EXTENSIONS OF HOPF ALGEBRAS

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Abstract. We investigate the notion of exact sequences of Hopf algebras. To two Hopf algebras $A$ and $B$, and a data consisting of an action of $B$ on $A$, a cocycle, a coaction of $A$ on $B$, and a co-cocycle we associate a short exact sequence of Hopf algebras $0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$. We define cleft short exact sequences of Hopf algebras and prove that their isomorphism classes are in a bijective correspondence with the quotient set of data as above such that the cocycle and the co-cocycle are invertible, modulo a natural action of a subgroup of $\text{Reg}(B, A)$.

§0. Introduction

The paper deals with extensions of Hopf algebras. By definition [D1], the category of quantum groups is the dual category to the category of Hopf algebras with bijective antipode. For this reason we can work mainly in this second category and all the results will translate to that of quantum groups in the obvious way. (Some of the results below are true with a weaker hypothesis about the antipode). It helps our intuition, however, to keep in mind that a Hopf algebra is the “algebra of functions on a quantum group”.

Let us fix, for simplicity, a commutative field $k$ and let us briefly say “Hopf algebra” for a Hopf algebra over $k$. We shall use the following notation: $m$, $\Delta$ (or $\delta$), $e$, $S$ mean respectively the multiplication, comultiplication, counit, antipode of a Hopf algebra (or an algebra or a coalgebra), specified with a subscript if necessary. The opposite (co)multiplication is betokened by a superscript “op”. We shall also use the following convention: if $c$ is an element of a tensor product $A \otimes B$, then we write $c = c_i \otimes c_j$, omitting the summation symbol. An exception is the case $c = \Delta(x)$, where we use Sweedler’s “sigma” notation but dropping again the summatory. The usual transposition $N \otimes N' \rightarrow N' \otimes N$ is denoted by $T$. If $g: N \otimes N \rightarrow N \otimes N$ is a morphism of $k$-modules then $g^{ij}: N^\otimes m \rightarrow N^\otimes m$ has the usual meaning, for example, $g^{i,i+1} = \text{id}_{N^t} \otimes g \otimes \text{id}_{N^m}$. The left (right) adjoint action of a Hopf algebra on itself is $\text{Ad}(b)a = b(1)aS(b(2))$ ($\text{Ad}^r(b)a = S(b(1))ab(2)$); the right (left) adjoint coaction is $\text{ad}(a) = a(2) \otimes S(a(1))a(3)$ ($\text{ad}^l(a) = a(2) \otimes a(1)S(a(3))$). Our main reference for the general theory of Hopf algebras is [Sw].

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We consider sequences of morphisms of Hopf algebras of the following type:

$$\mathbb{k} \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \mathbb{k}. \quad (C)$$

We shall say that (C) is exact if

1. $\iota$ is injective. Then we identify then $A$ with its image,
2. $\pi$ is surjective,
3. $\pi \iota = \varepsilon$,
4. $\ker \pi = CA^+$. ($A^+$ is the augmentation ideal, i.e., the kernel of the counit).
5. $A = \{ x \in C : (\pi \otimes \text{id}) \Delta (x) = 1 \otimes x \}$.

(3) follows either from (4) or from (5) and is included only to emphasize the relation with the classical notion. A "categorical" justification for this definition can be found in Section 1. Indeed, (4) implies that $\iota$ is conormal and (5), that $\pi$ is normal (cf. Definitions 1.1.5 and 1.1.9). That is, this definition enjoys the duality inherent to the theory of Hopf algebras. Clearly, given $C$ and $A$, there exists at most one $B$ (up to an isomorphism) making (C) exact; and reciprocally, $C$ and $B$ determine $A$. In fact, given a conormal injective morphism of Hopf algebras $A \xrightarrow{\iota} C$, the unique possible $B$ is the "Hopf cokernel" $C/CA^+$. Now condition (5) above can be dropped if $A \xrightarrow{\iota} C$ is faithfully flat and $A$ is stable with respect to the adjoint action of $C$. In such case, $B$ completes the exact sequence. We do not know if any inclusion of Hopf algebras is faithfully flat. A similar analysis proceeds for a normal, surjective morphism $C \xrightarrow{\pi} B$ of Hopf algebras; but the role of the "faithfully flat" requirement is played now by "faithfully coflat" (cf. Proposition 1.2.11).

These questions have already several antecedents in the literature. They were studied first in the setting of Hopf algebras graded by nonnegative integers, such that the 0 component is isomorphic to $\mathbb{k}$. In this subcategory, the case "A commutative, B cocommutative" was treated in [S] (see also [G]). The definition of extension given in [S] is justified by a result from [MM]. In the general situation, a definition of extensions of quantum groups was recently proposed in [PaW]. In terms of Hopf algebras, they say that the sequence (C) is exact if $\iota$ is a monomorphism and $B = C/C\iota(A^+)C$, i.e., $B$ is the cokernel of $\iota$. With this definition, two problems arise: given $C$ and $B$, on one hand $A$ is not unique, and on the other hand it is not proved that such an $A$ exists. As for the first problem, a counterexample is provided in [PaW, 6.3.3]; here is another one. Take a group $G$ with a nontrivial subgroup $H$ such that the union of all the conjugates of $H$ is $G$. Let $A=C$ be the group algebra of $H$ (of $G$), and let $\iota$ be the canonical inclusion. The augmentation ideal of a group algebra is the vector subspace generated by the elements $e_g - 1$ with $g$ nontrivial. Therefore, $B$ above is the trivial Hopf algebra. (Note also that the inclusion has no Hopf image, cf. Definition 1.2.0). This situation is not possible in our approach. Indeed, if (C) is exact in our sense and $B$ is trivial, then by condition (5) in Proposition 1.2.3 we have $A = C$.

Still another definition of exact sequences is given in [Sch], remedying the inconveniences in [PaW]; it was pointed out to the first author by S. Montgomery after receiving the first draft of this paper. In loc. cit. a short sequence of Hopf algebras like (C) is said to be exact if it satisfies either one of the two following sets of axioms:

1. $A$ is a conormal faithfully flat Hopf subalgebra of $C$, which is a submodule for...
the adjoint actions, and \( B = \text{HCoker } \iota \).

(2) \( B \) is a faithfully coflat quotient Hopf algebra of \( C \), which is a quotient comodule
for the adjoint coactions, and \( A = \text{HKer } \pi \).

The equivalence of the preceding requirements is proved by using a result from [T3].
By Corollaries 1.2.5 and 1.2.14, the definition in [Sch] agrees with ours, in the faithfully
flat case.

We wish to answer the following standard questions: given \( A \) and \( B \), which additional
data produce an exact sequence \((C)\); reciprocally, when an exact sequence \((C)\) can be
obtained in such way; how isomorphisms of exact sequences are translated in terms of
such data.

Let us discuss the first question. The construction of an algebra \( C \) out from a Hopf
algebra \( B \), an algebra \( A \) weakly acted upon by \( B \), and a "cocycle" \( B \otimes B \rightarrow A \) was
undertaken in [BCM] and independently in [DT] (see also [Sw2]). With these results
at hand, our strategy is simple: first, to obtain a dual statement, i.e., to show how to
construct a coalgebra \( C \) out from a coalgebra \( B \), a Hopf algebra \( A \) weakly co-acting
on \( B \) and a "co-cocycle" \( A \rightarrow B \otimes B \). Second, to analyze whether the algebra and
coalgebra structures on the vector space \( C = A \otimes B \) provided by the Hopf algebras \( A \),
\( B \), and the four data (weak action, weak coaction, cocycle, co-cocycle) give rise to a
bialgebra, or more precisely to a Hopf algebra. We say that the data is compatible, in the
"bialgebra" case; we say that a compatible data is Hopf if the corresponding bialgebra
has an antipode. We obtain a complete answer for the bialgebra question (see Theorem
2.20). These considerations are carried out in Section 2. The existence of the antipode
is proved later (see Lemma 3.2.17).

The isomorphism problem is treated in Section 3. Again, we take profit of what is
known in the algebra case [D]; again, we obtain the coalgebra version and then look for
the Hopf algebra case. In 3.1, we define an action of the group of invertible morphisms
from \( B \) to \( A \) (with respect to the convolution product) which preserve the unit and
counit, on the set of Hopf and compatible data. One has a mapping from the quotient
sets to isomorphism classes of extensions of Hopf algebras, resp. bialgebras. In the
algebra case [D] the related mapping is an isomorphism if one restricts, on one hand,
to data with invertible cocycle, and on the other to cleft extensions. An analogous result
is true in the coalgebra case. In 3.2, we define cleft extensions of Hopf algebras. In this
setting, we have complete answers to the problems stated above: any cleft extension of
Hopf algebras is isomorphic to one obtained from a data with invertible cocycle and
co-cocycle; the bialgebras constructed from data with invertible cocycle and co-cocycle
always have an antipode; and the corresponding quotient set classifies cleft extensions
up to isomorphism. Conversely, it follows from a remark in [By] that if \( C = A \otimes B \) with
the introduced bialgebra structure has an antipode, then the cocycle should be invertible.

