| \pmod{n}$. Further, consider a tableau $T \in STY(\lambda, \nu \setminus \mu)$ and its descent set $D(T)$. We recall that if $T \in STY(\lambda, \mu)$, then $D(T)$ is the maximal subset among those $\{x_1, \ldots, x_l\} \subset T$ that satisfy the following condition: there exists a sequence of entries $y_1, \ldots, y_l$ of the tableau $T$, all lying in different boxes, such that $y_i = x_i + 1$ and $n(x_i) < n(y_i)$, $1 \leq i \leq l$. Here $n(x)$ is the index of the row...
that contains \( x \). So, let \( D(T) = \{ x_1, \cdots, x_t \} \) be the descent set of \( T \). We define the set of exponents for \( T \) as a collection of integers \( \{ d_1(T), d_2(T), \cdots, d_{n-1}(T) \} \), where

\[
d_i(T) := \mu_{i+1} - |\{ x_j \in D(T) \mid x_j = i \}|, \quad 1 \leq i \leq n - 1.
\]

**Proposition 2.7** [GZ2, KR]. The LR-number \( c_{\lambda \mu}^\nu \) is equal to the number of standard Young tableaux \( T \) of shape \( \lambda \) and content \( \nu \setminus \mu \) such that

\[
d_i(T) \leq \mu_i - \mu_{i+1}, \quad 1 \leq i \leq n - 1.
\]

We denote the set of Young tableaux mentioned in Proposition 2.7 by \( \text{STY}(\lambda, \nu \setminus \mu) \).

**Proposition 2.8.** The involution \( q_{n-1} \) gives rise to a bijection

\[
q_{n-1}: \text{STY}(\lambda, \nu \setminus \mu) \rightarrow \text{STY}(\lambda, \mu^* \setminus \nu^*),
\]

where for a partition \( \mu = (\mu_1, \cdots, \mu_n) \) we put \( \mu^* := (\mu_1, \mu_1 - \mu_{n-1}, \cdots, \mu_1 - \mu_2) \).

Now we want to extend the bijection (2.5) to a bijection between some convex polytopes in the GT-pattern cone \( K_n \), but first we give the necessary definitions. Given a triangle \( x \in X_n \), we put

\[
d_j^{(i)}(x) = \sum_{1 \leq k \leq j} (x_{k,i+1} - 2x_{k,i} + x_{k,i-1}) + x_{j,i+1} - x_{j,i},
\]

where \( 1 \leq j \leq i \leq n - 1 \). As in [GZ1], we call the numbers \( d_j^{(i)} \) the exponents of a triangle \( x \in X_n \). Note (see [KR]) that if \( \lambda \) is a partition, \( \beta \) is a composition, and \( x \in K_{\lambda}^{\beta}(\beta) \), then

\[
\max\{ d_j^{(i)}(x) \mid 1 \leq j \leq i \} = d_i(x(T)), \quad 1 \leq i \leq n - 1.
\]

Finally, for \( \lambda, \beta, \gamma \in \mathbb{R}^n \) we define a convex polytope

\[
K^{\lambda}(\beta, \gamma) = \{ x \in K^{\lambda}(\beta) \mid d_j^{(i)}(x) \leq \gamma_i - \gamma_{i+1}, \quad 1 \leq j \leq i \leq n \}
\]

(see [GZ1]).

**Proposition 2.9.** Let \( V_\lambda, V_\mu, V_\nu \) be three irreducible finite-dimensional \( sl_n \)-modules with highest weights \( \lambda, \mu, \nu \). Then

i) **(the Gelfand–Zelevinsky theorem)**

\[
c_{\lambda \mu}^\nu = |K^{\lambda}(\nu \setminus \mu, \mu) \cap (X_n)_{\mathbb{Z}}|.
\]

ii) The involution \( q_{n-1} \) gives rise to a bijection

\[
q_{n-1}: K^{\lambda}(\nu \setminus \mu, \mu) \rightarrow K^{\lambda}(\mu^* \setminus \nu^*, \nu^*).
\]
The bijection (2.5) gives a combinatorial explanation of the well-known equality $c_{\lambda^*} = c_{\lambda^*}$. In order to better understand other symmetries of LR-numbers, it is convenient to use an interpretation of the LR-rule in terms of the Berenstein–Zelevinsky's triangles (BZ-triangles) [BZ3]. We briefly review the corresponding construction from [BZ3, C]. Denote by $T_n$, $n \geq 3$, the set of vertices of a regular triangular lattice filling the regular triangle with vertices $(2n - 3, 0, 0)$, $(0, 2n - 3, 0)$ and $(0, 0, 2n - 3)$; this triangle is decomposed into the union of elementary triangles having all three vertices in a set $Q_n$, and of elementary hexagons centered at points of a set $H_n$, where

$$H_n := \{(i, j, k) \in T_n \mid \text{all } i, j, k \text{ are odd}\},$$
$$Q_n = T_n \setminus H_n.$$

We consider the vector space $L$ consisting of families $(z(\xi))$, $\xi \in Q_n$, of real numbers such that for any elementary hexagon the sums over two vertices in every pair of opposite sides are equal; also, let $K = K_n := L \cap \mathbb{R}^n_+$ and $K_Z := L \cap \mathbb{Z}^n_+$. We define a linear projection $pr : L \rightarrow \mathbb{R}^{3n-3}$ by the formulas

$$pr(z) = (l_1, \ldots, l_{n-1}; m_1, \ldots, m_{n-1}; k_1, \ldots, k_n),$$

where

$$l_p = z(2(n - p) - 1, 2p - 2, 0) + z(2(n - 1 - p), 2p - 1, 0);$$
$$k_p = z(0, 2(n - p) - 1, 2p - 2) + z(0, 2(n - 1 - p), 2p - 1);$$
$$m_p = z(2p - 2, 0, 2(n - p) - 1) + z(2p - 1, 0, 2(n - 1 - p)).$$

Now, let $V_\lambda$, $V_\mu$, $V_\nu$ be three irreducible finite-dimensional $sl_n$-modules with highest weights $\lambda$, $\mu$, and $\nu$. We recall that the triple multiplicity $c_{\lambda \mu \nu}$ is defined as $\dim(V_\lambda \otimes V_\mu \otimes V_\nu)^0$. Evidently, we have $c_{\lambda \mu \nu} = c_{\lambda^* \mu^* \nu^*}$, where $\nu^*$ is the highest weight of the module $V_\nu^*$ dual to $V_\nu$ (see also Proposition 2.8). We denote by $BZ(\lambda, \mu, \nu)$ the set of families $z(\xi) \in K_n$ such that $pr(z(\xi)) = (l, m, k)$, where $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$ and $\nu_i = k_i - k_{i+1}$ for all $i$, $1 \leq i \leq n - 1$ (we assume that $l_n = m_n = k_n = 0$). We define a mapping

$$\theta_{\lambda \mu \nu} : K^\lambda(\nu^* \setminus \mu, \mu) \rightarrow BZ(\lambda, \mu, \nu)$$

in the following way: given a triangle $x \in K^\lambda(\nu^* \setminus \mu, \mu)$, we put $\theta_{\lambda \mu \nu}(x) = z(\xi)$, where

$$z(2i - 2, 2(j - i) - 1, 2(n - j)) = x_{ij} - x_{i,j-1}, 1 \leq i < j \leq n,$$

and observe that the values of $z(\xi)$ at all other points $\xi \in Q_n$ are uniquely determined by the condition $z(\xi) \in BZ(\lambda, \mu, \nu)$ (see [BZ3]).

**Proposition 2.10.** Assume that $V_\lambda$, $V_\mu$, $V_\nu$ are as before.

i) (Berenstein–Zelevinsky [BZ3]). The mapping (2.9) is a bijection. In particular,

$$c_{\lambda \mu \nu} = |BZ(\lambda, \mu, \nu)|.$$
ii) The transformation \( \tilde{q}_{n-1} := \theta_{\lambda \mu}^{-1} \cdot q_{n-1} \cdot \theta_{\lambda \mu} \) defines a bijection

\[
\begin{align*}
BZ(\lambda, \mu, \nu) \quad &\xrightarrow{\tilde{q}_{n-1}} \quad BZ(\lambda, \mu, \nu) \\
K^\lambda(\nu \setminus \mu, \mu) \quad &\xrightarrow{\tilde{q}_{n-1}} \quad K^\lambda(\nu \setminus \mu, \mu)
\end{align*}
\]

(2.11)

Now we summarize our discussion of symmetries of the Littlewood-Richardson numbers, or, what is the same, symmetries of the triple multiplicities \( c_{\lambda, \mu, \nu} \). Clearly, the coefficients \( c_{\lambda, \mu, \nu} \) must be invariant under all 6 permutations of \((\lambda, \mu, \nu)\), and also under the replacement of \((\lambda, \mu, \nu)\) by \((\lambda^*, \mu^*, \nu^*)\). These transformations generate a group of 12 symmetries. Obviously, the sets \( Q_n \) and \( H_n \) are invariant under all permutations of indices \((i, j, k)\). Therefore, we have a natural action of \( S_3 \) on \( R^{Q_n} \), and it is evident that \( L, K, \) and \( K_Z \) are invariant under this action. Formulas (2.8) imply at once that, if \( pr(z) = (\lambda, \mu, \nu) \), then \( pr(s_1(z)) = (\lambda^*, \nu^*, \mu^*) \) and \( pr(s_2(z)) = (\mu^*, \lambda^*, \nu^*) \), where \( z := z(\xi) \in BZ(\lambda, \mu, \nu) \), and \( s_1 = (1, 2) \) and \( s_2 = (2, 3) \) are two standard generators of \( S_3 \). We see that the expression (2.10) for triple multiplicities (the Berenstein-Zelevinsky theorem [BZ3]) makes evident the following symmetries:

\[
c_{\lambda, \mu, \nu} = c_{\lambda^*, \mu^*, \nu^*} = c_{\mu^*, \lambda^*, \nu^*} = c_{\nu^*, \mu^*, \lambda^*} = c_{\nu^*, \lambda^*, \mu^*}.
\]

(2.12)

The remaining symmetries can be derived from Proposition 2.10, ii), namely, the transformation \( \tilde{q}_{n-1} \) (see (2.11)) defines a bijection \( BZ(\lambda, \mu, \nu) \to BZ(\lambda, \nu, \mu) \), which gives a combinatorial proof of the equality \( c_{\lambda, \mu, \nu} = c_{\lambda, \nu, \mu} \). All other symmetries follow from (2.12) for the triple \((\lambda, \nu, \mu)\).

