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EXIT LAWS AND EXCESSIVE FUNCTIONS FOR SUPERPROCESSES¹)

Пусть ξ есть марковский процесс с переходной функцией p(r, x; t, dy), а X — соответствующий суперпроцесс Доусона–Ватанабэ (т.е. суперпроцесс с характеристикой ветвления ψ(u) = ρu²). Обозначим через P переходную функцию X и положим

\[ p_n(r, x; t, dy) = \prod_{i=1}^{n} p(r, x_i; t, dy_i). \]

Любому закону выхода \( t \) для \( p_n \) соответствует закон выхода \( L_t \) для \( P \) такой, что \( L_t^\mu(\mu) \) есть полином степени n по \( \mu \) со старшим членом \( (t, \mu^n) \) для любого \( t \). Любой полиномиальный закон выхода для \( P \) допускает единственное представление в виде \( L_{t_1} + \cdots + L_{t_n} \), где \( t_k \) есть закон выхода для \( p_k \).

Ключевые слова и фразы: марковский процесс, суперпроцесс Доусона–Ватанабэ, полиномиальный закон выхода для \( P \).

Dedicated to Yu. V. Prokhorov on his 70th birthday

1. Action of the Dawson–Watanabe semigroup on polynomials

1.1. The Dawson–Watanabe superprocess. With every measurable space \((E, \mathcal{B})\) another measurable space \((\mathcal{M}, \mathcal{B}_\mathcal{M})\) is associated: \(\mathcal{M}\) is the set of all finite measures on \((E, \mathcal{B})\) and \(\mathcal{B}_\mathcal{M}\) is the \(\sigma\)-algebra in \(\mathcal{M}\) generated by the functions \(F(\mu) = \mu(B)\) with \(B \in \mathcal{B}\). To every Markov transition function \(p\) in \((E, \mathcal{B})\) there corresponds a Markov transition function \(P\) in \((\mathcal{M}, \mathcal{B}_\mathcal{M})\) such that, for every \(f \in p\mathcal{B}, \quad \int_{\mathcal{M}} P(t, \mu; s, dv) e^{-f(u,v)} = e^{-f(u,v)}, \quad (1.1) \)

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²) This writing means that \(f\) is a \(\mathcal{B}\)-measurable function from \(E\) to \([0, \infty]\). We write \(f \in p\mathcal{B}\) if \(f \in p\mathcal{B}\) and \(f\) is bounded.
where
\[ u(r, x) + \gamma \int_r^t p(r, x; s, dy) u^2(s, y) \, ds = \int_E p(r, x; t, dy) f(y) \]  
(1.2)
(\( \gamma \) is a positive constant). If \( p \) is the transition function of a right Markov process \( \xi = (\xi_t, \Pi_{r,x}) \), then \( \mathcal{P} \) is the transition function of a right Markov process \( X = (X_t, P_{r,\mu}) \).

We call \( X \) the Dawson–Watanabe superprocess. The Markov semigroups of \( \xi \) and \( X \) are denoted \( T_t \) and \( \mathcal{P}_t \).

Both transition functions \( p \) and \( \mathcal{P} \) are defined on the same time interval \( I \). Without any loss of generality we can assume that \( I = \mathbb{R}_+ \).

1.2. Martingale characterisation of \( X \). We use the following property of the Dawson–Watanabe superprocess: for all \( r < u \in \mathbb{R}_+ \), \( \mu \in \mathcal{M} \) and \( \varphi \in \mathbf{p} \),
\[ \langle T_u \varphi, X_t \rangle = \int X_t(dx)p(t, x; u, dy)\varphi(y) \]  
(1.3)
is a continuous square integrable martingale on \((r, u)\) relative to \( \mathcal{P} \) with the quadratic variation \( \gamma((T_u \varphi)^2, X_t)dt \) (see, e.g., [2]; in a slightly different form this was proved earlier in [5]).