This is a revised version of the preprint MPI/93-53 (May 1993). After putting in
circulation this preprint, we noticed several papers related to our results, mostly under
suitable restrictions. The notion of cleft extensions is also discussed in [By], where also a
classification theorem is obtained in the "\( A \) commutative--\( B \) cocommutative" setting (in
what follows, the abelian case). For algebraic groups, cleftness was already discussed in
[Sch4]. Also the procedure leading to theorem 2.20 was outlined in [Mj], in the general
case; we remarked this only recently (previous work was done in [Ma]). Details of this
construction can be found in [MjS] (the explicit formula for the antipode is not given, however). In [Hf] the translation of Singer's cohomology to the non-graded case is worked out in detail, as well as the classification theorem—in the abelian case. Also previous work—still in the abelian case and with trivial cocycles—was done in [T4].

At various key points (e.g., Lemmas 1.1.12, 1.2.6, 1.2.7), we have used solutions of the Yang–Baxter equation found in [W2]. Thanks to them, any Hopf algebra is generalized commutative and generalized cocommutative. This remark, due to the first author, is completely new and has turned out to be very useful (see the Appendix).

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§1. The category of quantum groups

This section is devoted to basic constructions in the category of quantum groups. Our aim is to give a definition of an exact sequence, and to prove that it is equivalent to that discussed in the Introduction. We shall freely use the terminology and results of [McL], of common use nowadays. To avoid confusions, all the categoric concepts we will work are in the category of Hopf algebras, unless explicitly stated.

1.1. Kernels. The first point we shall touch is the existence of Hopf kernels, or more generally, of equalizers in the category of Hopf algebras. Let \( f, g : A \to B \) be two morphisms of Hopf algebras. Let us consider the following conditions on element \( x \in A \):

\[
\begin{align*}
(f \otimes \text{id})\Delta(x) &= (g \otimes \text{id})\Delta(x), \\
(\text{id} \otimes f)\Delta(x) &= (\text{id} \otimes g)\Delta(x), \\
(\text{id} \otimes f \otimes \text{id})(\Delta \otimes \text{id})\Delta(x) &= (\text{id} \otimes g \otimes \text{id})(\Delta \otimes \text{id})\Delta(x).
\end{align*}
\]

If \( x \) satisfies (1.1.0), then \( f(x) = g(x) \). Indeed, \( f(x) = (f \otimes \varepsilon)\Delta(x) \). The same is true if \( x \) satisfies (1.1.1) or (1.1.2). Clearly, the product of elements satisfying one of these three conditions again does. Let

\[ \text{HEqual}(f, g) = \{ x \in A : x \text{ satisfies (1.1.2)} \}. \]

We shall also consider the algebras (which include \( \text{HEqual}(f, g) \))

\[ \text{LEqual}(f, g) = \{ x \in A : x \text{ satisfies (1.1.0)} \}, \]
\[ \text{REqual}(f, g) = \{ x \in A : x \text{ satisfies (1.1.1)} \}. \]

Lemma 1.1.3.

1. \( S(\text{LEqual}(f, g)) = \text{REqual}(f, g) \) and \( S(\text{REqual}(f, g)) = \text{LEqual}(f, g) \).
2. \( \Delta(\text{LEqual}(f, g)) \subseteq \text{LEqual}(f, g) \otimes A \) and \( \Delta(\text{REqual}(f, g)) \subseteq A \otimes \text{REqual}(f, g) \).
3. \( \text{HEqual}(f, g) \) is a Hopf subalgebra of \( A \).
4. \( \text{HEqual}(f, g) \) is the equalizer of \( f \) and \( g \).
Proof. $S$ and $S^{-1}$ are anticomultiplicative, which implies (1); (2) follows from the coassociativity of $\Delta$, cf. [Sw, Lemma 16.1.1]. (3) is easy to verify and the condition (1.1.2) is included to guarantee that $\text{HEqual}(f, g)$ is a subcoalgebra.

Let $h: C \to A$ be a morphism of Hopf algebras such that $fh = gh$; then the image of $h$ is contained in $\text{HEqual}(f, g)$, because $h$ is a morphism of Hopf algebras. 

The following lemma is a generalization (and extension) of [BCM, Prop. 1.19].

Lemma 1.1.4. The following conditions are equivalent:

1. $\text{LEqual}(f, g) = \text{HEqual}(f, g)$.
2. $\text{LEqual}(f, g)$ is a Hopf subalgebra of $A$.
3. $\text{LEqual}(f, g) = \text{REqual}(f, g)$.
4. $\text{REqual}(f, g)$ is a Hopf subalgebra of $A$.
5. $\text{REqual}(f, g) = \text{HEqual}(f, g)$.

Proof. (1) $\implies$ (2) is trivial; (2) $\implies$ (3) and (3) $\implies$ (4) follow from Lemma 1.1.3; (4) $\implies$ (5) is a consequence of universality. Interchanging right with left, we obtain the proof of (5) $\implies$ (4) $\implies$ (3) $\implies$ (2) $\implies$ (1). 

The base field $k$ is a zero object in the category of Hopf algebras, i.e., it is initial (the unit $1: k \to A$ is the unique morphism of Hopf algebras with such domain and codomain) and final (idem for the counit $\varepsilon: A \to k$). Thus, the zero morphism between Hopf algebras $A'$ and $B$ is $1_{B} e_{A'}$. As usual, the kernel $\text{HKer} f$ of a morphism of Hopf algebras is merely the equalizer of it and the zero morphism. Similarly, we have the notions of $\text{RKer}$, $\text{Lker}$.

Remark. The notions of left and right Hopf kernels (sets of elements satisfying (1.1.0) or (1.1.1)) appear at least in [Sw, BCM]. The condition (1.1.2) was signaled to the first author by B. Enriquez in the course of discussions about [L, 9.2]. The notion of Hopf kernel also appears in [Sch], as was communicated to us by S. Montgomery after reading the first version of this paper.

Definition 1.1.5. A morphism of Hopf algebras $f: A \to B$ is normal if the pair $f, g = 1_{B} e_{A}$ satisfies the equivalent conditions of Lemma 1.1.4.

Example. We extend an example from [BCM]. Let $G$ be a finite group, $H \hookrightarrow G$ a subgroup, $A$ (resp. $B$) the algebra of functions on $G$ (resp. $H$), and $R: A \to B$ the restriction morphism. Then it is easy to see that

- $\text{Lker}(R) = \{ f \in A : f \text{ is constant on the left coset } Hx, \ \forall x \in G \}$,
- $\text{Rker}(R) = \{ f \in A : f \text{ is constant on the right coset } xH, \ \forall x \in G \}$,
- $\text{HKer}(R) = \{ f \in A : f \text{ is constant on the left-right coset } xHy, \ \forall x, y \in G \}$.

It is now clear that $R$ is a normal morphism if and only if $H$ is a normal subgroup of $G$.

The category of Hopf algebras is not abelian, but, as in the case of groups, we still have relations between kernels and monomorphisms. If a morphism $h: A \to B$ of Hopf algebras is a monomorphism and $\iota: \text{HKer } h \to A$ is the inclusion, as then $h \iota = h \varepsilon$ implies that $\iota = \varepsilon$ and hence $\text{HKer } h = k$. 

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Lemma 1.1.6. If a morphism \( h : A \to B \) of Hopf algebras is injective, then \( \text{H} \text{Ker} h = k \) and \( h \) is normal. Conversely, if \( \text{H} \text{Ker} h = k \) and \( h \) is normal, then \( h \) is a monomorphism (in the categorical sense) of Hopf algebras.

Proof. The first claim is clear: since \( h \) is injective, \( (h \otimes \text{id}) \Delta(x) = 1 \otimes x \) implies \( \Delta(x) = 1 \otimes x \), and therefore \( x \in k \). That is, \( \text{H} \text{Ker} h = \text{L} \text{Ker} h = k \).

Let us prove the converse. Let \( f, g : C \to A \) be morphisms of Hopf algebras such that \( hf = hg \). Then we want to conclude that \( f = g \). To do this we need the algebra structure on \( \text{hom}_k(C, A) \) [Sw]; explicitly

\[
(f \ast g)(c) = f(c(1))g(c(2)).
\]

Since \( f \) is a morphism of Hopf algebras, it is invertible in this algebra and, in fact, \( f^{-1} = f S_C \) [Sw, 4.0.4]. The equality \( f = g \) is thus equivalent to \( 1 = f^{-1}g \), that is, to

\[
\varepsilon(c)1_A = fS_C(c(1))g(c(2)), \quad \forall c \in C.
\]

Let \( t \) be an element in the right-hand side of the equation above. It suffices to prove that \( t \) belongs to \( \text{H} \text{Ker} h \); then we have \( t = \varepsilon(t) = \varepsilon(c) \). But this is easy to check, as \( \text{H} \text{Ker} h = \text{L} \text{Ker} h \);

\[
(h \otimes \text{id})\Delta(t) = [Shf(c(2)) \otimes Sf(c(1))][hg(c(3)) \otimes g(c(4))]
= Shf(c(2))hf(c(3)) \otimes Sf(c(1))g(c(4))
= 1 \otimes t. \quad \bullet
\]

A quotient Hopf algebra of a Hopf algebra is a quotient vector space provided with a Hopf algebra structure such that the projection is a morphism of Hopf algebras.

Lemma 1.1.7. Let \( \pi : C \to B \) be a projection of Hopf algebras. Suppose that \( B \) is a right \( C \)-quotient comodule for the right adjoint coaction \( \text{ad}(c) = c(2) \otimes S(c(1))c(3) \). Then the projection \( \pi \) is normal.