Finally, we consider an explanatory example. Assume \( n = 4, \lambda = (6, 2, 1), \mu = (5, 2, 1), \nu = (6, 5, 2) \). It is easy to see that \( c_{\lambda, \mu, \nu} = 4 \). Consider the following tableaux:

\[
T : \begin{array}{ccccc}
1 & 1 & 2 & 2 & 4 \\
3 & & & & \\
\end{array} \in STY(\lambda, \nu \setminus \mu \| \mu), \quad T' : \begin{array}{cccc}
1 & 2 & 2 & 3 \\
3 & 3 & & \\
4 & & & \\
\end{array} \in STY(\lambda, \nu^* \setminus \mu^* \| \mu^*).
\]

It is easy to verify that \( T' = S(T) \), where \( S \) is the Schützenberger involution (see Remark 2.3). Futher, we construct the corresponding GT-patterns:

\[
x(T) : \begin{array}{cccc}
6 & 2 & 1 & 0 \\
5 & 2 & & \\
5 & & 1 & \\
\end{array} \in K^\lambda(\nu \setminus \mu, \mu), \quad x(T') : \begin{array}{cccc}
6 & 2 & 1 & 0 \\
5 & & 0 & \\
3 & & & \\
\end{array} \in K^\lambda(\nu^* \setminus \mu^*, \mu^*).
\]

It is easy to see that \( x(T') = g_3(x(T)) \), \( g_3 = t_1 t_2 t_1 t_3 t_2 t_1 \). Finally, we construct the BZ-triangles corresponding to the GT-patterns \( x(T) \) and \( x(T') \). For clarity, we present the
construction in several steps:

\[
\begin{array}{cccc}
\bullet & 6 & 2 & 1 \\
\bullet & 5 & 2 & 1 \\
\bullet & 5 & 1 & 0 \\
\bullet & 2 & 3 & 0
\end{array}
\rightarrow
\begin{array}{cccc}
\bullet & 1 & 0 & 0 \\
\bullet & 0 & 1 & 0 \\
\bullet & 0 & 1 & 3 \\
\bullet & 1 & 0 & 1
\end{array}
\rightarrow
\begin{array}{cccc}
\bullet & * & * & 1 \\
\bullet & 3 & 1 & 3 \\
\bullet & 3 & 3 & 1 \\
\bullet & 1 & 0 & 0
\end{array}
\rightarrow
\begin{array}{cccc}
\bullet & 0 & * & * \\
\bullet & 1 & 0 & 1 \\
\bullet & 0 & 1 & 3 \\
\bullet & 1 & 0 & 0
\end{array}
\]

\[K^\lambda(\nu \setminus \mu, \mu) \quad BZ(\lambda, \mu, \nu^*)\]

The second triangles are composed from the differences \(\{x_{i,j} - x_{i,j-1}, \ 1 \leq i < j \leq n\}\) for the corresponding triangle \(x \in X_n\).

D) The action of the involutions \(u_i \in G_n, 1 \leq i \leq n - 1\).

Let a tableau \(T \in STY(\lambda \setminus \nu, \beta)\) be given. Note that the involution \(u_{n-i}, 1 \leq i \leq n-1\), acts nontrivially only on the skew tableau \(T_{p_{n-i}}\), where \(T_{p_{n-i}}\) is the part of \(T\) filled by the numbers \(i, i+1, \ldots, n\). So, it is sufficient to describe the action of the involution \(u_{n-1}\) on the set \(STY(\lambda \setminus \nu, \beta) \subset K^\lambda(\nu, \mu)\).

**Theorem 2.2.** Assume \(\nu = \phi\). On the set \(STY(\lambda, \beta)\), the involution \(u_{n-1}\) coincides with the dual Schützenberger involution \(U\), where \(U := P^{n-1}_{\nu} \cdot S\).

Now we continue the construction of natural operations on the set \(STY(\lambda \setminus \nu, \beta)\) (see subsection C). We remind the reader (see subsection B) that \(p_{n-1}(T) = \hat{T}\), where \(\hat{T} \in STY(\lambda \setminus \nu, \beta), \beta = (\beta_n, \beta_{n-1}, \ldots, \beta_0)\). In fact, the tableau \(\hat{T}\) is a disjoint union of the tableau \(\overline{T}\) of content \(\overline{\beta} = (\overline{\beta}_n, \ldots, \overline{\beta}_0)\), where \(\overline{\beta}_0 = 0, \overline{\beta}_i = \beta_{i-1}, 2 \leq i \leq n\), and the filled horizontal strip \(\theta_1, |\theta_1| = \beta_n\), with the number 1 in all boxes. As in subsection C, we may proceed in two ways. The first way consists in applying the same algorithm to the tableau \(\overline{T}\), and so on. As a result we obtain a sequence of horizontal strips \(\theta_1, \theta_2, \ldots, \theta_n\) such that \(\theta_i = \beta_{n-i+1}, 1 \leq i \leq n\), which determines a standard Young tableau \(T_\ast \in STY(\lambda \setminus \nu, \overline{\beta})\). Consequently, the following mappings arise:

\[U : STY(\lambda \setminus \nu, \beta) \rightarrow STY(\lambda \setminus \nu, \overline{\beta}), T \rightarrow T_\ast,\]

\[U : K^\lambda(\nu, \mu) \rightarrow K^\lambda(\nu, \overline{\nu}),\]  

(2.13)
The mapping $U$ is called the dual Schützenberger involution. If the weight $\beta$ has equal parts, we obtain an involution $U: STY(\lambda, \beta) \rightarrow STY(\lambda, \beta)$.

Another way is to apply the transformation $p_{n-1}^{-1}$ to the tableau $T$ $n$ times, i.e., to consider the tableaux $p_{n-1}^{-n}(T)$. It is possible to show that, in fact, both definitions of the involution $U$ coincide. So, $p_{n-1}^{-n} = SU$. From the definition of the dual Schützenberger involution $U$, it is clear that $u_{n-1}|_{STY(\lambda, \beta)} = U$. This completes the proof of Theorem 2.2.

Remark 2.5. In subsections C and D we have given combinatorial definitions for the transformations $S, U, \sigma^n = SU$, and $\sigma^{-n} = US$.

Our notation differ from that in [Sch1], [Sch2] and [EG] in the case $\beta = (1^n)$. We use the words “jeu de taquin” instead of “promotion transformation $T \rightarrow T^{\geq n}$” (see [Sch1, Sch2]). In fact, using the notation from [EG], we have (on the set $STY(\lambda)$):

$$S(T) = T^\beta + 1, \quad U(T) = T^S, \quad SU = p.$$ 

As an illustration of the operations $S, U, \sigma^n$, and $\sigma^{-n}$, consider the standard Young tableaux, along with corresponding rigged configurations (see Remark 2.8 below).

\[
\begin{align*}
T &= \begin{array}{cccccc}
1 & 2 & 3 & 8 & 10 \\
4 & 6 & 7 & 13 \\
5 & 9 & 12 \\
11
\end{array} \\
S(T) &= \begin{array}{cccccc}
1 & 2 & 3 & 8 & 9 \\
4 & 5 & 7 & 10 \\
6 & 12 & 13 \\
11
\end{array} \\
U(T) &= \begin{array}{cccccc}
1 & 4 & 6 & 10 & 11 \\
2 & 5 & 8 & 12 \\
3 & 7 & 13 \\
9
\end{array} \\
SU(T) &= \begin{array}{cccccc}
1 & 4 & 5 & 7 & 11 \\
2 & 6 & 9 & 12 \\
3 & 8 & 13 \\
10
\end{array}
\]

\[
\begin{array}{c}
0 \quad 0 \\
1 \quad 1 \\
0 \\
0 \\
1 \quad 1 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
1 \\
0 \\
0 \\
3 \\
2 \\
3 \\
3
\end{array}
\]

\[
\begin{array}{c}
1 \\
0 \\
0 \\
3 \\
1 \\
3 \\
3
\end{array}
\]
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US(T) =  

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Note that the tableaux T and S(T) have the same configuration and complementary quantum numbers, i.e.,

\[ J^{(k)}_{\alpha,n}(T) + J^{(k)}_{m_n(\nu(\lambda)) - \alpha + 1,n}(S(T)) = P^{(k)}_{n}(\nu, \beta), \]

see Remark 2.8. This is a general property of the Schützenberger involution, [Ki1].

Remark 2.6. In general, the involutions S and U do not commute. It is an interesting combinatorial task to find the order of the transformation SU on the set STY(\lambda, \beta). In some special cases the answer is known (see [Schl, EG]). Using this answer and formula (1.20), we find

Proposition 2.11.

a) If \( \lambda \) is a rectangular diagram, \( l(\lambda) \leq n \), then

\[ \sigma^n|_{K_2^\lambda} = SU = Id_{K_2^\lambda}. \] (2.14)

b) If \( \lambda = (n - 1, \ldots, 1,0) \) is a staircase diagram, then

\[ \sigma^{2n}|_{STY(\lambda)} = (SU)^2 = Id_{STY(\lambda)}. \] (2.15)

In fact, if \( T \in STY(\lambda) \), then \( \sigma^n(T) \) is a conjugating of \( T \).

Assertion a) was proved by M.-P. Schützenberger [Sch1] and later rediscovered by T. Miwa [Mi] in the context of the theory of crystal base (see [Ka1]). Assertion b) was proved by P. Edelman and C. Greene [EG] in the context of the theory of balanced tableaux. The proofs from [Sch1] and [EG] depend on the consideration of the “geometric” properties of the involutions S and U. It is desirable to find algebraic proofs of (2.14) and (2.15).

Note that we have also another distinguished involution \( W_0 \in G_n \), where \( w_0 \) is the element of maximal length in the symmetric group \( S_n \):

\[ W_0 = s_1 s_2 s_3 \cdots s_{n-1} s_{n-2} \cdots s_2 s_1; \]

\[ W_0 : K^\lambda \to K^\lambda, \ \beta(W_0(x)) = w_0\beta(x) = \overline{\beta(x)}. \]

It seems plausible that the involutions S and \( W_0 \) commute on the set \( K_2^\lambda \) (but not on entire \( X_n \)). In any case, it is interesting to understand the structure of the group generated by S, U, and \( W_0 \).
E) The action of the symmetric group $S_n$ on the set $STY(\lambda, \beta)$ (compare with [LS2, LS3]).

Given a tableau $T \in STY(\lambda, \beta)$ and an involution $s_i \in G_n$, consider the word $w(T)$ that corresponds to the tableau $T$. Recall that, by definition, the word $w(T)$ may be obtained from the tableau $T$ in the following way: let us read the elements of the tableau $T$ consecutively from right to left, starting from the upper row. We obtain the word $w(T)$ under consideration (see, e.g., [Ma]). For convenience, we shall denote $i$ by $a$ and $i + 1$ by $b$. First, we extract from $w(T)$ a subword $w'$ that contains only letters $a$ and $b$. Second, in the word $w'$ we remove consecutively all pairs $ab$. As a result we obtain a subword of type $b^m a^n$. Replace it by the word $b^n a^m$ and after this recover all removed pairs and all letters which differ from $a$ and $b$. As a result, we obtain a new word $\tilde{w}$. This word corresponds to some Young tableau $\tilde{T}$. We denote this tableau by $\sigma_i(T) := \tilde{T}$.