1.3. Polynomials on the space \( \mathcal{M} \). Denote by \( S^k \) the cone of finite positive measurable symmetric functions on the product space \((E^k, \mathcal{B}^k)\) of \( k \) replicas of \((E, \mathcal{B})\). To every \( \mu \in \mathcal{M} \) there corresponds a measure
\[ \mu^k(dx_1, \ldots, dx_k) = \mu(dx_1) \cdots \mu(dx_k) \]  
on \((E^k, \mathcal{B}^k)\). We deal with functions on \( \mathcal{M} \) of the form
\[ F(\mu) = \sum_1^n (f_k, \mu^k) \quad \text{with } f_k \in S^k. \]  
(1.4)
We call them polynomials in \( \mu \). Note that the coefficients \( f_k \) are uniquely determined by \( F \). Indeed, \( k! f_k(x_1, \ldots, x_k) \) is the coefficient at \( \lambda_1 \cdots \lambda_k \) in the polynomial \( F(\lambda_1 \delta_{x_1} + \cdots + \lambda_k \delta_{x_k}) \).

Operators \( B_k \): \( S^k \to S^{k-1} \) are defined by the formula
\[ B_k f(x_1, \ldots, x_k) = \frac{k\gamma}{2} \sum_\sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \]  
(1.5)
where \( \sigma \) runs over all monotone increasing mappings from \( \{1, \ldots, k\} \) onto \( \{1, \ldots, k-1\} \). For instance,
\[ B_3 f(x_1, x_2) = \frac{3\gamma}{2} [f(x_1, x_1, x_2) + f(x_1, x_2, x_2)]. \]

Formula
\[ p_k(r, x; t, dy) = \prod_1^k p(r, x_i; t, dy_i) \]  
(1.6)
defines a transition function in \((E^k, \mathcal{B}^k)\). The corresponding semigroup \( T_k(r, t) \) preserves cone \( S^k \).

1.4. Action of \( \mathcal{P}_t^k \) on polynomials.

Theorem 1.1. For every \( r < t \) and every \( f \in S^k \)
\[ P_{r,\mu}(f, X^k_t) = \sum_{j=1}^k \langle R_{j,k}(r, t) f, \mu^j \rangle, \]  
(1.7)
where
\[ R_{kk}(r, t) = T_k (r, t), \quad R_{jk}(r, t) = \int_r^t R_{j,k-1}(r, s) B_k T_k(s, t) \, ds \quad \text{for } j < k. \]  
(1.8)

3) The definition of a right process in the nonhomogeneous setting and the construction of \( X \) can be found, e.g., in [3].
Proof. Formula \((1.7)\) follows by induction in \(k\) from the equation

\[
Pr,\mu(f, X^k_t) = (T_k(r, t) f, \mu^k) + Pr,\mu \int_r^t (B_k T_k(s, t) f, X^{k-1}_s) \, ds.
\]  \((1.9)\)

To establish \((1.9)\) we note that if \(\eta\) and \(\tilde{\eta}\) are two measures on \((E^k, \mathcal{G}^k)\) and if

\[
\int f \, d\eta = \int f \, d\tilde{\eta}
\]  \((1.10)\)

for all \(f\) of the form

\[
f(x) = \varphi(x_1) \cdots \varphi(x_k)\quad \text{with} \quad \varphi \in \mathbb{P} \mathcal{G} \]  \((1.11)\)

then \((1.10)\) holds for all \(f \in S^k\). Therefore it is sufficient to prove \((1.9)\) for functions of the form \((1.11)\). For these functions

\[
(f, X^k_t) = (\varphi, X_t)^k.
\]  \((1.12)\)

Fix \(t\) and put \(h^t_T = T_t \varphi\). Consider the martingale \(Y_s = (h^s, X_s)\) on the interval \([r, t)\). As we know, its quadratic variation is equal to \(\gamma(h^s)^2, X_s) \, ds\) and, by the Itô formula,

\[
Y^k_t = Y^k_r + \int_r^t k Y^{k-1}_s \, dY_s + \frac{1}{2} k(k - 1) \int_r^t Y^{k-2}_s \gamma(h^s)^2, X_s) \, ds.
\]

Therefore,

\[
Pr,\mu Y^k_t = Pr,\mu Y^k_r + \frac{1}{2} \gamma k(k - 1) Pr,\mu \int_r^t Y^{k-2}_s \gamma(h^s)^2, X_s) \, ds
\]

which implies \((1.9)\). Theorem 1.1 is proved.