Proof. The assumption reads: if \( \pi(c) = 0 \), then \( \pi(c(2)) \otimes S(c(1))c(3) = 0 \). Let \( c \in \text{R} \text{Ker} \pi \); then \( \pi(c(2)) \otimes S(c(1))c(3) = \varepsilon(c) \). Now \( \pi(c(1)) \otimes c(2) = \pi(c(2)) \otimes c(1)S(c(2))c(4) = 1 \otimes c(1)\varepsilon(c(2)) \) because of Lemma 1.1.3 (2). That is, \( \text{R} \text{Ker} \pi \subseteq \text{L} \text{Ker} \pi \). By Lemma 1.1.3 (1) we get the equality. \( \bullet \)

The definitions of coequalizers and cokernels are rather easy: if \( f, g : A \to B \) are two morphisms of Hopf algebras, let \( J \) denote the set of the all elements of the form \( f(x) - g(x) \), \( x \in A \). Then the quotient of \( B \) by the two-sided ideal generated by \( J \) is the corresponding coequalizer, denoted \( \text{H} \text{Coeq}(f, g) \). It is also useful to consider the quotients

\[
\text{L} \text{Coeq}(f, g) = B / BJ, \quad \text{R} \text{Coeq}(f, g) = B / JB,
\]

which are actually quotient coalgebras of \( B \). Indeed, to see that \( BJ, JB, BJB \) are actually coideals of \( B \), one uses a standard trick:

\[
\Delta(f(x) - g(x)) = [f(x(1)) - g(x(1))] \otimes f(x(2)) + g(x(1)) \otimes [f(x(2)) - g(x(2))].
\]
Lemma 1.1.8. The following statements are equivalent:

1. \( BJ = BJB, \)
2. \( BJ \) is a two-sided ideal,
3. \( BJ = JB, \)
4. \( JB \) is a two-sided ideal,
5. \( JB = BJB. \)

Proof. (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) are obvious. We have (2) \( \Rightarrow \) (3) because \( JB = S(BJ). \)

Taking as \( g \) the zero morphism of Hopf algebras, we have the definitions of Hopf, left and right cokernels. Observe that the set \( J \) in the above definitions specializes in this case to \( f(A^+), \) where \( A^+ = \{ x \in A : \varepsilon(x) = 0 \} \) is the augmentation ideal of \( A. \)

Definition 1.1.9. We shall say that a morphism \( f : A \rightarrow B \) of Hopf algebras is conormal if \( HCoker f = LCoker f. \)

Remark. The preceding definition is known; it appears, e.g., in [Sch3].

Example. Let \( k(X) \) denote the group-like coalgebra on the set \( X \) [Sw, p. 6]. Consider a morphism of groups \( \phi : G \rightarrow H \) and extend it to the group algebras \( A = k(G), B = k(H). \) In this case \( RCoker(\phi) = k(\text{Im}(\phi)\backslash H), \) \( LCoker(\phi) = k(H/\text{Im}(\phi)) \) and \( HCoker(\phi) = k(G/N), \) where \( N \) is the (normal) subgroup of \( G \) generated by \( ghg^{-1}, g \in G, h \in \text{Im}(\phi). \)

If \( h \) is an epimorphism, then the projection to \( HCoker h \) is the zero morphism and hence \( HCoker h = k. \)

Lemma 1.1.10. If a morphism \( h : A \rightarrow B \) of Hopf algebras is surjective, then \( HCoker h = k \) and \( h \) is conormal. Conversely, if \( HCoker h = k \) and \( h \) is conormal, then \( h \) is an epimorphism of Hopf algebras.

Proof. If \( h \) is surjective, the relation \( bh(a) = b(1)h(a)Sb(2)b(3) \) implies the conormality of \( h. \)

The proof of the second statement is similar to that of 1.1.6. Indeed, if \( f, g : B \rightarrow C \) are Hopf algebra morphisms such that \( fh = gh, \) then it suffices to prove that

\[ \varepsilon_B(b)1_C = S_C f(b(1))g(b(2)), \quad \forall b \in B. \]

Now it is easy to see that

\[ (f^{-1} * g)(h(a)) = \varepsilon(a), \]
\[ (f^{-1} * g)(xy) = S f(y(1))(f^{-1} * g)(x)g(y(2)). \]

Therefore, using that \( h \) is conormal, we are done. •

The following Lemma is a dual analogue of Lemma 1.1.7 and relates our definition of conormality with [PaW, 1.5].
Lemma 1.1.11. The inclusion morphism of a Hopf subalgebra stable by either the right or the left adjoint actions is conormal.

Proof. Suppose that \( A \hookrightarrow B \) is \( \text{Ad}_r \)-stable. Then, for \( a \in A^+ \), \( b \in B \), one has \( ab = b(1)S(b(2))ab(3) \), i.e., \( BA^+ \) is a two-sided ideal. •

We finish this subsection with additional remarks. A variant of the following Lemma is proved in [Sch, 1.3].

Lemma 1.1.12. (i) If \( h : A \to B \) is normal, then \( \text{HKer} \ h \hookrightarrow A \) is conormal.

(ii) If \( h \) is conormal, then \( B \to \text{HCoker} \ h \) is normal.

Proof. (i) First we claim that we always have

\[
(L\text{Ker} \ h)^+ A \subseteq A(L\text{Ker} \ h)^+.
\]

(1.1.13)

It is not difficult to see that \( L\text{Ker} \ h \) is \( \text{Ad}_r \)-stable [BCM]. Thus if \( x \in L\text{Ker} \ h \) and \( a \in A \), we use

\[
xa = (a(1))S(a(2))xa(3) = a(1)x \text{Ad}_r(a(2))
\]

and \( e(x \text{Ad}_r a) = e(x)e(a) \) to prove (1.1.13). But now if \( h \) is normal, then \( \text{HKer} \ h \hookrightarrow A \) is conormal. (Observe that the restriction of the antipode to \( \text{HKer} \ h \) is bijective).

(ii) Let \( S_0 : B \otimes B \to B \otimes B \) be the map given by

\[
S_0(b \otimes c) = c(2) \otimes bS(c(1))c(3).
\]

(1.1.14)

We claim that there exists a map \( S_0 \) making the following diagram commutative:

\[
\begin{array}{ccc}
B \otimes B & \xrightarrow{S_0} & B \otimes B \\
\text{id} \otimes \pi & & \Downarrow \pi \otimes \text{id} \\
B \otimes \text{RCoker} \ h & \xrightarrow{S_0} & \text{RCoker} \ h \otimes B.
\end{array}
\]

For proving this, it suffices to check that

\[
0 = (\pi \otimes \text{id})(c(2) \otimes S(c(1))c(3)), \quad \text{if } c = h(a)d \in h(A^+)B.
\]

But

\[
(\pi \otimes \text{id}) (h(a(2))d(2) \otimes S(d(1))S(h(a(1))h(a(3)))d(3))
\]

\[
= e(a(2))\pi(d(2)) \otimes S(d(1))S(h(a(1))h(a(3)))d(3)
\]

\[
= 0.
\]

Next, we claim that the following diagram commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{\Delta} & B \otimes B \\
\Downarrow \quad & & \Downarrow \pi \otimes \text{id} \\
\quad & & \text{RCoker} \ h \otimes B.
\end{array}
\]

\[
\begin{array}{ccc}
B & \xrightarrow{\Delta} & B \otimes B \\
\Downarrow S_0 & & \Downarrow S_0 \\
B & \xrightarrow{\pi \otimes \text{id}} & \text{RCoker} \ h \otimes B.
\end{array}
\]

\[
\begin{array}{ccc}
B & \xrightarrow{\Delta} & B \otimes B \\
\Downarrow & & \Downarrow S_0 \\
B & \xrightarrow{\pi \otimes \text{id}} & \text{RCoker} \ h \otimes B.
\end{array}
\]
We merely need to show that $\Delta = S_0 \Delta$, but this is a routinary computation.

Now assume that $h$ is conormal and let $b \in R\ker \pi$, where $\pi : B \to H\coker h = R\coker h$ is the projection. Then

$$(\pi \otimes \text{id}) \Delta(b) = S_0 (\text{id} \otimes \pi) \Delta(b) = S_0 (b \otimes 1) = 1 \otimes b,$$

and hence $B \to H\coker h$ is normal. •

1.2. Exact sequences. Here we shall be concerned with the “image” of a morphism of Hopf algebras, $f : A \to B$. First, observe that $f$ factorizes in the following way:

$$
\begin{array}{cccc}
H\ker f & \to & A & \overset{f}{\to} B & \overset{\pi}{\to} H\coker f \\
\downarrow & & \downarrow & & \uparrow \\
H\coker (H\ker f) & \to & H\ker (H\coker f)
\end{array}
$$

This is proved in two steps. First, $A(H\ker f)^+ A \subseteq \ker f$, because $f_0 = \epsilon$, and hence $f$ factorizes through $H\coker (H\ker f)$. Second, $\text{Im } f \subseteq H\ker (H\coker f)$, because $\pi f = \epsilon$.