**Theorem 2.3.** We have

$$s_i(T) = \sigma_i(T), \; 1 \leq i \leq n - 1.$$  

**Remark 2.7.** The fact that the involutions $\sigma_i, \; 1 \leq i \leq n - 1$, generate the symmetric group was proved by A. Lascoux and M.-P. Schützenberger [LS2] using a subtle analysis of properties of the plactic monoid and the Robinson–Schensted correspondence. The same action was rediscovered by M. Kashiwara [Ka2] in the context of the theory of crystal base, as an action of the Weyl group $W$ (in our case $S_n \simeq W$) on the crystal base, corresponding to an irreducible representation (irrep) $V_\lambda$ with highest weight $\lambda$ of the Lie algebra $\mathfrak{gl}_n$. Our definition of the involution $s_i$ also appears as an attempt to understand the action of the Weyl group $W$, but on the Gelfand–Tsetlin basis (see [GZ3]) of the irrep $V_\lambda$. In fact, it is possible to show that, if $s_i \in W \simeq S_n, \; 1 \leq i \leq n - 1$, then for $x \in K^2$ we have

$$s_i \cdot |x| = |s_i(x)| + \sum_{y < x} \alpha_{xy}|y| >$$

with respect to some ordering on the set $K^2$. Here for $x \in K^2$ the symbol $|x|$ means a base vector in the space $V_\lambda$ which corresponds to a GT-pattern $x$.

**Remark 2.8.** Another description of the action of the symmetric group on the set of standard Young tableaux can be given, based on the concept of rigged configurations, [Ki1, Ki2]. To begin with, we remind the reader some definitions from [Ki1]. Given a partition $\lambda$ and a composition $\beta$, $l(\beta) \leq N$, a configuration $\nu$ of type $(\lambda, \beta)$ is, by definition, a collection of partitions $\nu^{(1)}, \nu^{(2)}, \ldots$ such that

(i) $|\nu^{(k)}| = \sum_{j \geq k + 1} \lambda_j$;

(ii) $P_n^{(k)}(\nu, \beta) := Q_n(\nu^{(k - 1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k + 1)}) \geq 0$ for all $k, n \geq 1$,

where $\nu^{(0)} := \beta$, and for any composition $\gamma$ we put $Q_n(\gamma) := \sum_k \min(n, k) \gamma_k$. 

In accordance with the ultimate traditions of the Bethe Ansatz [KR], the numbers $P_n^{(k)}(\nu, \beta)$ are called the vacancy numbers. We define the cocharge $c(\nu)$ of the configuration $\{\nu\}$ in the following way:

$$c(\nu) = \sum_{n \geq 1} \left( \frac{(\nu(1))'_n}{2} + 1 \right) + \sum_{k,n \geq 1} \left( \frac{(\nu(k+1))'_n - (\nu(k))'_n}{2} \right).$$  \hfill (2.16)

Now we want to define the set of rigged configurations $QM(\lambda, \beta)$ of type $(\lambda, \beta)$. By definition, a rigged configuration of type $(\lambda, \beta)$ is a configuration of type $(\lambda, \beta)$ together with a set of integers $J_{\alpha,n}^{(k)}, k, n \geq 1, 1 \leq \alpha \leq s, s := m_n(\nu(k))$, that satisfy the following inequalities:

$$0 \leq J_{1,n}^{(k)} \leq J_{2,n}^{(k)} \leq \cdots \leq J_{s,n}^{(k)} \leq P_n^{(k)}(\nu, \beta),$$

where $m_n(\nu(k)) = (\nu(k))'_n - (\nu(k))'_{n+1}$.

The main property of rigged configurations is that they give another way to describe the set of Young tableaux, namely, there exists a bijection [KR, Ki1]

$$\Phi_\beta : STY(\lambda, \beta) \rightarrow QM(\lambda, \beta).$$

This bijection has many interesting properties [KR, Ki1, Ki2], but now we only remark that the set $QM(\lambda, \beta)$ depends only on the partition corresponding to the composition $\beta$. Thus, we can define an action of the symmetric group $S_N$ on the set of standard Young tableaux $STY(\lambda, \leq N)$ using the following commutative diagram ($\sigma \in S_N$):

$$\begin{array}{ccc}
STY(\lambda, \beta) & \xrightarrow{\Phi_\beta} & QM(\lambda, \beta) \\
\Phi_\sigma & | & \\
STY(\lambda, \sigma \beta) & \xleftarrow{\Phi_\sigma^{-1}} & QM(\lambda, \sigma \beta)
\end{array} \hfill (2.17)
$$

**Proposition 2.12.** Let $\lambda$ be a partition, $l(\lambda) \leq N$, and $T \in STY(\lambda, \leq N)$. Then

$$s_i(T) = \Phi_{(i,i+1)}(T),$$

where the map $\Phi_{(i,i+1)}$ corresponds to the vertical arrow in (2.17) if $\sigma = (i,i+1)$ is a simple transposition.

It follows from Proposition 2.12 that all Young tableaux lying in the same orbit with respect to the action of the symmetric group generated by the involutions $s_i$ (see (1.16)), $1 \leq i \leq N-1$, have the same rigged configuration. In particular, the cocharge of a Young tableau (see (2.16)) is an invariant of the action of the symmetric group.

**F** Combinatorial interpretation of the transformations $f_i$ and $e_i$, and the crystal graph (see [Ka1, Ka2]).

First, we recall the construction of the action of the symmetric group $S_n$ on the set $K^n_\lambda$ (see subsection E or [LS2]). Given a tableau $T \in STY(\lambda, \beta)$, consider the word
For $T \in STY(\lambda, \leq n)$ we have:

$$f_i T = \tilde{T}, \quad e_i T = \overline{T}, \quad 1 \leq i \leq n - 1.$$ 

Now we consider a colored graph $\Gamma_{Z}(\lambda)$. The vertices of this graph correspond to all standard Young tableaux $T$ of shape $\lambda$ with all entries $\leq n$. Two vertices $T_1$ and $T_2$ are connected by an edge of color $i$, $T_1 \xrightarrow{i} T_2$, if and only if $T_2 = f_i T_1$ (or, equivalently, $T_1 = e_i T_2$).

Theorem 2.5. The graph $\Gamma_{Z}(\lambda)$ coincides with the crystal graph corresponding to the irreducible representation $V_{\lambda}$ of the Lie algebra $gl_n$ with highest weight $\lambda$.

Remark 2.9. For the first time, a description of the edges of the crystal graph $B(V_{\lambda})$ was obtained by M. Kashiwara and T. Nakashima [KN, N] in purely combinatorial terms. Essentially, we have presented the same description. The difference is that M. Kashiwara and T. Nakashima in [KN] gave a description of the action of the generators $f_1$ and $\overline{e}_1$ on the set $STY(\lambda, \leq n)$. Our result gives an opportunity to extend this action to the Gelfand–Tsetlin polytope $K^\lambda$ and to obtain some subdivision of the convex polytope $K^\lambda$ in colored parts.

Remark 2.10. Let $U = U_q(sl(n))$ be the $q$-analog of the universal enveloping algebra of the Lie algebra $sl(n)$. The algebra $U$ is the algebra over $\mathbb{Q}(q)$ generated by $E_i$, $F_i$ and $K_i^{\pm 1}$, $1 \leq i \leq n - 1$ (see, e.g., [J, Ka1]). We denote by $U^*$ the multiplicative monoid in $U$ generated by $F_1, \ldots, F_{n-1}$. Let $G^*$ be the multiplicative monoid in $Aut(X_n)$ generated by all transformations $s_1^{(1)}, \ldots, s_{n-1}^{(1)}$ (see (1.16)).

Conjecture 2.2. The correspondence $F_i \rightarrow s_i^{(1)}$, $1 \leq i \leq n - 1$, generates an isomorphism $U^*_q \simeq G^*$ of multiplicative monoids.

It should be noted that for any $k, l \in \mathbb{Z}_+$ we have the following relation (see [Lu2]):

$$F_i^k F^{k+1}_{i+1} F_i^l = F^l_{i+1} F^{l+k}_{i+1} F^k_i, \quad 1 \leq i \leq n - 2.$$ 

§3. The cocharge and Gelfand-Tsetlin patterns

First we recall the definition of the charge and the cocharge of a tableau, according to A. Lascoux and M.-P. Schützenberger [LS1, Ma]. Let $\lambda$ and $\mu$ be partitions, $T \in STY(\lambda, \mu)$. Consider the word $w(T)$ (which corresponds to the tableau $T$ (see [Ma])). We define the charge $c(T)$ of $T$ as the charge of corresponding word $w(T)$. Now we define the charge of a word $w$. Recall that the weight $\mu$ of a word $w$ is a sequence
\[ \mu = (\mu_1, \mu_2, \ldots), \] where \( \mu_i \) is the number of \( i \)'s occurring in the word \( w \). We assume that the weight \( \mu \) of a word \( w \) is dominant, i.e., \( \mu_1 \geq \mu_2 \geq \ldots \).

(i) First, we assume that \( w \) is a standard word, i.e., its weight is \( \mu = (1^N) \). We index all elements of \( w \) as follows: the index of 1 is equal to 0, and if the index of \( k \) is \( i \), the index of \( k+1 \) is either \( i \) or \( i+1 \), in accordance with the location of \( k+1 \) either to the right or to the left of \( k \). The charge \( c(w) \) of \( w \) is the sum of all its indices. (ii) Now, assume that \( w \) is a word of weight \( \mu \) and \( \mu \) is a partition. We extract a standard subword from \( w \) in the following way. Reading \( w \) from left to right, we choose the first occurrence of 1, then the first occurrence of 2 to the right of the 1 chosen and so on. If at some step there is no \( s+1 \) to the right of the \( s \) chosen before, we come back to the beginning of the word. This operation extracts from \( w \) a standard subword \( w_1 \). Let us delete the word \( w_1 \) from \( w \) and repeat the operation, thus obtaining \( w_2 \), etc.

The charge of \( w \) is defined as the sum of the charges of the standard subwords obtained in this way: \( c(w) = \sum c(w_i) \). We note that the charge of \( w \) is zero if and only if \( w \) is a lattice word.

Now let \( \lambda \) and \( \mu \) be partitions, and let \( T \in \text{STY}(\lambda, \mu) \). We define the cocharge of the tableau \( T \) as

\[ \overline{c}(T) = n_{\mu} - n_{\lambda} - c(T). \] (3.1)

We give another method of computing the cocharge of a tableau \( T \in \text{STY}(\lambda, \mu) \). First, assume that \( T \in \text{STY}(\lambda) \). For any \( x \in T \), we denote by \( n(x) \) the index of the row which contains \( x \). For any \( x \in T \) we define \( \overline{c}(x) \) by induction : \( \overline{c}(1) = 0 \),

\[ \overline{c}(x+1) = \overline{c}(x) + \begin{cases} n(x) - n(x+1) & \text{if } n(x) \geq n(x+1) \\ n(x) - n(x+1) + 1 & \text{if } n(x) < n(x+1), \end{cases} \] (3.2)

and put \( \overline{c}(T) = \sum_{x \in T} \overline{c}(x) \).

**Proposition 3.1.** Assume that \( T \in \text{STY}(\lambda) \). Then

(i) \( \overline{c}(x) \geq 0 \) for all \( x \in T \).