1.5. Properties of \(R_{jk}\). It follows from \((1.8)\) that, for all \(j < k\),

\[
R_{jk}(r, t) = \int_{r < s_j < s_{j+1} < \cdots < s_{j-1} < t} T_j(r, s_j) B_{j+1} T_{j+1}(s_j, s_{j+1}) \cdots B_k T_k(s_{k-1}, t) \, ds_j \cdots ds_{k-1}.
\]  \((1.13)\)

For instance,

\[
R_{j,j+2}(r, t) = \int_{r < s_j < s_{j+1} < t} T_j(r, s_j) B_{j+1} T_{j+1}(s_j, s_{j+1}) B_{j+2} T_{j+2}(s_{j+1}, t) \, ds_j \, ds_{j+1}.
\]

Formula \((1.3)\) implies that, for all \(r < s < t, i \leq j \leq k\),

\[
\sum_{i=1}^k R_{ij}(r, s) R_{jk}(s, t) = R_{ik}(r, t).
\]  \((1.14)\)

(\(\frac{}{}\))

(The semigroup property of operators \(R_{ii}(r, t) = T_i(r, t)\) is a particular case of \((1.14)\) corresponding to \(i = j = k\).)

1.6. Time-homogeneous setting. In this setting, it is convenient to assume that \(I = \mathbb{R}\). If the transition function \(p\) is stationary (that is, if \(p(r,x; t, dy) = p_t(x, dy)\)), then the Dawson-Watanabe superprocess has also a stationary transition function \(\mathcal{P}_t(\mu, d\nu)\). The operator

\[
G^{k}_\lambda = \int_0^\infty e^{-\lambda t} T_k(t) \, dt
\]

is the resolvent of the semigroup \(T_k(t) = T_k(r, t + r)\). We denote resolvent corresponding to \(\mathcal{P}\) by \(G^{k}_\lambda\).

Theorem 1.2. \(\text{In the time-homogeneous case,}\)

\[
\int \mathcal{P}_t(\mu, d\nu) \langle f, \nu^k \rangle = \sum_{j=1}^{k} \langle R_{jk}(t) f, \mu^j \rangle,
\]  \((1.16)\)

where

\[
R_{kk}(t) = T_k(t), \quad R_{jk}(t) = \int_0^t R_{j,k-1}(s) B_k T_k(t - s) \, ds \quad \text{for} \ j < k.
\]  \((1.17)\)
If
\[ F(\nu) = (f, \nu^k) \text{ with } f \in S^k, \] (1.18)
then
\[ \mathcal{G}_\lambda F(\mu) = \sum_{j=1}^{k} \langle G_\lambda^j B_{j+1} G_\lambda^{j+1} \ldots B_k G_\lambda^k f, \mu^j \rangle. \] (1.19)

Proof. Formulae (1.16)–(1.17) follow from (1.7)–(1.8). Clearly, they imply (1.18)–(1.19).

2. Exit laws

2.1. Construction of polynomial \( \mathcal{P} \)-exit laws. Suppose that \( p \) is a Markov transition function in a measurable space \((E, \mathcal{B})\). Let \( 0 < \beta \leq \infty \) and let a function \( \ell^t \in p\mathcal{B} \) not identically equal to \( \infty \) be given for every \( t \in (0, \beta) \). We say that \( \ell \) is a \( p \)-exit law at time \( \beta \) and we write \( \ell \in \mathcal{L}_\beta(p) \) if \( \ell_r^t = \ell^t \) for all \( r < t < \beta \).

Theorem 2.1. To every \( \ell \in \mathcal{L}_\beta(p) \) there corresponds a \( \mathcal{P} \)-exit law
\[ L_\ell^t(\mu) = \langle \ell^t, \mu^n \rangle + \sum_{j=1}^{n-1} \int_{t}^{\beta} \langle R_{j,n-1}(t,u) B_n \ell^u, \mu^j \rangle du. \] (2.1)

If \( L \in \mathcal{L}_\beta(\mathcal{P}) \) and if, for every \( t \), \( L^t \) has the form (1.4), then
\[ L = \sum_{j=1}^{n} L_{\ell_j}, \] (2.2)
where \( \ell_j \in \mathcal{L}_\beta(p_j) \).