Definition 1.2.0. We shall say that a morphism $f : A \to B$ has a Hopf image if the canonical map $H\coker (H\ker f) \to H\ker (H\coker f)$ is an isomorphism, and in such case we write $H\text{Im } f = H\ker (H\coker f) \simeq H\coker (H\ker f)$. Thus, $f$ has a Hopf image if and only if the following two conditions hold:

$$
\begin{align*}
\ker f & \subseteq A(H\ker f)^+ A, & (1.2.1) \\
H\ker (H\coker f) & \subseteq \text{Im } f. & (1.2.2)
\end{align*}
$$

Now we are ready to define exact sequences. We shall say that a sequence of morphisms of Hopf algebras

$$
A \overset{f}{\to} B \overset{g}{\to} C
$$

is exact if

1. $f$ is conormal and has a Hopf image,
2. $g$ is normal,
3. $H\text{Im } f = H\ker g$.

As always, a sequence

$$
\ldots A_i \overset{f_i}{\to} A_{i+1} \overset{f_{i+1}}{\to} A_{i+2} \ldots
$$

is exact if and only if so is each “segment” $A_i \overset{f_i}{\to} A_{i+1} \overset{f_{i+1}}{\to} A_{i+2}$.

Now let us consider a short sequence

$$
0 \to A \overset{i}{\to} C \overset{\pi}{\to} B \to 0,
$$

where 0 is of course the Hopf algebra $k$. 
Proposition 1.2.3. The sequence $(C)$ is exact if and only if the following conditions hold:

1. $i$ is injective. Identify then $A$ with its image,
2. $\pi$ is surjective,
3. $\ker i = \varepsilon$,
4. $\ker \pi = CA^+$,
5. $A = \{ x \in C : (\pi \otimes \text{id})\Delta(x) = 1 \otimes x \}$.

Proof. Let us first assume that $(C)$ is exact.

1. The exactness of $0 \to A \to C$ is equivalent to "$i$ is normal and $\text{HKer } i = k$".
2. Since $i$ has a Hopf image, by (1.2.1) we have $\ker i = A(\text{HKer } i)^+ A = 0$ and hence $i$ is injective.
3. The exactness of $C \to A \to 0$ is equivalent to "$\pi$ is surjective and $\ker \pi \subseteq C(\text{HKer } \pi)^+ C$".
4. Since $A = \text{HKer } \pi$, one has $\pi \varepsilon = \varepsilon$. But, moreover, $\pi$ is conormal and this together with (iii) implies (4); (5) holds because $\pi$ is normal.

Assume now that the conditions (1), ..., (5) are true. By Lemma 1.1.6 and (i), we have exactness at $0 \to A \to C$.

6. Since $A$ is a Hopf algebra, (5) and Lemma 1.1.6 imply that $\pi$ is normal. By (2), (4), and (iii) we also have exactness at $C \to B \to 0$.
7. (4) also shows that $i$ is conormal. Now we show that $i$ has a Hopf image: (1.2.1) is clear, by the injectivity of $i$. Again, $C \to B$ is the cokernel of $i$ and hence (1.2.2) follows from (5). Therefore we have exactness also at $A \to C \to B$.

Note that $A = \text{Lker } \pi = \text{Rker } \pi$ is a submodule of $C$ for the adjoint actions; and dually, $B$ is a quotient comodule for the adjoint coactions.

Remark. Note that the bridge between the Hopf- and set-theoretic conditions is the requirement on the Hopf image. We have not deepened our understanding of this question. Though, let us mention a related fact. Let

\[
0 \longrightarrow A \overset{i}{\longrightarrow} C \overset{\pi}{\longrightarrow} B \longrightarrow 0
\]

be a morphism of exact sequences. Let us try to prove that $\Theta$ is an isomorphism. Let $c \in \text{Lker } \Theta$. Then $(\pi \otimes \text{id})\Delta c = (\pi_1 \Theta \otimes \text{id})\Delta c = 1 \otimes c$ and therefore $c \in A$. But then $(\Theta \otimes \text{id})\Delta c = c(1) \otimes c(2) = 1 \otimes c$ and hence $c \in k$; i.e., $\Theta$ is normal and $\text{HKer } \Theta = k$. It is also not difficult to prove that $\Theta$ is conormal and $\text{Hcoker } \Theta = k$. Thus, $\Theta$ is a monomorphism and epimorphism of Hopf algebras. We do not know if it is then an isomorphism. We shall prove this if the extension is cleft, see Lemma 3.2.19. We also ignore if a monomorphism (resp., epimorphism) of Hopf algebras is injective (resp., surjective).

It seems that the above conditions are slightly redundant. Here is a result in this direction.
Proposition 1.2.4. Assume that \( A \xrightarrow{\gamma} C \) is faithfully flat and stable by the left adjoint action. Then \((C)\) is exact if and only if (1) to (4) hold.

Proof. We need to prove that (5) follows from (1), \ldots, (4) in presence of faithful flatness.

First assume only (1), \ldots, (4). Let \( R \) be an algebra and \( \gamma : H \to C \) a morphism of Hopf algebras with \( \pi \gamma = \varepsilon \).

Claim. Let \( x, y \in \text{Hom}_{\text{alg}}(C, R) \) be such that \( x \gamma = y \gamma \). Then \( x \gamma = y \gamma \).

Proof of the Claim. First we show that \( x * y^{-1} \) factorizes through \( B \). Indeed, by (4), we need to check that \( x * y^{-1}(ca) = 0 \), for \( c \in C \), \( a \in A^{\perp} \). But \( x * y^{-1}(ca) = x(c(1)a(1))y(Sa(2)Sc(2)) = 0 \). Now \( x \gamma *(y \gamma)^{-1}(h) = x * y^{-1}(\gamma h) = \varepsilon(h) \). Thus, \( x \gamma = y \gamma \).

Let \( H = \text{HKer } \pi \). By (2), (4), and Lemma 1.12 (ii) (applied to \( A \to C \), \( H = \text{LKer } \pi \); by (3) \( A \subseteq H \). Now apply the Claim to the inclusion \( H \to C, R = C \otimes_A C \) (see below), \( x : c \mapsto c \otimes 1, y : c \mapsto 1 \otimes c \). By faithful flatness, \( A \supseteq H \).

Corollary 1.2.5. Assume that \( A \xrightarrow{\gamma} C \) is faithfully flat and stable by the left adjoint action. Set \( B = C/CA^{\perp} \) and let \( \pi \) be the natural projection. Then \((C)\) is exact.

Remark. We do not know if any inclusion of Hopf algebras is faithfully flat. In the commutative setting, this is well known; a purely algebraic proof is contained in [T]. More generally, it is shown in [Sch] that any inclusion of a central Hopf algebra is faithfully flat. See also [MW].

Since \( A \) is not central in \( C \), one should be careful with the algebra structure of \( C \otimes_A C \).

Lemma 1.2.6. Let \( C \) be a Hopf algebra. The multiplication in \( C \otimes C \) defined by

\[
(c \otimes d)(x \otimes y) = cd(1)xSc(2) \otimes d(3)y \tag{1.2.7}
\]

is associative with unit \( 1 \otimes 1 \). The mappings \( c \mapsto c \otimes 1 \) and \( c \mapsto 1 \otimes c \) are morphisms of algebras, with respect to (1.2.7). Assume in addition that \( A \) is an Ad-invariant Hopf subalgebra of \( C \). Then \( C \otimes_A C \) inherits an algebra structure from (1.2.7).

Proof. We shall check that the kernel of \( C \otimes C \to C \otimes_A C \) is a two-sided ideal and leave the rest of the proof to the reader. From the left:

\[
(c \otimes d)(xa \otimes y - x \otimes ay) = cd(1)xaSc(2) \otimes d(3)y - cd(1)xSc(3) \otimes d(3)ay = cd(1)xSc(2)d(3)aSc(4) \otimes d(5)y - cd(1)xSc(2) \otimes d(3)aSc(4)d(5)y.
\]

From the right:

\[
(xa \otimes y - x \otimes ay)(c \otimes d) = xay(1)cSc(2) \otimes y(3)d - x(a(1)y(1)cSc(2)a(2) \otimes a(3)y(3)d = x(a(1)y(1)cSc(2)a(2)a(3)y(3)d - x(a(1)y(1)cSc(2)a(2) \otimes a(3)y(3)d. \]

Now let us recall that the cotensor product of a right comodule $M$ and a left comodule $N$ over a coalgebra $C$ is

$$M \boxtimes_C N = \ker \left( M \otimes N \xrightarrow{c \otimes \text{id} - \text{id} \otimes c} M \otimes C \otimes N \right).$$

Here is the dual version of the preceding Lemma.

**Lemma 1.2.8.** Let $C$ be a Hopf algebra. The comultiplication in $C \otimes C$ defined by

$$\widetilde{\Delta}(c \otimes d) = c_{(1)} \otimes d_{(2)} \otimes c_{(3)}Sd_{(3)}d_{(4)}$$

(1.2.9) is coassociative with counit $1 \otimes 1$. The mappings $c \otimes d \mapsto c(c)d$ and $c \otimes d \mapsto cc(d)$ are morphisms of coalgebras, with respect to (1.2.9). Assume in addition that $C \twoheadrightarrow B$ is a surjective morphism of Hopf algebras satisfying the following property: there exists a $\psi : B \twoheadrightarrow C$ such that $\psi(\pi b) = \pi(b(2)) \otimes Sb_{(1)}b_{(3)}$. Then $C \boxtimes_B C$ inherits a coalgebra structure from (1.2.9).