(ii) \( \overline{c}(T) \) is equal to the cocharge of the tableau \( T \), as it was defined by (3.1).

(iii) Define the descent set \( D(T) \) of a tableau \( T \) as

\[ D(T) = \{ x \in T \mid n(x) < n(x+1) \}, \] (3.3)

and put

\[ \text{des}(T) = \sum_{x \in D(T)} x, \quad p = \sum_{x \in D(T)} 1. \]

Then

\[ \overline{c}(T) = pN - \text{des}(T) - n_{\lambda}, \]

\[ c(T) = \binom{N}{2} - pN + \text{des}(T), \]

---

1By definition, \( \text{STY}(\lambda) = \text{STY}(\lambda, 1^N) \) (ed.)
where $N = |\lambda|$.

It is convenient to consider the tableau $\overline{C}(T)$ obtained from $T$ by replacement each occurrence of $x \in T$ by $c(x)$. It is possible to show that, in fact, $\overline{C}(T)$ is a reverse plane partition.

We consider a clarifying example.

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In our case $D(T) = \{2, 4, 5, 8, 10, 12\}$, $p(T) = 6$, $\operatorname{des}(T) = 41$, $n(\lambda) = 20$ and

$$c(T) := \sum_{c \in \overline{C}(T)} c = 23 = 6 \cdot 14 - 41 - 20.$$

Second, if $T \in \text{STY}(\lambda, \mu)$, $\mu$ is a partition, we act as in step (ii) of Lascoux–Schrützenberger’s algorithm: consider the most left and upper 1 in the tableau $T$ and compute $c(1) = n(1) - 1$, after that consider the most left 2 such that $n(2) > n(1)$ and compute $c(2)$ in accordance with (3.2) and so on. If at some step there is no $x + 1$ satisfying $n(x + 1) > n(x)$, then consider the most left and upper $x + 1 \in T$, and compute $c(x + 1)$ (using (3.2)). This operation extracts a subset $w_1$ from the tableau $T$. We delete this subset $w_1$ from $T$ and repeat the previous operation, thus obtaining $w_2$, etc.. The cocharge of the tableau $T$ is defined as the sum

$$\overline{c}(T) = \sum_{i} \sum_{x \in w_i} \overline{c}(x).$$

We note that, in general, some of the numbers $\overline{c}(x)$, $x \in T$, may be negative.

Thus, we have defined of the charge and the cocharge of a tableau $T \in \text{STY}(\lambda, \mu)$, where $\mu$ is a partition. We know (see Section 1, or [GZ1, BZ1]) that $\text{STY}(\lambda, \mu)$ is the set of integral points in the convex polytope $K^\lambda(\mu) \subset K^\lambda$, so it is natural to ask if it is possible to extend the cocharge (or the charge) from the set $\text{STY}(\lambda, \mu)$, $\mu$ being a partition, to the entire convex polytope $K^\lambda$ in a natural manner? It is easy to see that the charge does not possess good properties with respect to the action of the symmetric group $S_n$ on the Gelfand–Tselin polytope $K^\lambda$.

The main result of this section, Theorem 3.1, asserts that such a continuation really exists and has nice properties (at least for $n \leqslant 5$). Thus, we want to construct a function $\overline{c}_n : X_n \to \mathbb{R}$ coinciding with the cocharge on the integral points of the cone $K_n$ (at least for $n \leqslant 5$ and, probably, for all $n$). To begin with, we define an embedding of the symmetric group $S_n$ into the set $(K_n)_{\mathbb{Z}}$:

$$p_n : S_n \hookrightarrow (K_n)_{\mathbb{Z}}.$$
Definition 3.1. Denote by $\mathcal{P}_n$ the set of all Gelfand-Tsetlin patterns $p := (p_{ij}) \in (K_n)^2$ that satisfy the following conditions:

i) $p_{n-1,n-1} = p_{n,n} = 0$;

ii) $p_{i,j} - p_{i+1,j+1} \leq 1$, $1 \leq i \leq j \leq n - 1$; \hfill (3.4)

iii) $p_{i,n} - 1 \leq p_{i,i} \leq p_{i+1,n}$, $1 \leq i \leq n - 1$.

Proposition 3.2. There exists a bijection\footnote{This is not correct as stated; e.g., $P_3 = 7$. An additional condition should be imposed, for instance, \[(iv) \ p_{i,j} - p_{i+1,j} \leq 1, \quad 1 \leq i < j \leq n.\]}

$$\pi : \mathcal{P}_n \simeq [0, n - 1] \times [0, n - 2] \times \cdots \times [0, 1] \times [0, 0].$$

In particular, $|\mathcal{P}_n| = n!$.

Proof. Let $p = (p^{(n)}, \ldots, p^{(1)}) \in \mathcal{P}_n$. We shall construct a vector $k = (k_1, \ldots, k_n)$ by the following rule. Observe that, for a given $i$, $1 \leq i \leq n$, it follows from the inequality $0 \leq p_{i,n} - p_{i,i} \leq 1$ that there exists an integer $j$, $0 \leq j \leq n - i$, such that

$$p_{i,i} = p_{i,i+1} = \cdots = p_{i,i+j} < p_{i,i+j+1} = \cdots = p_{i,n}.$$ 

In this case we put $k_i = j$. The correspondence $p \to k$ is the desired bijection. \hfill •

As an illustrating example, we take $n = 4$ and $k = (3,1,0,0)$. First, we observe that $p_{3,4} = 1$. Secondly, from inequalities (3.4) we find that $p_{3,4} \leq p_{2,3} = p_{2,2} \leq p_{3,4}$. Consequently, $p_{2,3} = p_{2,2} = p_{3,4} = 1$ and $p_{2,4} = 2$. Further, using the same trick, we obtain that $p_{2,4} \leq p_{1,4} = p_{1,3} = p_{1,2} = p_{1,1} \leq p_{2,4}$. Thus,

$$p = \begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 \end{pmatrix}$$

Proceeding further, we consider the following linear functionals on the space of triangles $X_{n+1}$:

(i) The functional $\varphi_{0,n} : X_{n+1} \to \mathbb{R}$ given by a formula

$$\varphi_{0,n}(x) = \sum_{j=1}^{n-1} (n - j)(x_{j,n+1} - x_{j,n}).$$ \hfill (3.5)

(ii) For every $p \in \mathcal{P}_n$, the functional $\varphi_p : X_{n+1} \to \mathbb{R}$ given by a formula

$$\varphi_p(x) = \sum_{1 \leq i < j \leq n} p_{i,j}^T \cdot \varphi_{n-j+1,n-i+1}(x),$$ \hfill (3.6)
where if \( p = (p_{ij}) \in P_n \), then \( p^T := (p_{ij})^{T} \), \( p^T_{ij} = p_{n-j+1, n-i+1} \), and the linear maps \( \varphi_{ij}: X_{n+1} \to \mathbb{R} \) are defined by (1.32).

Finally, we define a function \( \widetilde{c}_{n+1}: X_{n+1} \to \mathbb{R} \) by the following inductive formula

\[
\begin{align*}
(i) \quad \widetilde{c}_{1}(x) &= 0 \quad \text{if} \ x \in X_1, \\
(ii) \quad \widetilde{c}_{n+1}(x) &= \widetilde{c}_{n}(x) + \varphi_0(x) + \min\{\varphi_p(x) \mid p \in P_n\},
\end{align*}
\]

where for any triangle \( x \in X_{n+1}, \ x = (x^{(n+1)}, \tilde{x}), \ \tilde{x} \in X_n \), we set, by definition, \( \widetilde{c}_{n}(x) = \widetilde{c}_{n}(\tilde{x}) \).

Now we are ready to formulate the main result of this section.

**Theorem 3.1.** The function \( \widetilde{c}_n: X_n \to \mathbb{R} \) is continuous, piecewise linear, and satisfies the following properties:

1) **Invariance:**

   (i) \( \widetilde{c}_n(\sigma x) = \widetilde{c}_n(x) \) for any \( \sigma \in S_n \);
   
   (ii) \( \widetilde{c}_n(I x) = \widetilde{c}_n(x) \), where the map \( I \) is defined in (1.38);
   
   (iii) \( \widetilde{c}_n(T_{+1} x) = \widetilde{c}_n(x) \) if \( T_{+1} x \neq 0 \) (resp., \( T_{-1} x \neq 0 \)).

2) **Stability:** if \( \varphi_\alpha: X_{n-1} \to X_n, \ \alpha = 1,2, \) are the embeddings given by (1.40), then

\[
\widetilde{c}_n(x) = \widetilde{c}_{n-1}(\tilde{x})
\]

3) **Positivity:**

\[
\widetilde{c}_n(x) \geq 0 \quad \text{for all} \ x \in K_n.
\]

We conjecture that the function \( \widetilde{c}_n(x) \) given by (3.7) possesses an additional property:

4) **Main Conjecture:** if \( \lambda \) and \( \mu \) are partitions, \( l(\lambda) \leq n, \ l(\mu) \leq n, \ T \in STY(\lambda, \mu) \) and \( x(T) \) is the image of \( T \) (see §2) in the set \( K^*_X \), then

\[
\widetilde{c}_n(x(T)) = \widetilde{c}_n(T).
\]

**Corollary 3.1** (to the Main Conjecture). Let \( \lambda \) and \( \mu \) be the dominant weights of the Lie algebra \( sl_n \), and let \( K_{\lambda, \mu}(q) \) be the Kazhdan–Lusztig polynomial for the affine Hecke algebra (see [Lu1, Ma]) corresponding to \( \lambda \) and \( \mu \). Then

\[
K_{\lambda, \mu}(q) = \sum_{x \in K(\lambda) \cap (X_n)_{x}} q^{\widetilde{c}_n(x)}.
\]

**Corollary 3.2** (to Theorem 3.1). Let \( \lambda \) be a dominant weight and \( \beta \) any (integral) weight of the Lie algebra \( sl_n \). Define a polynomial

\[
K_{\lambda, \beta}(q) := \sum_{x \in K(\beta) \cap (X_n)_{x}} q^{\widetilde{c}_n(x)}.
\]

Then \( K_{\lambda, \beta}(q) = K_{\lambda, \omega(\beta)}(q) \) for any \( \omega \in S_n \).
Problem 3.1. Given \( \lambda, \beta \in \mathbb{R}^n \), compute the following integral

\[
K_{\lambda, \beta}^{\text{cont}}(q) := \int_{K^\lambda(\beta)} \exp(h \cdot \bar{c}(x))d\mu(x), \quad q = \exp(h),
\]

(3.10)

where \( \mu(x) \) is the Lebesque measure on \( K^\lambda(\beta) \) induced from \( \mathbb{R}^{n(\lambda^T + 1)} \).

It is clear that \( K_{\lambda, \beta}^{\text{cont}}(1) = \text{Vol}(K^\lambda(\beta)) \). We consider the functions \( K_{\lambda, \beta}^{\text{cont}}(q) \) as a continuous analog of the Kostka–Foulkes polynomials.