Proof. 1°. By using (1.7), we establish that
\[ L^t(\mu) = \sum_{k=1}^{n} \langle f_k^t, \mu^k \rangle \text{ with } f_k^t \in S^k \] (2.3)
satisfies the condition
\[ \mathcal{P}_t L^t = L^t \] (2.4)
if and only if
\[ f_k^t = \sum_{j=k}^{n} R_{k,j}(r,t) f_j^t \text{ for } k = 1, \ldots, n. \] (2.5)

2°. Function (2.1) has the form (2.3) with
\[ f_n^t = \ell^t, \quad f_j^t = \int_{t}^{\beta} R_{j,n-1}(t,u) B_n \ell^u du \text{ for } j < n. \] (2.6)

We claim that these function satisfy (2.5). This is obvious for \( k = n \). If \( k < n \), then, by (2.6),
\[ \sum_{j=k}^{n} R_{k,j}(r,t) f_j^t = R_{k,n}(r,t) f_n^t + \sum_{j=k}^{n-1} \int_{t}^{\beta} R_{k,j}(r,t) R_{j,n-1}(t,u) B_n \ell^u du. \] (2.7)

By (1.8), the first term in the right-hand side of (2.7) is equal to
\[ \int_{t}^{\ell} R_{k,n-1}(r,s) B_n T_n(s,t) f_n^s ds = \int_{t}^{\ell} R_{k,n-1}(r,u) B_n \ell^u du. \] (2.8)

By (1.14), the second term is equal to
\[ \int_{t}^{\beta} R_{k,n-1}(r,u) B_n \ell^u du. \] (2.9)
3°. By 1°, if $L$ given by (2.3) is a $\mathcal{P}$-exit law, then the coefficients $f_k^t$ satisfy conditions (2.5). By (1.8), this implies

$$f_n^r = T_n(r, t) f_n^t, \quad f_k^r = T_k(r, t) f_k^t + \sum_{j=k+1}^{n} \int_r^t R_{k,j-1}(r, s) B_j T_j(s, t) f_j^t \, ds \quad \text{for } k < n. \quad (2.10)$$

We conclude from this formula that $T_k(r, t) f_k^t \leq f_k^r$ for all $k$ and all $r < t < \beta$. Therefore $T_k(r, t) f_k^t$ is monotone decreasing in $t$, a limit $\ell_k^t = \lim_{t \to \beta} T_k(r, t) f_k^t$ exists and it does not exceed $T_k(r, t) f_k^t$ for all $k$.

By passing to the limit in formula $T_k(r, s) T_k(s, t) f_k^t = T_k(r, t) f_k^t$, we prove that $\ell_k^t \in \mathcal{L}_\beta(p_k)$. (The dominated convergence theorem is applicable since $T_k(r, t) f_k^t \leq f_k^t$ and $T_k(r, s) f_k^t \leq f_k^t < \infty$.)

Since $\ell_j^t \leq T_j(s, t) f_j^t$, formula (2.10) implies

$$\int_r^t R_{k,j-1}(r, s) B_j \ell_j^t \, ds \leq \int_r^t R_{k,j-1}(r, s) B_j T_j(s, t) f_j^t \, ds \leq f_k^t$$

for all $r < t < \beta$ and all $k < j < n$. \quad (2.11)

Therefore,

$$\int_r^\beta R_{k,j-1} B_j \ell_j^t \, ds \leq f_k^t \quad \text{for all } r < t < \beta \quad \text{and all } k < j < n. \quad (2.12)$$

By passing to the limit in (2.10), we get

$$f_n^r = \ell_n^r, \quad f_k^r = \ell_k^r + \sum_{j=k+1}^{n} \int_r^\beta R_{k,j-1}(r, s) B_j \ell_j^t \, ds \quad \text{for } k < n. \quad (2.13)$$

By (2.3), (2.10) and (2.13),

$$L^t(\mu) = \sum_{k=1}^{n} (f_k^t, \mu_k^t) = \sum_{k=1}^{n} (\ell_k^t, \mu_k) + \sum_{1 \leq k < j \leq n} \int_t^\beta (R_{k,j-1}(t, s) B_j \ell_j^t, \mu_k^t) \, ds \quad (2.14)$$

which implies (2.2). Theorem 2.1 is proved.