**Proof.** Again we prove only the last statement. Assume for simplicity that $a \otimes b \in C \boxtimes_B C$, i.e., that $a \otimes \pi(b_{(1)}) \otimes b_{(2)} = a_{(1)} \otimes \pi(a_{(2)}) \otimes b$. Applying $S_0^34m^{24}(\Delta \otimes \psi \otimes \Delta)$ to the both sides we deduce that $\widetilde{\Delta}(a \otimes b) \in (C \boxtimes_B C) \otimes C$. (Here $S_0$ is defined in (1.1.14), the superscript 34 indicates in which copy of $C \otimes C \otimes C$ acts, and $m^{24}(x \otimes y \otimes z \otimes u \otimes v) = x \otimes z \otimes yu \otimes v$).

Applying instead $S_0^3S_0^4(\Delta \otimes \text{id} \otimes \Delta)$, we obtain $\widetilde{\Delta}(a \otimes b) \in C \otimes (C \boxtimes_B C)$ and we are done. •

Recall that with an epimorphism $C \twoheadrightarrow B$ of coalgebras one can associate an "extension of coefficients" functor, $\boxtimes_B C$, from $B$-left comodules to $C$-right comodules. The next Definition and Propositions are known, see [Sch2].

**Definition 1.2.10.** We shall say that $C$ is $B$-coflat, if whenever $M \twoheadrightarrow N$ is an epimorphism of $B$-comodules then $M \boxtimes_B C \twoheadrightarrow N \boxtimes_B C$ is also an epimorphism of $C$-comodules.

We shall be interested in a stronger property defined by the three equivalent conditions of the following

**Proposition 1.2.11.** Let $C \twoheadrightarrow B$ be coflat. Then the following conditions are equivalent:

(a) $M \boxtimes_B C \twoheadrightarrow M$ defined by $m \otimes c \mapsto mc(c)$ is surjective for any $B$ comodule $M$,

(b) $M \boxtimes_B C = 0$ implies $M = 0$ for any $B$-comodule $M$,

(c) If $M \twoheadrightarrow N$ is a morphism of $B$-comodules and $M \boxtimes_B C \xrightarrow{\tau \boxtimes \text{id}} N \boxtimes_B C$ is an epimorphism, then also $\tau$ is an epimorphism.

If these conditions hold, then we say that $C \twoheadrightarrow B$ is faithfully coflat.

**Proof.** (a) $\implies$ (b) is clear. (b) $\implies$ (c): Let $L = \text{coker } \tau$. Then $N \boxtimes_B C \twoheadrightarrow L \boxtimes_B C$ is an epimorphism by the coflatness of $C$. The surjectivity of $\tau \boxtimes \text{id}$ implies that $L \boxtimes_B C = 0$ and by (2) this means that $L = 0$.

(c) $\implies$ (a): By (c), it is enough to show that the morphism $M \boxtimes_B C \boxtimes_B C \xrightarrow{\tau} M \boxtimes_B C$ defined by $\pi(m \otimes c \otimes d) = m \in (c) \otimes d$ is surjective, but this follows from the existence of a morphism $\tau : M \boxtimes_B C \twoheadrightarrow M \boxtimes_B C \boxtimes_B C$, $\tau(m \otimes c) \mapsto m \otimes c_{(1)} \otimes c_{(2)}$ which is a right inverse of $\pi$. •

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Proposition 1.2.12. Let $C \to B$ be faithfully coflat. Then for any comodule $M$, $M \boxtimes_B C \xrightarrow{\rho} M$, $\rho(m \otimes c) = mc(c)$ is the coequalizer of

$$M \boxtimes_B C \boxtimes_B C \rightrightarrows M \boxtimes_B C$$

where $p_1(m \otimes c \otimes d) = m \otimes ce(d)$ and $p_2(m \otimes c \otimes d) = m \otimes (c)d$.

Proof. Let $M \boxtimes_B C \to L$ be the coequalizer. Then there is a comodule epimorphism $L \to M$. Let $K = \ker(L \to M)$. Now consider

$$M \boxtimes_B C \boxtimes_B C \boxtimes_B C \rightrightarrows M \boxtimes_B C \boxtimes_B C.$$  

If we can show that the new equalizer is $M \boxtimes_B C$, then by the left exactness of the $\boxtimes_B C$ functor $L \boxtimes_B C = M \boxtimes_B C$ and therefore $K = 0$. The comultiplication $\Delta_C$ induces $\tau' : M \boxtimes_B C \boxtimes_B C \to M \boxtimes_B C \boxtimes_B C \boxtimes_B C$, $\tau'(m \otimes c \otimes d) = m \otimes c \otimes d(1) \otimes d(2)$. If $x = \sum_i m_i \otimes c_i \otimes d_i \in \ker\{M \boxtimes_B C \xrightarrow{\rho} M\}$, then

$$p_1 \tau(x) = x$$

and

$$p_2 \tau(x) = 0,$$

so that $x = p_1 \tau(x) - p_2 \tau(x)$. This shows that $L \boxtimes_B C \to M \boxtimes_B C$ is injective. •

Now we present another result concerning the redundancy of the conditions in Proposition 1.2.3.

Proposition 1.2.13. Consider a sequence $(C)$. Assume that $C \to B$ is faithfully coflat and a quotient comodule for the adjoint coaction. Then $(C)$ is exact if and only if $(1), (2), (3), (5)$ hold.

Proof. First assume only $(1), (2), (3), (5)$. Let $R$ be a coalgebra, and $\gamma : C \to H$ a morphism of Hopf algebras such that $\gamma_1 = \varepsilon$.

Claim. Let $x, y \in \text{Hom}_{\text{coalg}}(R, C)$ be such that $\pi x = \pi y$. Then $\gamma x = \gamma y$.

Proof of the Claim. First we show that $x^{-1} * y$ factorizes through $A$. Indeed, by $(5)$, we need to check that $(\pi \otimes \text{id}) \Delta(x^{-1} * y(r)) = 1 \otimes x^{-1} * y(r)$, for $r \in R$. But the left-hand side is

$$\pi \left( S(x(r_{(1)}(2)))\pi(y(r_{(2)}(1))) \otimes S(x(r_{(1)}(1)))\pi(y(r_{(2)})) \right)$$

$$= S\pi(x(r_{(2)}))\pi(y(r_{(3)})) \otimes S(x(r_{(1)}))\pi(y(r_{(4)}))$$

$$= 1 \otimes S(x(r_{(1)}))\pi(y(r_{(2)})).$$

Now $(\gamma x)^{-1} * \gamma y = \gamma \circ x^{-1} * \gamma e = \varepsilon$. Thus, $\gamma x = \gamma y$. •

Let $H = C/CA^+$. By $(5)$, $C \to B$ is normal and hence (by $(1)$ and Lemma 1.1.12 (i)), $H$ is a Hopf algebra. By $(3)$, $C \to B$ factorizes through $H$. Now apply the claim to the projection $C \to H$, $R = C \boxtimes_B C$, $x, y$ the restrictions of $c \otimes d \mapsto ce(d)$, $c \otimes d \mapsto e(c)d$. By faithful coflatness, $B \simeq H$, i.e., $(4)$ holds. •
Corollary 1.2.14. Assume that $C \xrightarrow{\pi} B$ is faithfully coflat and a quotient comodule for the adjoint coaction. Set $A = \text{H}Ker\, \pi$ and let $\iota$ be the inclusion. Then $(C)$ is exact. •

If one restricts the consideration to the category of finite dimensional Hopf algebras, then the existence problems mentioned in the Introduction both have positive answers. Indeed, it is known that finite dimensional Hopf algebras are free over its Hopf subalgebras [NZ] and the dual statement is easy to deduce and well known. Therefore, it is natural to define a simple quantum finite group by any of the two following conditions.

Definition 1.2.15. Let $H$ be a finite dimensional Hopf algebra. We shall say that $H$ is simple if it satisfies any of the two following equivalent conditions.

(a) Let $K$ be a quotient Hopf algebra of $H$ which is also a quotient comodule for the adjoint coactions. Then $K = H$ or $K = k$.

(b) Let $K$ be a Hopf subalgebra of $H$ stable by the adjoint actions. Then $K = H$ or $K = k$.

Proof of the equivalence. (a) $\implies$ (b). Let $B = \text{H}coker\, \iota$. Then, by Lemma 1.1.12 (ii), the morphism $H \rightarrow B$ is normal. So by (a), $B = k$ and $K = H$, or $B = H$ and $K = k$.

(b) $\implies$ (a). Let $C = \text{H}ker\, \pi$. By Lemma 1.1.12 (i), $C = H$ and then $K = k$, or $C = k$ and $K = H$. •

Remark. A statement more general than the above equivalence between (a) and (b) is [Sch, Thm. 1.4].

§2. Short exact sequences

Our purpose now is to study short exact sequences. More precisely, given Hopf algebras $A$ and $B$, we wish (as usual) to study extensions of $A$ by $B$, i.e., short exact sequences like $(C)$. First we recall some important facts from [BCM, DT].