We postpone the proof of Theorem 3.1 till the end of § 3 (see Remark 3.1), while right now we consider in more detail the particular cases \( n = 3 \) and \( n = 4 \). Our goal is to prove the Main Conjecture for these cases.

We start with the case \( n = 3 \). Given a triangle \( x \in X_3 \), we recall that

\[
\varphi_{22}(x) := x_{12} + x_{22} - x_{13} - x_{23}
\]

\[
\varphi_{12}(x) := x_{12} + x_{22} - x_{13} - x_{23}.
\]

It is easy to see that

\[
x_{12}(s_2(x)) = x_{12} - (\varphi_{12})_+ - (\varphi_{22})_+,
\]

\[
x_{22}(s_2(x)) = x_{22} - (\varphi_{12})_+ - (\varphi_{22})_-,
\]

\[
\varphi_{12}(s_2(x)) = -\varphi_{22}(x), \quad \varphi_{12}(s_1(x)) = -\varphi_{12}(x),
\]

\[
\varphi_{22}(s_1(x)) = x_{11} - x_{23}, \quad \varphi_{22}(s_1(x)) = \varphi_{22}(x).
\]

Lemma 3.1. In the case \( n = 3 \) we have

\[
\bar{c}_3(x) = \min(x_{13} - x_{12}, x_{22} - x_{33}),
\]

(3.11)

\[
\bar{c}_3(q_2(x)) = \min(x_{23} - x_{22}, x_{12} - x_{11}) + (x_{11} - x_{23})_-,
\]

(3.12)

where \( q_2 = t_1 t_2 t_1 \in G_3 \), \( x \in X_3 \) and \((a)_+ := \max(x, 0), \ (a)_- := \min(x, 0)\).

Proof. By Definition 3.1, we have

\[
P_2 = \left\{ \begin{array}{c} 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \\ 0 \ 0 \ 1 \end{array} \right\}, \quad \pi(P_2) = \{(0,0), (1,0)\},
\]

\[
\varphi_{12}(x) = p_{12}^T \cdot \varphi_{12}(x) = \varphi_{12}(x),
\]

\[
\varphi_{02}(x) = 0, \quad \varphi_{02}(x) = x_{13} - x_{12}.
\]

Consequently,

\[
\bar{c}_3(x) = x_{13} - x_{12} + (\varphi_{12})_- = \min(x_{13} - x_{12}, x_{22} - x_{33}).
\]
Formula (3.12) follows from the description of the Schützenberger involution $q_2$:

$$
\begin{align*}
    x_{12}(q_2(x)) &= x_{13} - \min(x_{11} - x_{22}, x_{12} - x_{23}), \\
    x_{22}(q_2(x)) &= x_{33} + \min(x_{12} - x_{11}, x_{23} - x_{22}), \\
    x_{11}(q_2(x)) &= \beta_3(x); \\
\end{align*}
$$

\[ \bullet \]

The function (3.11) satisfies properties 1)-4) of Theorem 3.1 (see Proposition 3.3). The function (3.12) satisfies properties 1)-3), except 1(iii) and, in particular, is invariant with respect to the action of the symmetric group $S_3$. Further, for the dual Schützenberger involution $u_2$ we have

$$
\begin{align*}
    x_{12}(u_2(x)) &= x_{12} - (\varphi_{12})_+ - (x_{11} - x_{23})_+, \\
    x_{22}(u_2(x)) &= x_{22} - (\varphi_{12})_+ - (x_{11} - x_{23})_-, \\
    x_{11}(u_2(x)) &= \beta_3(x).
\end{align*}
$$

It is easy to see that $\overline{c}_3(u_2(x)) = \overline{c}_3(x)$, and

$$
q_2 \cdot u_2 = u_2 \cdot q_2. \tag{3.13}
$$

Equality (3.13) is a consequence of the identity $(t_1 t_2)^6 = 1$, since $q_2 = t_1 t_2 t_1$ and $u_2 = t_2 t_1 t_2$. In general, $\overline{c}_n(u_{n-1}(x)) \neq \overline{c}_n(x)$, and $SU \neq US$.

Now we consider the case $n = 4$. Given a triangle $x \in X_4$, we recall that

$$
\begin{align*}
    \varphi_{13}(x) &: = x_{13} + x_{33} - x_{14} - x_{44}, \\
    \varphi_{23}(x) &: = x_{13} + x_{23} - x_{12} - x_{24}, \\
    \varphi_{33}(x) &: = x_{23} + x_{33} - x_{22} - x_{24}.
\end{align*}
$$

It can be checked (see Theorem 1.3) that

$$
\begin{align*}
    x_{13}(s_3(x)) &= x_{13} - (\varphi_{23} + (\varphi_{33})_-)_+ - (\varphi_{13} + (\varphi_{33})_+)_-, \\
    x_{23}(s_3(x)) &= x_{23} - (\varphi_{33} + (\varphi_{13})_-)_+ - (\varphi_{23} + (\varphi_{13})_+)_-, \\
    x_{33}(s_3(x)) &= x_{33} - (\varphi_{13} + (\varphi_{23})_-)_+ - (\varphi_{33} + (\varphi_{23})_+)_- \tag{3.14}
\end{align*}
$$

Proposition 3.3. Assume that $x \in X_4$, then the function

$$
\overline{c}_4(x) = \min(x_{13} - x_{12}, x_{22} - x_{33} + x_{14} - x_{13} + x_{33} - x_{44} \\
+ \min(x_{23} - x_{34}, x_{24} - x_{23}, x_{13} - x_{12}, x_{22} - x_{33}, \beta_4(x) - x_{34}, x_{24} - \beta_4(x)) \tag{3.15}
$$

satisfies properties 1)-4) of Theorem 3.1.
Proof. First of all, we check that our previous definition (3.7) of the cocharge $c_n$ coincides with (3.15). In accordance with Definition 3.1, we have

$$P_3 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\pi(P_3) = \{(000), (010), (110), (100), (200), (210)\}.$$

Using (3.6), we find

$$\varphi_{110}(x) = \varphi_{12} + \varphi_{13} + \varphi_{23}, \quad \varphi_{100}(x) = \varphi_{13} + \varphi_{23},$$

$$\varphi_{210}(x) = \varphi_{12} + 2\varphi_{13} + \varphi_{23} + \varphi_{33}, \quad \varphi_{100}(x) = \varphi_{13},$$

$$\varphi_{110}(x) = \varphi_{12} + \varphi_{13} + \varphi_{23} + \varphi_{33}, \quad \varphi_{000}(x) = 0.$$

Furthermore, it follows from (3.5) that

$$\varphi_{03}(x) = 2(x_{14} - x_{13}) + (x_{24} - x_{23}) = x_{14} - x_{13} + x_{33} - x_{44} + x_{24} - x_{23} - \varphi_{13}.$$

After some not very complicated calculations we can obtain (3.15) from (3.7).

Secondly, we check that the function $c_\pm(x)$ is invariant with respect to the action of the symmetric group $S_4$. Note that invariance under the action of the generators $s_1$ and $s_2$ is clear from (3.5). In order to prove the $s_3$-invariance of $c_4(x)$, we use some lemmas.

**Lemma 3.2.** Assume that $x \in X_4$, then

$$(\varphi_{\alpha3}(s_3(x)))_- = x_{\alpha3} - (s_3(x))_- + (\varphi_{\alpha3}(x))_-, \quad \alpha = 1, 2, 3,$$  \hspace{1cm} (3.16)

where for any $a \in \mathbb{R}$, $(a)_- := \min(a, 0)$, $(a)_+ := \max(a, 0)$.

**Proof.** Using formulas (3.14) for the action of $s_3$ on the space of triangles $X_4$, we find

$$\bar{\varphi}_{13} = -\varphi_{23} - \varphi_{33} + (\varphi_{23} + (\varphi_{13})_-) + (\varphi_{33} + (\varphi_{13})_+),$$

where for any function $\varphi$ on the space of triangles we exploit the notation $\bar{\varphi} := \varphi(s_3(x))$. In order to calculate $(\bar{\varphi}_{13})_-$, we make use of the following identity, which may be verified by direct computation:

$$[-a - b + (a + (c)_-)_+ + (b + (c)_+)_-]_+ = (c)_- - (b + (a)_-)_+ - (c + (a)_+)_-.$$

Consequently,

$$(\bar{\varphi}_{13})_- = (\varphi_{13})_- - (\varphi_{23} + (\varphi_{33})_-)_+ - (\varphi_{13} + (\varphi_{33})_+)_- = \bar{x}_{13} - x_{13} + (\varphi_{13})_-.$$

Similarly, we may prove the corresponding formulas for $\alpha = 2$ or 3. •

*The triangle $x = \begin{bmatrix} 2 & 1 & 0 & 2 \end{bmatrix}$ satisfies all the conditions (3.4), too. Cf. footnote on page 135 (ed.)
Corollary 3.3 (to Lemma 3.2). The following functions are $s_3$-invariant

$$
\begin{align*}
\min(x_{14} - x_{13}, x_{33} - x_{44}), \\
\min(x_{13} - x_{12}, x_{24} - x_{23}), \\
\min(x_{23} - x_{22}, x_{34} - x_{33}).
\end{align*}
$$

(3.17)

Lemma 3.3. The expression for $\bar{c}_4(x)$ (given below) is equivalent to (3.15):

$$
\bar{c}_4(x) = x_{14} - x_{13} - x_{44} + \bar{c}_3(x) + \bar{c}_3(s_3 x) + \bar{x}_{33} + (\varphi_{33} + (\varphi_{13})_-).
$$

The proof is based on direct computation.

Thus, to reduce the prove of the $s_3$-invariance of the cocharge $\bar{c}_4(x)$, it suffices to check the $s_3$-invariance of the function

$$(\varphi_{33} + (\varphi_{13})_-) - x_{13} - x_{33}.$$  

The latter can easily be deduced from Lemma 3.2.

Finally, we must prove that if $T \in STY(\lambda, \mu)$, then $\bar{c}_4(x(T))$ coinsides with the cocharge of the tableau $T$.

For this purpose we use a description of Young tableaux in terms of rigged configurations [Ki1, Ki2]. More precisely, let $T$ be a standard Young tableau

$$
T := \begin{array}{cccc}
  d_{11} & d_{12} & d_{13} & d_{14} \\
  d_{22} & d_{23} & d_{24} \\
  d_{33} & d_{34} \\
  d_{44}
\end{array}
$$

where $d_{ij}$ is the number of $j$'s lying in the $i$-th row of $T$. We shall construct the corresponding rigged configuration step by step:

$$
\begin{array}{cccccc}
  0 & d_{22} & 0 & 0 & d_{33} & 0 & 0 & d_{33} & 0 & P & 0 & d_{33} & 0 \\
  0 & & & & 0 & 0 & d_{33} & 0 & 0 & d_{33} & 0 \\
  \bar{c}_3(\nu) = 0 & \bar{c}_3(\nu) = 0 & \bar{c}_3(\nu) = 0 \\
\end{array}
$$

$$
\begin{array}{cccc}
  0 & d_{22} + d_{23} & P + J & 0 & d_{33} & 0 \\
  0 & d_{33} & 0 \\
  \bar{c}_3(\nu) = 0 \\
\end{array}
$$

where $P = \min(d_{11} - d_{22}, d_{23}) + (d_{12} - d_{23})_-$, $J = \min(d_{13}, d_{22} - d_{33})$. 