2.2. Examples. Fix $\beta < \infty$. To every function $f \in S$ there corresponds $\ell_{\beta, f}$ given by the formula

$$\ell_{\beta,f}(x) = \int \prod_{i=1}^{n} p(t, x_i; \beta, dy_i) f(y_1, \ldots, y_n). \quad (2.15)$$

We say that $m$ is a reference measure for $p$ if the measure $p(r, x; t, \cdot)$ is absolutely continuous with respect to $m$ for all $r < t$ and for all $x \in E$. By [4], the density function $\rho$ can be chosen to satisfy the equation

$$\int \rho(r, x; s, y) m(dy) p(s, y; t, z) = \rho(r, x; t, z) \quad (2.16)$$

for all $r < s < t$ and for all $x, z \in E$. To every symmetric measure $\eta$ on $(E^n, \mathcal{B}^n)$ there corresponds $\ell_{\beta, \eta} \in \mathcal{L}_\beta(p_n)$ defined by the formula

$$\ell_{\beta, \eta}(x) = \int \prod_{i=1}^{n} \rho(t, x_i; \beta, y_i) \eta(dy_1, \ldots, dy_n). \quad (2.17)$$

We denote by $L_{\beta,f}$ and by $L_{\beta,\eta}$ the $\mathcal{P}$-exit laws which correspond to $\ell_{\beta,f}$ and to $\ell_{\beta,\eta}$ by Theorem 2.1. Clearly, $L(\beta, \eta) = L(\beta, f)$ if

$$\eta(dy_1, \ldots, dy_n) = f(y_1, \ldots, y_n) m(dy_1) \cdots m(dy_n).$$
3. Excessive functions

Let $T(t)$ be the semigroup corresponding to a stationary transition function $p$. We say that a function $h \in \mathcal{H}(p)$ is $p$-excessive and we write $h \in \mathcal{H}(p)$ if $h(x) < \infty$ for some $x$, $T(t)h \leq h$ for all $t$ and $T(t)h \rightarrow h$ as $t \rightarrow 0$. A function $h \in \mathcal{H}(p)$ is called purely excessive if $T(t)h \rightarrow 0$ as $t \rightarrow \infty$ and it is called invariant if $T(t)h = h$ for all $t$. We denote the set of all purely excessive functions by $\mathcal{H}^0(p)$ and the set of all invariant functions by $\mathcal{H}_{inv}(p)$.

Every $h \in \mathcal{H}(p)$ has a unique representation $h = h_1 + h_2$, where $h_1 \in \mathcal{H}_{inv}(p)$ and $h_2 \in \mathcal{H}^0(p)$.

Note that $\mathcal{H}_{inv}(p)$ coincides with the class of $t$-independent $p$-exit laws at $\infty$.

To every $\ell \in L_0(p)$ there corresponds a function
\[
h = \int_{-\infty}^{0} \ell^t \, dt
\]
which is purely excessive unless it is equal to $\infty$ identically. The set of all purely excessive functions $h$ of the form (3.1) will be denoted by $\mathcal{H}^0(p)$.

If there exists a reference measure for $p$, then, by [1] (Section 7.6), $\mathcal{H}(p) = \mathcal{H}^0(p)$.

Theorem 3.1. To every $h \in \mathcal{H}^0(p_n)$ there corresponds $H_h \in \mathcal{H}^0(p)$ defined by the formula
\[
H_h(\mu) = \langle h, \mu^n \rangle + \sum_{j=1}^{n-1} \langle G^j B_{j+1} G^{j+1} \cdots B_n h, \mu^j \rangle,
\]
where $G^j$ is given by (1.15) with $\lambda = 0$.

Proof. Let $\ell$ correspond to $h$ by (3.1) and let $L \in L_0(\mathcal{D})$ correspond to $\ell$ by Theorem 2.1. Then
\[
H_h(\mu) = \int_{-\infty}^{0} L^t(\mu) \, dt
\]
belongs to $\mathcal{H}^0(p)$. Formula (3.2) follows from (3.3), (2.1), (1.9), (1.17).

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