Definition 2.0. Let $H$ be a Hopf algebra and $A$ an algebra. A morphism $\rightarrow: H \otimes A \rightarrow A$ of vector spaces, $h \otimes a \mapsto h \rightarrow a$, is a weak action if the following conditions are fulfilled, for all $a, b \in A, h \in H$:

(2.1) $h \mapsto ab = (h_1 \mapsto a)(h_2 \mapsto b),$
(2.2) $h \mapsto 1 = \varepsilon(h)1,$
(2.3) $1 \mapsto a = a.$

We say that a weak action is an action if, in addition,

(2.4) $h \mapsto (\ell \mapsto a) = h\ell \mapsto a$ for all $a \in A, h, \ell \in H.$

Let us fix an algebra $A$ with a weak action of a Hopf algebra $H$. For each bilinear map $\sigma: H \times H \rightarrow A$ we can define a (not necessarily associative) algebra structure on the vector space $A \otimes H$ (denoted $A \#_{\sigma} H$) as follows:

$$(a \otimes h)(b \otimes \ell) = a(h_1 \mapsto b)\sigma(h_2, \ell_{(1)}) \otimes h_{(3)}\ell_{(2)}.$$ (2.5)

When emphasis on the algebra structure is needed, the element $a \otimes h$ is denoted $a \# h.$
Proposition 2.6. (i) $1 \# 1$ is the identity of $A \#_\sigma H$ if and only if
\[ \sigma(1, h) = \sigma(h, 1) = \epsilon(h)1, \quad \forall h \in H. \] (2.7)

(ii) Assume that $\sigma(h, 1) = \epsilon(h)1$ for all $h \in H$. Then $A \#_\sigma H$ is associative if and only if for any $h, l, m \in H$ and $a \in A$ the following conditions hold:

1. (cocycle condition)
\[ [h(1) \rightarrow \sigma(l(1), m(1))]\sigma(h(2), l(2)m(2)) = \sigma(h(1), l(1))\sigma(h(2)l(2), m); \] (2.8)

2. (twisted module condition)
\[ (h(1) \rightarrow (l(1) \rightarrow a))\sigma(h(2), l(2)) = \sigma(h(1), l(1))(h(2)l(2) \rightarrow a). \] (2.9)

Now we pass to the dual notion.

Definition 2.10. Let $C$ be a coalgebra with comultiplication $\delta$, $H$ a Hopf algebra. We say that a map $\rho: C \rightarrow C \otimes H$ is a weak coaction if the following conditions are fulfilled:

1. (2.11) $(\delta \otimes \text{id})\rho = m^{24}(\rho \otimes \rho)\delta$, where $m^{24}: C \otimes C \otimes C \otimes H \rightarrow C \otimes C \otimes H$ is the map $c \otimes h \otimes d \otimes k \mapsto c \otimes d \otimes hk$.
2. (2.12) $(\epsilon_c \otimes \text{id})\rho = \epsilon_c \otimes 1$.
3. (2.13) $(\text{id} \otimes \epsilon_H)\rho = \text{id}_C$.

As usual, a weak coaction is a coaction if, in addition,

4. (2.14) $(\text{id} \otimes \Delta)\rho = (\rho \otimes \text{id})\rho$.

Example. The trivial coaction is the map $\rho: C \rightarrow C \otimes H$, $\rho(c) = c \otimes 1$.

It is well known that an extension of groups $H \rightarrow G \rightarrow K$ with $H$ abelian gives rise in a natural way to an action of $K$ on $H$. This is still the case for quantum groups. Let $1 \rightarrow A \rightarrow C \rightarrow B \rightarrow 1$ (C) be an exact sequence of Hopf algebras and assume in addition that $B$ is cocommutative. Then the adjoint action $\text{ad}: C \rightarrow C \otimes C$, $\text{ad} c = c(2) \otimes \mathcal{S}(c(1))c(3)$, induces $\text{ad}_\pi: C \rightarrow B \otimes C$ by composing with $(\pi \otimes 1)$. Since $A \rightarrow C$ is conormal, $\text{ad}_\pi(\ker \pi) = \text{ad}_\pi(A^+C) = 0$. So there is a well-defined coaction $\gamma: B \rightarrow B \otimes C$.

Now the hypothesis "$B$ cocommutative" implies that $\gamma$ lifts to $\phi: B \rightarrow B \otimes A$. Indeed, by condition (5) in Proposition 1.2.3, $A$ is the kernel of $U: C \rightarrow B \otimes C$, $U = ((\pi \otimes \text{id}) - (\epsilon \otimes \text{id}))\Delta$. Therefore, $B \otimes A = \ker(\text{id} \otimes U)$, and so we need to show that $(\text{id} \otimes U)\gamma(c) = 0$. Now

\[ (\text{id} \otimes U)\gamma(c) = \pi(c_{(3)}) \otimes \pi(\mathcal{S}(c_{(2)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} \]
\[ - \pi(c_{(3)}) \otimes \epsilon(\mathcal{S}(c_{(2)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} \]
\[ = \pi(c_{(2)}) \otimes \pi(\mathcal{S}(c_{(3)})c_{(4)}) \otimes \mathcal{S}(c_{(1)})c_{(5)} - \pi(c_{(2)}) \otimes 1 \otimes \mathcal{S}(c_{(1)})c_{(3)} \]
\[ = 0. \]
Clearly, \( \phi \) is a coaction.

Examples of extensions with \( B \) cocommutative are the cocentral extensions. One says that \((C)\) is cocentral if the following equivalent conditions hold for any \( c \in C\):

1. \( \pi(c_{(1)}) \otimes c_{(2)} = \pi(c_{(2)}) \otimes c_{(1)}, \)
2. \( \pi(c_{(2)}) \otimes S(c_{(1)})c_{(3)} = \pi(c) \otimes 1. \)

**Proof of the equivalence.** Let \( x, y: C \to B \otimes C \) be the mappings \( x(c) = \pi(c) \otimes 1, \) \( y(c) = 1 \otimes c. \) Consider the usual multiplication in \( B \otimes C. \) Then (1) reads \( x \ast y = y \ast x, \) and (2), \( y^{-1} \ast x \ast y = x. \)

Let us fix a coalgebra \( C \) with a weak coaction \( \rho \) of a Hopf algebra \( H. \) For each linear map \( \tau: C \to H \otimes H, \) we can define a (not necessarily coassociative) comultiplication \( \delta^\tau: H \ast \# C \to H \ast \# C \otimes H \ast \# C \) (here \( H \ast \# C \) denotes, for the sake of differentiating from the usual product coalgebra structure, the vector space \( H \otimes C \) as follows:

\[
\delta^\tau(h \otimes c) = h_{(1)}\tau(c_{(1)})_j \otimes \rho(c_{(2)})_i \otimes h_{(2)}\tau(c_{(1)})^j \rho(c_{(2)})^i \otimes c_{(3)}. \quad (2.15)
\]

**Proposition 2.16.** (i) \( \varepsilon_H \ast \# C := \varepsilon_H \otimes \varepsilon_C \) is a counit of \( H \ast \# C \) if and only if

\[
(\varepsilon_H \otimes \text{id})\tau(c) = \varepsilon_C(c)1_H = (\text{id} \otimes \varepsilon_H)\tau(c). \quad (2.17)
\]

(ii) Assume that \( \varepsilon_C(c)1_H = (\varepsilon_H \otimes \text{id})\tau(c). \) Then the coproduct \( \delta^\tau \) is coassociative if and only if the following two conditions hold:

1. (co-cocycle condition)
   \[
   m_{H \otimes H}(\Delta \otimes \text{id} \otimes \tau \otimes \text{id})(\tau \otimes \rho)\delta = (\text{id} \otimes m_{H \otimes H})(\text{id} \otimes \Delta \otimes \text{id} \otimes \text{id})(\tau \otimes \tau)\delta; \quad (2.18)
   \]
2. (twisted comodule condition)
   \[
   (\text{id} \otimes m_{H \otimes H})(\text{id} \otimes \Delta \otimes \text{id} \otimes \text{id})(\rho \otimes \tau)\delta = m_{H \otimes H}^{13}(\text{id} \otimes \text{id} \otimes \rho \otimes \text{id})(\tau \otimes \rho)\delta, \quad (2.19)
   \]
   where \( m_{H \otimes H}^{13}: H \otimes H \otimes C \otimes H \otimes H \to C \otimes H \otimes H \) sends \( h \otimes k \otimes c \otimes h \otimes k \mapsto c \otimes h \otimes h \otimes k. \)

The co-cocycle condition reads

\[
(\tau(c_{(1)})_j)_{(1)} \tau(\rho(c_{(2)})_i)_j \otimes (\tau(c_{(1)})_j)_{(2)} \tau(\rho(c_{(2)})_i)_j \otimes (\rho(c_{(2)})_i)_j \otimes (\tau(c_{(1)})_j)_{(1)} \tau(c_{(2)})_h \otimes (\tau(c_{(2)})_h)_{(2)} \tau(c_{(2)})^h,
\]
and the twisted comodule condition is

\[
\rho(c_{(1)})_i \otimes (\rho(c_{(1)})_i)_j \tau(c_{(2)})_h \otimes (\rho(c_{(2)})_i)_j \otimes (\rho(c_{(2)})_i)_j \otimes (\tau(c_{(1)})_j)_{(1)} \tau(c_{(2)})^h;
\]

**Proof.** We omit the superscript \( \tau \) in what follows.
(i) It is easy to see that \((\text{id} \otimes c) \delta(h \otimes c) = h(\text{id} \otimes \varepsilon)\tau(c(1)) \otimes c(2)\) and \((\varepsilon_H \otimes \text{id}) \delta(h \otimes c) = h(\varepsilon \otimes \text{id})\tau(c(1)) \otimes c(2)\). Thus (2.17) implies that \(\varepsilon_H \otimes \text{id}\) is a counit. The reciprocal is trivial.