Consequently, $\bar{c}_3(T) = \bar{c}(\nu) + J = \min(x_{13} - x_{12}, x_{22} - x_{33}) = \bar{c}_3(x(T))$. Proceeding further, we start to join successively the parts $\begin{bmatrix} d_{44} \\ d_{33} \\ d_{34} \\ d_{44} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ d_{35} \\ d_{34} \\ 0 \end{bmatrix}$. 

\[
\begin{array}{c|cccc}
 & d_{22} + d_{23} & P + J & \begin{bmatrix} 0 \\ d_{35} \\ d_{34} \end{bmatrix} & \begin{bmatrix} 0 \\ d_{44} \\ 0 \end{bmatrix} \\
\hline
0 & d_{33} & d_{34} & 0 & 0 \\
0 & d_{44} & 0 & 0 & 0
\end{array}
\]

where $Q = \min(d_{11} - d_{33}, d_{14}) + \min(d_{12} + d_{22}, d_{33} + d_{34}) - 3d_{34}$.

Finally, we add the parts $\begin{bmatrix} d_{24} \\ d_{14} \end{bmatrix}$ and $\begin{bmatrix} d_{24} \\ d_{34} \end{bmatrix}$. As a result, we obtain the following rigged configuration:

\[
\begin{array}{c|cccc}
 & d_{24} + d_{23} & d_{24} - M & P & \begin{bmatrix} d_{33} + d_{34} \\ d_{44} \\ 0 \end{bmatrix} \\
\hline
\tilde{J}_1 & 0 & d_{24} - M & 0 & 0 \\
\tilde{J}_2 & \begin{bmatrix} d_{24} - M \\ d_{34} \end{bmatrix} & \begin{bmatrix} 0 \\ d_{44} \\ 0 \end{bmatrix} & 0 & 0 \\
0 & d_{44} & 0 & 0 & 0
\end{array}
\]

where

\[
\begin{align*}
M &= \min(d_{22} + d_{23} - d_{33} - d_{34}, d_{13}, d_{22} - d_{33}), \\
\tilde{J}_1 &= \min(d_{14}, d_{22} + d_{23} - d_{34} - d_{44} - M) \\
&\quad + [\min(d_{13}, d_{22} - d_{33}) - d_{24}] \cdot e, \\
\tilde{J}_2 &= \min(d_{33} - d_{44}, d_{14} + d_{24} - M) \\
&\quad + \min(d_{13}, d_{22} - d_{33}) - M - [\min(d_{13}, d_{22} - d_{33}) - d_{24}] \cdot e, \\
\end{align*}
\]

and $e := e(T)$ is equal to 1 if $d_{24} \leq \min(d_{22} + d_{23} - d_{33} - d_{34}, d_{22} - d_{33}, d_{13})$, and to 0 otherwise.

The exact values of the vacancy numbers $\tilde{P}$ and $\tilde{Q}$ are not essential for the computation of the cocharge. Consequently,

\[
\bar{c}_4(T) = \bar{c}(\nu) + \tilde{J}_1 + \tilde{J}_2 = \bar{c}_3(x) + d_{14} + d_{33} - d_{34} + \min(M, d_{14} + d_{24} - d_{33} + d_{44}) \\
&\quad + \min(M, d_{22} + d_{23} - d_{14} - d_{34} - d_{44}) - M.
\]

It is easy to check that this expression for $\bar{c}_4(T)$ coincides with $\bar{c}_4(x(T'))$. The proof of Proposition 3.3 is finished.

We also write a formula for the action of the Schützenberger involution $g_3 = t_1t_2t_3t_4t_4t_3t_2t_1$ on the space of triangles $X_4$: if $x \in X_4$ and $\tilde{x} = g_3(x)$, then

\[
\tilde{x}_{13} = x_{14} + x_{24} - x_{13} + (\varphi_{23} + (\varphi_{22})+)/+ ,
\tilde{x}_{23} = x_{24} + x_{34} - x_{23} + (\varphi_{23} + (\varphi_{22})+)/+ + (\varphi_{33} + (\varphi_{31})+)/+ ,
\tilde{x}_{33} = x_{34} + x_{44} - x_{33} + (\varphi_{33} + (\varphi_{22})+)/- ,
\tilde{x}_{12} = x_{14} + x_{24} + x_{34} - x_{13} - x_{23} + (\varphi_{33} + (\varphi_{23})+)/+ ,
\tilde{x}_{22} = x_{24} + x_{34} - x_{23} - x_{33} + (\varphi_{33} + (\varphi_{31})+)/-, \\
\tilde{x}_{11} = \beta_4(x), \quad \tilde{x}_{\alpha} = x_{\alpha} \quad \text{if } \alpha = 1, 2, 3, 4.
\]
From the description of the action of the symmetric group $S_n$ on the set $STY(\lambda, \leq n)$ one can deduce that all Young tableaux belonging to one and the same $S_n$-orbit have the same rigged configuration. Thus, we see that all parameters of a rigged configuration (for example, its quantum numbers or vacancy numbers) are $S_n$-invariant. The computations contained in Proposition 3.3 seem to be interesting, because they shed some light on certain deep invariants of Young tableaux, namely, its quantum numbers. At this moment we do not know whether or not the quantum numbers (3.18) regarded as functions on the space $X_n$ are $S_n$-invariants. Anyway, it seems very interesting to find a “complete list” of continuous, piecewise linear invariants (cpl-invariants) for a fixed involution $s_j$. In this direction, we are going to prove the following result.

**Theorem 3.2.** Let $j$ be an integer, $2 \leq j \leq n - 1$. Then the following functions on the space $X_n$ are $s_j$-invariants:

1. $\psi_{ij}(x) = \min(x_{i,j+1} - x_{i,j}, x_{j,j} - x_{j+1,j+1})$
2. $\psi_{ij}(x) = \min(x_{i-1,j} - x_{i-1,j-1}, x_{i,j+1} - x_{ij})$, $2 \leq i \leq j \leq n - 1$,

$$\psi_{ij}(x) = (\min(\varphi_{jj}(x), -\varphi_{ij}(x)))_+, \quad 2 \leq i \leq j \leq n - 1.$$  

**Conjecture 3.2.** The group of all continuous, piecewise linear automorphisms of the space $X_n$ having the same set of invariants (3.19) and (3.20) coincides with the cyclic group of second order $< 1, s_j >$.

It seems plausible that the set of $s_j$-invariants (3.19) and (3.20) is a fundamental system of continuous, piecewise linear $s_j$-invariants, i.e., any cpl-invariant with respect to the action of $s_j$ on the space $X_n$ is a min–max linear combination of those from (3.19) and (3.20).

**Proof** (of Theorem 3.2). We must prove that for $2 \leq j \leq n - 1$ the functions $\psi_{ij}$ and $\bar{\psi}_{ij}$, $1 \leq i \leq j$, are $s_j$-invariants. We shall prove this using the inductive formula (1.29) for $s_j$:

$$s_j = R_{j-1,j}[j-1,j][j,j+1]s_{j-1}[j,j+1][j-1,j]R_{j-1,j}.$$  

(3.21)

The case $j = 1$ is almost evident, namely, it is necessary to check the $s_1$-invariance of a single function $\psi_{11}(x) = \min(x_{12} - x_{11}, x_{11} - x_{22})$, but the latter is clear from the definition of $s_1$. So, we may assume that $\psi_{i,j-1}$ and $\bar{\psi}_{i,j-1}$ ($i = 1, \ldots, j-1$) are invariant with respect to the action of $s_{j-1}$. Now, using (3.21), we are going to reduce the proof of the $s_j$-invariance of the functions (3.19) and (3.20) to the same for simpler functions. First, we define a map $J : X_n \to X_n$ in the following way (compare with (1.38)):

$$(J(x))_{ij} := -x_{j-i+1,j}, \quad 1 \leq i \leq j \leq n.$$
Lemma 3.4. We have

(i) 
\[ J^2 = 1, \ t_j J = J t_j, \ s_j J = J s_j, \quad 1 \leq j \leq n - 1, \]

(ii) 
\[ \psi_{1,j}(J(x)) = \psi_{1,j}(x), \quad \overline{\psi}_{1,j}(J(x)) = \overline{\psi}_{2,j}(x), \]
\[ \psi_{i,j}(J(x)) = \psi_{j-i+2,j}(x), \quad 2 \leq i \leq j, \]
\[ \overline{\psi}_{i,j}(J(x)) = \overline{\psi}_{j-i-3,j}(x), \quad 3 \leq i \leq j. \]

The proof of this lemma is based on direct computation.

Lemma 3.5. If \( \psi : X_n \to \mathbb{R} \) is an \( s_j \)-invariant function (i.e., \( \psi(s_j(x)) = \psi(x) \) for all \( x \in X_n \)), then so is \( \overline{\psi}(x) = \psi(J(x)) \).

Secondly, we introduce the notation
\[ \psi'_{ij}(x) := \psi_{ij}(R_{j-1,j}(x)), \quad \overline{\psi}'_{ij}(x) := \overline{\psi}_{ij}(R_{j-1,j}(x)), \]
\[ \psi''_{ij}(x) := \psi'_{ij}([j,j+1][j-1,j](x)), \quad \overline{\psi}''_{ij}(x) := \overline{\psi}'_{ij}([j-1,j][j+1,j](x)). \]

By (3.21) and (3.22) below, it suffices to prove that the functions \( \psi''_{ij} \) and \( \overline{\psi}''_{ij} \) are \( s_{j-1} \)-invariant.

Lemma 3.6 (Reduction formulas). We have

a) \( \psi'_{ij} = \psi_{ij}(x) \) and \( \psi''_{ij} = \psi_{ij-1}(x) \) if \( 2 \leq i \leq j - 2 \);

b) \( \overline{\psi}'_{ij}(x) = \overline{\psi}_{ij}(x) \) and \( \overline{\psi}''_{ij}(x) = \overline{\psi}_{ij-1}(x) \) if \( 3 \leq i \leq j - 2 \).

The proof is based on direct computation, and the following formulas for the action of the involution \( R_{j-1,j} \) (see Definition 1.2): if \( \tilde{x} := R_{j-1,j}(x) \), then
\[ \tilde{x}_{j-1,j} = x_{j-1,j+1} - \min(x_{j,j+1} - x_{j,j}, x_{j-1,j} - x_{j-1,j+1}) = x_{j-1,j+1} - \psi_{j,j}(x), \]
\[ \tilde{x}_{j,j} = x_{j,j} - x_{j-1,j+1} + x_{j-1,j} + \min(x_{j,j+1} - x_{j,j}, x_{j-1,j} - x_{j-1,j+1}), \]
\[ \tilde{x}_{\alpha,\beta} = x_{\alpha,\beta} \text{ if } (\alpha,\beta) \neq (j,j) \text{ or } (j-1,j). \]

Further, using the induction assumptions and Lemma 3.6, we see that it only remains to prove that the following functions are \( s_j \)-invariant:
\[ \psi_{1,j}, \quad \overline{\psi}_{1,j}, \quad \psi_{2,j}, \quad \psi_{j-1,j}, \quad \psi_{j,j}, \quad \overline{\psi}_{j-1,j}, \quad \overline{\psi}_{j,j}. \]

But by Lemmas 3.4 and 3.5, the latter will be achieved if we prove the \( s_j \)-invariance of the following functions:
\[ \psi_{1,j}, \quad \overline{\psi}_{1,j}, \quad \psi_{2,2}, \quad \psi_{33}, \quad \psi_{34}, \quad \overline{\psi}_{33}, \quad \overline{\psi}_{34}, \quad \overline{\psi}_{45}. \]

In order to finish the inductive step and, thus, to prove our theorem, we are going to give certain expressions for the functions under consideration that are \( s_j \)-invariant either by the inductive assumption or by evident reasons. The following lemmas contain all necessary formulas.
Lemma 3.7 (formula (3.22)). We have

\[
\begin{align*}
\psi_{1j}''(x) & = \min(x_{1,j+1} - x_{1,j}, x_{j,j+1} - x_{j,j+1} + x_{j-1,j} - x_{j-1,j+1}, 2x_{j-1,j} - x_{j-1,j+1} - x_{j-1,j-1} + x_{j,j} - x_{j+1,j+1}), \\
\psi_{1j}''(x) & = \min(\psi_{1,j-1}(x), x_{j-1,j+1} - x_{j,j}).
\end{align*}
\]

b) \(\psi_{jj}''(x) = x_{j-1,j+1} - x_{j-1,j},\)
\(\psi_{jj}''(x) = x_{j,j} - x_{j,j+1}.
\]

c) \(\psi_{34} = \min(x_{24} - x_{23}, x_{34} - x_{33}, x_{45} - x_{44}),\)
\(\psi_{34}'(x) = \min(\psi_{33}(x), x_{34} - x_{35}).\)

d) \(\overline{\psi}_{jj}''(x) = (\min(x_{j-2,j} - x_{j-2,j-1} - x_{j-1,j} - x_{j-1,j-1}, x_{j,j+1} - x_{j,j} - x_{j-1,j} - x_{j-1,j-1})),\)
\(\overline{\psi}_{jj}''(x) = (\psi_{j-1,j-1}(x) + x_{j-1,j+1} - x_{j-1,j}).\)

e) \(\overline{\psi}_{45}(x) = (\min(x_{14} - x_{13} - x_{25} + x_{24}, x_{34} - x_{33} - x_{24} + x_{23}, x_{45} - x_{44} - x_{24} + x_{33}))+,\)
\(\overline{\psi}_{45}'(x) = \min(\overline{\psi}_{44}(x), (x_{45} - x_{46} + x_{33} - x_{34}))+.\)

f) \(\overline{\psi}_{1j}''(x) = (\min(x_{1,j+1} - x_{1,j} + x_{j-1,j+1} - x_{j,j+1}, x_{j-1,j} - x_{j-1,j-1} - x_{j,j+1} + x_{j,j}))+, \quad j \geq 3,\)
\(\overline{\psi}_{1j}''(x) = (\psi_{j-1}(x) + x_{j,j} - x_{j-1,j+1})+, \quad j \geq 3.\)

g) \(\overline{\psi}_{12}(x) = \overline{\psi}_{12}(x), \quad \overline{\psi}_{12}(x) = (\overline{\psi}_{11} + x_{22} - x_{13}).+\)

Lemma 3.8. a) The function \(\overline{\psi}_{45}(x)\) is \(s_4\)-invariant if and only if \(\overline{\psi}_{45}''(R_{34}([34][45])(x))\) is \(s_3\)-invariant.

b) We have the following relation:

\[
\overline{\psi}_{45}''(R_{34}([34][45])(x)) = (\min(\psi_{33}(x) + x_{25} - x_{24}, x_{45} - x_{44} - x_{36} + x_{35})).+
\]

The proofs of Lemma 3.7 and Lemma 3.8 are based on direct computation. We need Lemma 3.8, because the \(s_4\)-invariance of \(\overline{\psi}_{45}\) is not evident from Lemma 3.7, e). •

As noted above, these lemmas complete the inductive step and so the proof of Theorem 3.2 is over. •

Corollary 3.4 (to Theorem 3.2). Let \(Q_n^1(a_1, \ldots, a_n)\) be the piecewise linear function defined by means of the recurrence relation (1.31). Then

\[
(a_1 + Q_n^1(a_1 \cdots a_n) + Q_n^1(a_2 \cdots a_na_1))_+ = (a_1)_+ + Q_n^1(a_2 \cdots a_na_1). \quad (3.24)
\]
Proof. Identity (3.24) is equivalent to the $s_n$-invariance of the function $\psi_{1n}(x)$ (see (3.19)) on the space $X_{n+1}$. For $n = 3$, we obtain the following identity (see Lemma 3.2):

$$[-a - b + (a + (c)_-)_+ + (b + (c)_+)_-] = (c)_-(b + (a)_+)_+ - (c + (a)_+)_-,$$

where $a_1 = c$, $a_2 = a$, $a_3 = b$. Note that, as a corollary to Theorem 3.2, we obtain a "geometric" proof of (3.24). It is an interesting task to find an algebraic proof of (3.24).

Remark 3.1. The proof of Theorem 3.1, i.e., the $s_n$-invariance of the cocharge $\zeta_{n+1}(x)$, is based on similar, but more refined technique. The crucial step is analogs of Lemmas 3.6 and 3.7. The details will appear elsewhere.

Remark 3.2. Note that Theorem 3.1 makes it possible to define the charge of a tableau $T \in STY(\lambda, \beta)$ in the case where the weight $\beta$ is a composition. In fact, we may define the charge $c(T)$ of a tableau $T \in STY(\lambda, \beta)$ as follows:

$$c(T) = n(\beta) - n(\lambda) - \bar{c}(T),$$

where $n(\beta) = \sum_i (i - 1)\beta_i$ and $\bar{c}(T)$ is the cocharge of a tableau $T \in STY(\lambda, \beta) \rightarrow K_2^\beta$. Now we give a direct method for calculating the charge, which is a generalization of the Lascoux–Schützenberger algorithm (see [Maj]). Given a standard tableau $T \in STY(\lambda, \beta)$, consider the corresponding word $w(T)$ and a subdivision

$$w(T) = w_1 \bigcup w_2 \bigcup \cdots \bigcup w_k$$

of $w(T)$ into standard subwords $w_i$ in accordance with the Lascoux–Schützenberger algorithm, but now the standardness of a word means that its weight $\beta$ satisfies the condition $\beta_i \leq 1$ for all $i$; i.e., the word $w = a_1 \cdots a_N$, $a_i \in [1, n]$, is called standard if all elements $a_i$ are different. We denote by $a(w)$ the minimal element in a word $w$ and by $a^- := \max \{ b \in w \mid b < a \}$. Now we define the index $\text{ind}_{w_k}(a)$ of a letter $a \in w_k$ for all $k \geq 1$. First, we want to attribute the indices to the minimal elements $a(w_k)$ of the standard subwords $w_k$. We shall use induction.

(i) We put $\text{ind}_{w_k}(a(w)) = a(w) - 1$;

(ii) Assume that we have already defined the indices of all minimal elements $< a(w_k)$ (for some $k \geq 1$). Let $k'$ be the maximal number such that $a(w_{k'}) < a(w_k)$; put $a := a(w_k)$, $a' := a(w_{k'})$. In this case we define

$$\text{ind}_{w_k}(a) = \begin{cases} \text{ind}_{w_{k'}}(a') + a - a' - 1 & \text{if } a' \text{ lies to the right of } a \text{ in } w; \\ \text{ind}_{w_{k'}}(a') + a - a' & \text{if } a' \text{ lies to the left of } a \text{ in } w. \end{cases}$$

Now, using the known values of the indices for $a(w_k)$, we shall find all other values by induction.

(iii) Assume that $a \in w_k$ and $a > a(w_k)$. Then there exists $a^- \in w_k$. In this case, we define

$$\text{ind}_{w_k}(a) = \begin{cases} \text{ind}_{w_k}(a^-) + (a - a^- - 1) & \text{if } a \text{ lies to the right of } a^- \text{ in } w_k; \\ \text{ind}_{w_k}(a^-) + (a - a^-) & \text{if } a \text{ lies to the left of } a^- \text{ in } w_k. \end{cases}$$
Finally, we define

\[ c(w) = \sum_k \sum_{a \in w_k} \text{ind}_{w_k}(a). \]

**Proposition 3.4.** If \( t \in STY(\lambda, \beta) \), then

\[ c(T) + \overline{c}(T) = n(\beta) - n(\lambda). \]

We give an explanatory example. Take \( \lambda = (11, 9, 6, 2) \), \( \beta = (3, 4, 4, 4, 4, 4, 4, 4) \) and consider the following tableau:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 \\
2 & 3 & 3 & 3 & 4 & 5 & 7 & 8 \\
T := 3 & 4 & 5 & 6 & 7 & 8 & 8 & 8 \\
5 & 5 & 7 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

\( \in STY(\lambda, \beta) \)

Our algorithm gives the following answer for the charge of the tableau \( T \):

\[
\begin{align*}
\omega_1(T) &= 84152637, \quad c(\omega_1) = 6, \\
\omega_2(T) &= 72183456, \quad c(\omega_2) = 9, \\
\omega_3(T) &= 42173856, \quad c(\omega_3) = 14, \\
\omega_4(T) &= 6284375, \quad c(\omega_4) = 16.
\end{align*}
\]

Consequently, \( c(T) = 45 \), and \( \overline{c}(T) := n(\beta) - n(\lambda) - c(T) = 112 - 38 - 45 = 29 \). We may use another method (see [Ki1]) for calculating the cocharge, based on an application of rigged configurations. First of all, we find the rigged configuration corresponding to the tableau \( T \):

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 4 & 1 & 1 & 4 \\
1 & 1 & 4 & 1 & 1 & 1 & 4 & 1 \\
3 & 3 & 4 & 1 & 4 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\( \begin{array}{c}
1 \quad 1 \quad 1 \quad 1 \quad 1 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\end{array} \)

\[
\beta = \begin{pmatrix}
3 & 3 & 4 & 4 & -2 & -1 \\
2 & 2 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
The charge of the configuration \( \beta \) is equal to \( c(\beta) := \sum_{i,j} \binom{\beta_{ij}}{2} = 26 \), and the charge of a unique tableau \( \bar{T} \) with dominant weight that lies in the \( S_8 \)-orbit corresponding to the tableau \( T \) is equal to

\[
c(\bar{T}) := c(\beta) + \{ \text{the sum of the quantum numbers} \} = 26 + 12 = 38.
\]

Consequently,

\[
e(\bar{T}) = n(\beta) - n(\lambda) - c(\bar{T}) = 105 - 38 - 38 = 29,
\]

where the dominant weight is \( \beta = (4^7, 3) \). Of course, our two methods give the same result.