(ii) Let us first assume that (2.18) and (2.19) are satisfied. Clearly, it suffices to check the coassociativity in an element of the form \(1 \otimes c\). Hence

\[
(\delta \otimes \text{id}) \delta(c) = (\tau(c(1)))_{(1)} \tau([\rho(c(2)))_{(1)}] \otimes \rho \left(\rho(c(2)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= \tau(c(1)))_{(1)} \tau\left(\rho(c(2)))_{(1)} \otimes \rho \left(\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= \tau(c(1)))_{(1)} \otimes (\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= \tau(c(1)))_{(1)} \otimes (\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= (\text{id} \otimes \delta) \delta(c)
\]

Here the first and the last equalities follow from the definition; the second, from

\[
\delta \otimes \text{id} = \rho \left(\rho(c(2)))_{(1)} \otimes \rho\left(\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= \tau(c(1)))_{(1)} \otimes (\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= \tau(c(1)))_{(1)} \otimes (\rho(c(3)))_{(2)} \otimes [\tau(c(1)))_{(2)}\right) \\
= (\text{id} \otimes \delta) \delta(c)
\]

which is an iteration of (2.11); the third, from the cocycle condition (2.18), and the fourth, from the twisted comodule condition (2.19).

Conversely, suppose that the coalgebra \(H \otimes C\) is coassociative. Applying \((\text{id} \otimes \varepsilon) \otimes \varepsilon\) to both sides of the equality expressing the coassociativity, we get the cocycle condition (resp., the twisted module condition; in this case we must use the assumption \(\varepsilon_H \otimes \text{id} \otimes \varepsilon\)). •

Now we consider the extension problem. First, combining Propositions 2.6 and 2.16, we show how to build extensions of Hopf algebras.

**Theorem 2.20.** Let \(A, B\) be two Hopf algebras, provided with a weak action \(\rightharpoonup\colon B \otimes A \to A\) and a weak coaction \(\rho\colon B \to B \otimes A\). Let us also fix a cocycle \(\sigma\colon B \otimes B \to A\) and a co-cocycle \(\tau\colon B \to A \otimes A\). Let \(C = A \otimes B\) denote the vector space \(A \otimes B\) provided with the multiplication (2.5) and the comultiplication (2.15) (denoted here \(\Delta\)).

Then \(C\) is a bialgebra if and only if the following conditions hold:

(i) \(\sigma\) satisfies the unitary condition (2.7), and \(\tau\) the co-unitary condition (2.17).

(ii) \(\sigma\) satisfies the cocycle condition (2.8) and the twisted module condition (2.9).

(iii) \(\tau\) satisfies the co-cocycle condition (2.18) and the twisted comodule condition (2.19).

(iv) (compatibility with the unit and counit) \(\rho(1) = \tau(1) = 1 \otimes 1, \varepsilon \circ \sigma = \varepsilon \otimes \varepsilon, \varepsilon(a \rightharpoonup b) = \varepsilon(a) \varepsilon(b)\).
(v) (compatibility between the product and coproduct)

\[
(b_{(1)} \rightarrow a)_{(1)} \sigma(b_{(2)} \otimes \widetilde{b}_{(1)})(b_{(3)} \widetilde{b}_{(2)})_j \otimes (b_{(1)} \rightarrow a)_{(2)} \sigma(b_{(2)} \otimes \widetilde{b}_{(1)})(b_{(3)} \widetilde{b}_{(2)})_j
\]

\[
= \tau(b_{(1)} h \left( \rho(b_{(2)})_i \rightarrow a_{(1)} \tau(\widetilde{b}_{(1)}) \right) \sigma \left( \rho(b_{(3)})_j \otimes \rho(\widetilde{b}_{(2)})_g \right)
\]

\[
\otimes \tau(b_{(1)} h \rho(b_{(2)})^i \rho(b_{(3)})^j \left( b_{(4)} \rightarrow a_{(2)} \tau(\widetilde{b}_{(1)}) \right) \rho(\widetilde{b}_{(2)})^p \sigma(b_{(3)} \otimes \widetilde{b}_{(3)}))
\]

\[
(2.21)
\]

\[
\rho(b_{(3)} \widetilde{b}_{(2)}); \otimes (b_{(1)} \rightarrow a) \sigma(b_{(2)} \otimes \widetilde{b}_{(1)}) \rho(b_{(3)} \widetilde{b}_{(2)})^i
\]

\[
= \rho(b_{(1)} h \rho(\widetilde{b}_{(1)}) \rho(b_{(3)} \widetilde{b}_{(2)})^i \left( b_{(2)} \rightarrow a \rho(\widetilde{b}_{(1)})^k \right) \sigma(b_{(3)} \otimes \widetilde{b}_{(2)}))
\]

\[
(2.22)
\]

for all \(a \in A, b, \widetilde{b} \in B\).

Further, assume that

(vi) (compatibility with the antipode)

\[
e(b) = S (\tau(b_j) \tau(b') = \tau(b_j) S (\tau(b')^j),
\]

\[
e(b) = \sigma (b_{(1)} \otimes S(b_{(2)}) = \sigma (S(b_{(1)}) \otimes b_{(2)}).
\]

Then \(A \# B\) is a Hopf algebra with the antipode given by

\[
S(a \# b) = \{ S[\rho(b)_{(1)}] \rightarrow S[\rho(b)_{(1)}] \otimes S[\rho(b)_{(1)}] \} S(a) \neq 1.
\]

Moreover, let \(i_A : A \rightarrow C\) and \(p_B : C \rightarrow B\) be the mappings \(a \mapsto a \otimes 1, a \otimes b \mapsto e(a)b\). Then

\[
0 \xrightarrow{i} A \rightarrow C \xrightarrow{p_B} B \rightarrow 0
\]

is an exact sequence.

**Proof.** It follows from Propositions 2.6 and 2.16 that \(C\) is an associative algebra and coassociative coalgebra if and only if (i), (ii), (iii) hold. It is clear that (iv) means that \(\epsilon_c\) is a morphism of algebras and that \(\Delta(1) = 1 \otimes 1\). Let us assume now that (2.21) and (2.22) are true. Let \(c = a \otimes \widetilde{b}, \bar{c} = \widetilde{a} \otimes \widetilde{b} \in C\). Then

\[
\Delta(\bar{c}c) = a_{(1)}(b_{(1)} \rightarrow \widetilde{a}_{(1)}) \sigma(b_{(2)} \otimes \widetilde{b}_{(1)})(b_{(3)} \widetilde{b}_{(2)})_j \otimes \rho(b_{(4)} \widetilde{b}_{(3)})_j
\]

\[
\otimes a_{(2)}(b_{(1)} \rightarrow \widetilde{a}_{(2)}) \sigma(b_{(2)} \otimes \widetilde{b}_{(1)})(b_{(3)} \widetilde{b}_{(2)})_j \rho(b_{(4)} \widetilde{b}_{(3)})_j \otimes b_{(5)} \widetilde{b}_{(4)}
\]

\[
= a_{(1)} \tau(b_{(1)} h \left( \rho(b_{(2)})_i \rightarrow \widetilde{a}_{(1)} \tau(\widetilde{b}_{(1)}) \right) \sigma \left( \rho(b_{(3)})_j \otimes \rho(\widetilde{b}_{(2)})_g \right)
\]

\[
\otimes \rho(b_{(4)} \widetilde{b}_{(4)})_i \otimes a_{(2)} \tau(b_{(1)})^p \rho(b_{(2)})^j \rho(b_{(3)})^p \rho(b_{(3)})^j
\]

\[
\left( b_{(4)} \rightarrow a_{(2)} \tau(\widetilde{b}_{(1)}) \rho(b_{(2)})^p \rho(b_{(3)})^p \sigma(b_{(6)} \otimes \widetilde{b}_{(6)}) \rho(b_{(5)} \widetilde{b}_{(4)})_i \otimes b_{(7)} \widetilde{b}_{(6)}
\right)
\]

\[
= a_{(1)} \tau(b_{(1)} h \left( \rho(b_{(2)})_i \rightarrow a_{(1)} \tau(\widetilde{b}_{(1)}) \right) \sigma \left( \rho(b_{(3)})_j \otimes \rho(\widetilde{b}_{(2)})_g \right)
\]

\[
\otimes \rho(b_{(4)} \widetilde{b}_{(4)})_i \rho(\widetilde{b}_{(3)})_i \otimes a_{(2)} \tau(b_{(1)})^p \rho(b_{(2)})^j \rho(b_{(3)})^p \rho(b_{(3)})^j
\]

\[
\left( b_{(5)} \rightarrow \widetilde{a}_{(2)} \tau(\widetilde{b}_{(1)}) \rho(b_{(2)})^p \rho(b_{(3)})^p \sigma(b_{(6)} \otimes \widetilde{b}_{(6)}) \otimes b_{(7)} \widetilde{b}_{(6)}
\right)
\]

\[
= \Delta(c) \Delta(\bar{c}).
\]
Here the first equality follows from the definitions, the second from (2.21), the third from (2.22), and the last is again by definition, complemented by iteration of (2.1). Conversely, (2.21) (resp., (2.22)) can be deduced from the equality expressing the multiplicative character of \( \Delta \) by taking \( a = 1 \) and applying \( \text{id} \otimes \varepsilon \otimes \text{id} \otimes \varepsilon \) (resp., \( \varepsilon \otimes \text{id} \otimes \text{id} \otimes \varepsilon \)).