Appendix. Proof of Theorem 1.2.

Here we give only the proof of statement (1.28a). Statement (1.28b) may be proved by a similar method. Let \( n \) be a fixed positive integer. Consider the space of triangles \( X_n \) (see § 1) and the space of partitions \( D_n \). By definition, a partition \( d \in D_n \) is a triangular array of real numbers \( d = (d_{ij}) \), \( 1 \leq i < j \leq n \). As a vector space, \( D_n \cong \mathbb{R}^{n(n-1)/2} \). We define the weight of a partition \( d \) as the vector \( \gamma = (\gamma_1, \ldots, \gamma_n) \) with the components

\[
\gamma_j = -\sum_{i<j} d_{ij} + \sum_{k>j} d_{jk}, \quad 1 \leq j \leq n.
\]

Also, we define a mapping \( \partial = \partial_n : X_n \to D_n \) by putting \( \partial(x) = d \), where \( d_{ij} = x_{ij} - x_{ij-1}, \quad 1 \leq i < j \leq n \).

Lemma A.1. (i) The map \( (\lambda \partial) : x \mapsto (\lambda(x), \partial(x)), \quad x \in X_n \), is a bijection of vector spaces \( X_n \to \mathbb{R}^n \oplus D_n \).

(ii) \( \gamma(\partial(x)) = \lambda(x) - \beta(x) \).

The bijection \( (\lambda \partial) \) allows us to transfer the action of any transformation of the space \( X_n \) to the space \( D_n \), and vice versa. We shall denote the operators \( F : X_n \to X_n \) and \( (\lambda \partial) \cdot F \cdot (\lambda \partial)^{-1} : \mathbb{R}^n \oplus D_n \to \mathbb{R}^n \oplus D_n \) by the same letter \( F \) if this does not lead to confusion.

Lemma A.2. Let \( [i, i+1] \) be the involution from Definition 1.3, then

\[
[i, i+1](\lambda, d) = (\lambda, \tilde{d}),
\]

where

\[
(i) \quad \tilde{d}_{i,i+1} = -d_{i,i+1}, \quad \tilde{d}_{i,j} = d_{i+1,j} \quad \text{if} \quad j > i+1, \quad \tilde{d}_{k,i+1} = d_{k,i} \quad \text{if} \quad k < i,
\]
and for all remaining elements \( \bar{x}_{\alpha \beta} = x_{\alpha \beta} \).

(ii) \( \gamma(\bar{d}) = (i, i + 1)\gamma(d) \).

It follows from Lemma 1.1 that the involutions \([i, i + 1]\) generate a group isomorphic to the symmetric group \( S_n \).

We fix a positive integer \( j, 1 < j < n \), and define an operator \( \partial^{(j)} : X_n \to D_n \) in the following way: \( \partial^{(j)}(x) = d \), where (see Fig.1)

- \( d_{ii} = x_{i,l} - x_{i,l-1} \) if \( l > j \) and \( l > i \),
- \( d_{ii} = x_{i-1,l-1} - x_{i-1,l-2} \) if \( 2 < i < l \leq j \),
- \( d_{ii} = x_{i-1,j-1} - x_{i,j} \) if \( 1 < l \leq j \).

![Fig. 1. \( \partial^{(k)} : X_6 \to D_6 \).](image)

It is clear that \( \partial^{(1)} = \partial \). Further, we define a mapping \( (\lambda \partial^{(j)}): X_n \to \mathbb{R}^n \otimes D_n \) by the rule

\[ x \rightarrow (\lambda(x), \partial^{(j)}(x)). \]

**Lemma A.3.** The map \( (\lambda \partial^{(j)}) \) is a vector space isomorphism.

Now we rewrite the involutions \( R_{ijk} \) and \( \varphi_i \) (see Definition 1.2) of the space \( X_n \) in terms of partitions.

**Lemma A.4.** We have:

(i) \( R_{ijk}(\lambda, d) = (\lambda, \bar{d}) \),
where

\[ \tilde{d}_{i,j} = d_{i,k} + \max(d_{ij} - d_{jk}, 0), \]

\[ \tilde{d}_{j,k} = d_{i,k} + \max(d_{jk} - d_{ij}, 0), \]

\[ \tilde{d}_{i,k} = \min(d_{ij}, d_{jk}), \]

and for all remaining elements \( \tilde{d}_{\alpha\beta} = d_{\alpha\beta}; \)

(ii) \( \varphi_i(\lambda, d) = (\lambda, \tilde{d}), \)

where

\[ \tilde{d}_{i,i+1} = d_{i,i+1} + \gamma_i - \gamma_{i+1} - \gamma_i + \gamma_{i+1}. \]

For all remaining elements \( \tilde{d}_{\alpha\beta} = d_{\alpha\beta}. \) Here \( (\gamma_1, \ldots, \gamma_n) = \gamma(d) \) is the weight of \( d. \)

Note that the involutions \( R_{ijk}, 1 < i < j < k \leq n, \) satisfy the relations of Proposition 1.5.

Finally, we define \( s_{\lambda} = (\lambda \partial) s_{\lambda(\lambda \partial)^{-1}} : R^n \oplus D_n \to R_n \oplus D_n, \) \( 1 \leq i \leq n-1, \) where the involutions \( s_i \) are given by (1.16).

**Theorem A.1.** We have the following equality:

\[ s_{\lambda} ^{\gamma} = R_i^{-1} \varphi_i R_i, \quad 1 \leq i \leq n-1, \]

where \( R_i := R_{1,i,i+1} \cdot R_{2,i,i+1} \cdots R_{i-1,i,i+1} \) if \( 2 \leq i \leq n-1, \) and \( R_1 = Id. \)

It is clear that identity (1.28a) follows from Theorem A.1.

**Proof of Theorem A.1.** We use formula (1.19) as inductive step. Note that Theorem A.1 is evident if \( i = 1. \) So, it is sufficient to prove that

\[ (\lambda \partial^{(i-1)}) s_i (\lambda \partial^{(i-1)})^{-1} = R_i^{-1} \varphi_i R_i. \]

Now we use the inductive hypothesis and formula (1.19). So, we have

\[ (\lambda \partial^{(i-1)}) s_i (\lambda^{(i-1)})^{-1} = \tilde{\tau}_{i-1} \tilde{s}_{i-1} (\tilde{\tau}_{i-1})^{-1} (\tilde{\tau}_{i-1})^{-1}, \quad (A.1) \]

where

\[ \tilde{s}_{i-1} = (\lambda \partial^{(i+1)}) s_{i-1} (\lambda \partial^{(i+1)})^{-1}, \]

\[ \tilde{\tau}_i = (\lambda \partial^{(i)}) t_i (\lambda \partial^{(i+1)})^{-1}, \]

\[ \tilde{\tau}_{i-1} = (\lambda \partial^{(i-1)}) t_{i-1} (\lambda \partial^{(i)})^{-1}. \]
Proposition A.1.

(i) If Theorem A.1 is valid for the involution $s_{i-1}^\varphi$, then

$$
\tilde{s}_{i-1} = (\tilde{R}_i)^{-1}\varphi_i \tilde{R}_i,
$$

(ii)

$$
\tilde{t}_i = R^{(i+1)}, \quad \tilde{t}_{i-1} = R^{(i)}, \quad \text{(A.2)}
$$

where

$$
R^{(i)} = R_{1,2,i} R_{1,3,i} \cdots R_{1,i-1,i},
$$

$$
\tilde{R}_i = R_{2,i,i+1} \cdots R_{i-1,i,i+1} = R_{i,i+1} \cdot \tilde{R}_i.
$$

Statement (ii) can be proved by direct computation. We prove statement (i). We have

$$
\tilde{s}_{i-1} = (\lambda \vartheta^{(i+1)})(\lambda \vartheta)^{-1}((\lambda \vartheta)s_{i-1}(\lambda \vartheta)^{-1})(\lambda \vartheta)(\lambda \vartheta^{(i+1)})^{-1}.
$$

By the inductive step, we have an identity

$$
(\lambda \vartheta)^{-1}(\lambda \vartheta)^{-1} = R_{i-1}^{-1}\varphi_{i-1} R_{i-1}.
$$

The following lemma completes the proof of statement (i).

Lemma A.5. The following identities are valid:

$$
(\lambda \vartheta^{(i+1)})(\lambda \vartheta)^{-1}R_{i-1}(\lambda \vartheta)(\lambda \vartheta^{(i+1)})^{-1} = \tilde{R}_i,
$$

$$
(\lambda \vartheta^{(i+1)})(\lambda \vartheta)^{-1}\varphi_{i-1}(\lambda \vartheta)(\lambda \vartheta^{(i+1)})^{-1} = \tilde{\varphi}_i.
$$

This lemma follows immediately from the next one.

Lemma A.6. Let $(\lambda, d) \in \mathbb{R}^n \oplus D_n$. Then $(\lambda \vartheta^{(i+1)})(\lambda \vartheta)^{-1}(\lambda \vartheta) = (\lambda, \tilde{d})$, where

$$
\tilde{d}_{k,j} = d_{k,j} \quad \text{if } j > i + 1,
$$

$$
\tilde{d}_{k,j} = d_{k-1,j-1} \quad \text{if } j \leq i + 1, \quad k > 1,
$$

$$
\tilde{d}_{1,j} = \lambda_{j-1} - d_{j-1,n} - d_{j-1,n-1}
$$

$$
- \cdots - d_{j-1,j} - \lambda_j + d_{j,n} + d_{j,n-1} + \cdots + d_{j,j+1} \quad \text{if } j \leq i + 1.
$$

We finish the proof of Theorem A.1 and, thus, that of Theorem 1.2 by means of the following relation between the transformations (A.2).
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Proposition A.2.

\[ R^i R^{i+1} (\overline{R}_i) = (\overline{R}_i) R^i R^{i+1} R^i. \] (A.3)

In fact, it follows from (A.1)–(A.2) that

\[ (\lambda \delta^{i-1}) s_i (\lambda \delta^{i-1})^{-1} = R^i R^{i+1} (\overline{R}_i) \varphi_i (R^i R^{i+1})^{-1} \overline{R}_i. \]

Now, using (A.3) we obtain

\[ (\lambda \delta^{i-1}) s_i (\lambda \delta^{i-1})^{-1} = R_i^{-1} R^{i+1} R_i R_i^{-1} \varphi_i (R^i R^{i+1})^{-1} R_i = R_i^{-1} \varphi_i R_i \]

because both \( R^i \) and \( R^{i+1} R_i R_i^{-1} \) commute with \( \varphi_i \).

The proof of Proposition A.2 consists of successive application of relations c) and b) of Proposition 1.5 to the quadruples \((1, k, i, i + 1)\) with \( k = i - 1, i - 2, \ldots, 2 \).

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