Now assume that (2.23) and (2.24) hold, and let us check that the \( S \) given by (2.25) satisfies the axioms of the antipode. One can suppose that \( \sigma = 1 \). One has

\[
m(S \otimes \text{id}) \Delta (1 \otimes b) = \left( (S[\rho(b(2)i)](b(2))i) \otimes S[\rho(b(2)i)](b(2))i) \right) S[\rho(b(2)i)](b(2))i \otimes b(3)
\]

Here the first equality follows from the definitions; the second from (2.1) and (2.11); the third from the twisted comodule condition; the fourth from condition (2.13), and the last, from (2.23) and (2.24). In a similar way, one proves \( m(\text{id} \otimes S) \Delta (1 \otimes b) = \varepsilon(b) \).

Conditions (1) to (5) of Proposition 1.2.3 are also easy to verify, and the Proposition follows.

Remark. Conditions (2.23) and (2.24) are sufficient but not necessary to have an antipode. A more satisfactory answer is given below (Lemma 3.2.17) in the context of cleft extensions. The general question, however, remains open.

Definition 2.26. Let \( A, B \) be two Hopf algebras. A data \( \mathcal{V} = (\sigma, \rho, \tau) \) is compatible if it satisfies conditions (i), \ldots, (v) in the Theorem above. If in addition, \( A \#_\sigma B \) is a Hopf algebra, then we say that \( \mathcal{D} \) is a Hopf data.

Claim. The requirements (2.21), (2.22) are equivalent to the following set of conditions

\[
\Delta(b(1) \rightarrow a)\tau(b(2)) = \tau(b(1)) \left( (\rho(b(2)i) \rightarrow a(1)) \otimes \rho(b(2)i)(b(3) \rightarrow a(2)) \right).
\]

(A)

\[
1 \otimes \sigma(b(1))(b(2)) = \rho(b(1))(1 \otimes (b(2) \rightarrow a))(1 \otimes \sigma(b(3))(b(2))).
\]

(B)

\[
\Delta \left( \sigma(b(1) \otimes b(2)) \right) \tau(b(2)) = \tau(b(1)) \left( \rho(b(2)i) \rightarrow \tau(b(1))p \otimes \rho(b(2)i)(b(3) \rightarrow \tau(b(1))p) \right) \sigma(\rho(b(4)i) \otimes \rho(b(4)i)(b(5) \rightarrow \rho(b(4)i)(b(6) \otimes b(3)))).
\]

(C)

\[
\Delta \left( \sigma(b(1) \otimes b(2)) \right) \tau(b(2)) = \tau(b(1)) \left( \rho(b(2)i) \rightarrow \tau(b(1))p \otimes \rho(b(2)i)(b(3) \rightarrow \tau(b(1))p) \right) \sigma(\rho(b(4)i) \otimes \rho(b(4)i)(b(5) \rightarrow \rho(b(4)i)(b(6) \otimes b(3)))).
\]

(D)
**Proof.** (A) (resp., (D)) follows letting $\tilde{b} = 1$ (resp., $a = 1$) in (2.21); (B) (resp., (C)) follows letting $a = 1$ (resp., $\tilde{b} = 1$) in (2.22). Conversely, applying consecutively (B) and then (C) to the left hand-side of (2.22), one arrives to the right one; similarly, (2.21) follows from (A), (D), and (C).

**Remark.** If $A$ is commutative and $B$ cocommutative and $\sigma$ (or $\tau$) is invertible, then (A) (resp, (B), (D)) is equivalent to [Si, (3.2)] (resp., [Si, (3.2)\*], [Si, (2.14)] = [Si, (4.8)]); moreover (C) is void in such case. On the other hand, (A), . . . , (D) were first found by Majid [Mj]. See also [Hf].

Now we shall give an example of reconstruction of a Hopf data from an exact sequence. A more general statement will be given in the next section. Let us fix an exact sequence

$$0 \rightarrow A \xrightarrow{i} C \xrightarrow{\pi} B \rightarrow 0.$$ 

In particular, we consider $C$ as a left $A$-module and $B$-comodule in the obvious way. In the rest of this section we shall assume the existence of a linear isomorphism $\mathcal{F}: C \rightarrow A \otimes B$ (whose inverse is denoted by $\mathcal{G}$) satisfying the following conditions:

1. $\mathcal{F}$ is a morphism of $A$-modules, i.e., $\mathcal{F}(ac) = (a \otimes 1)\mathcal{F}(c)$,
2. $\mathcal{F}$ is morphism of $B$-comodules, i.e., the following diagram commutes:

$$\begin{array}{ccc}
C & \xrightarrow{(\text{id} \otimes \pi)\Delta} & C \otimes B \\
\mathcal{F} & & \downarrow \mathcal{F} \otimes \text{id} \\
A \otimes B & \xrightarrow{(\text{id} \otimes \Delta)} & A \otimes B \otimes B.
\end{array}$$

3. $\mathcal{F}(1) = 1 \otimes 1$,
4. $(\varepsilon_A \otimes \varepsilon_B)\mathcal{F} = \varepsilon_C$,
5. $\mathcal{G}(1 \otimes b_{(1)})\mathcal{G}(1 \otimes Sb_{(2)}) = \varepsilon(b) = \mathcal{G}(1 \otimes Sb_{(1)})\mathcal{G}(1 \otimes b_{(2)})$,
6. $(p_A\mathcal{F}(c_{(1)})S(p_A\mathcal{F}(c_{(2)})) = \varepsilon(c) = S(p_A\mathcal{F}(c_{(1)}))(p_A\mathcal{F}(c_{(2)}))$.

Here $p_A: A \otimes B \rightarrow A$ is the canonical projection, $p_A = \text{id} \otimes \varepsilon_B$; in the same vein, $i_A: A \rightarrow A \otimes B$ is the canonical inclusion $a \mapsto a \otimes 1$, and similarly for $i_B, p_B$. One easily deduces from the first four above axioms that $\mathcal{F}i = i_A, p_B\mathcal{F} = \pi$.

Thanks to such $\mathcal{F}$, we shall obtain the exact sequence (C) as in the preceding proposition. Let us first introduce $\bar{\sigma}: B \otimes B \rightarrow C$, $\bar{\tau}: C \rightarrow A \otimes A, \bar{\rho}: C \rightarrow B \otimes A$ by

$$\bar{\sigma}(b \otimes \tilde{b}) = \mathcal{G}(1 \otimes b_{(1)})\mathcal{G}(1 \otimes \tilde{b}_{(1)})\mathcal{G}(1 \otimes S\tilde{b}_{(2)}Sb_{(2)}),$$

$$b = a = \mathcal{G}(1 \otimes b_{(1)})a\mathcal{G}(1 \otimes Sb_{(2)}),$$

$$\bar{\tau}(c) = (S\mathcal{F}(c_{(1)}))_{(1)}p_A\mathcal{F}(c_{(2)}) \otimes (S\mathcal{F}(c_{(1)}))_{(2)}p_A\mathcal{F}(c_{(3)}),$$

$$\bar{\rho}(c) = \pi(c_{(2)}) \otimes S\mathcal{F}(c_{(1)})p_A\mathcal{F}(c_{(3)}).$$
Lemma 2.33. These maps give rise to $\sigma : B \otimes B \to A, \quad \tau : B \to A \otimes A, \quad \rho : B \to B \otimes A$.

Proof. It is straightforward. For example, let us show that the image of $\tilde{\sigma}$ is contained in $A$:

$$(id \otimes \pi)\Delta \tilde{\sigma}(b \otimes \tilde{b})$$

$$= (id \otimes \pi)\Delta \sigma(1 \otimes b(1))\Delta \tilde{\sigma}(1 \otimes \tilde{b}(1))$$

$$= (G(1 \otimes b(1)) \otimes b(2)) (G(1 \otimes \tilde{b}(1)) \otimes \tilde{b}(2)) (G(1 \otimes \tilde{b}(2)) S b(4)) \otimes S b(3) S b(3))$$

$$= \tilde{\sigma}(b \otimes \tilde{b}) \otimes 1.$$ 

The rest is similar. •

Proposition 2.34. $F$ is an isomorphism of Hopf algebras from $C$ onto $A^\# \#_\sigma B$.

Proof. $\rho$ (resp. $\rho$) is a weak coaction (resp. action) because of (2.32) (resp. (2.31)).

These two axioms also imply $\sigma(b \otimes 1) = \sigma(1 \otimes b) = \epsilon(b)$ and $\epsilon \otimes \epsilon \tau = (id \otimes \epsilon)\tau$.

From this, we deduce $(a \otimes 1)(1 \otimes b) = a \otimes b$.

So let us prove that $G$ is a morphism of algebras:

$$G((a \otimes b)(\tilde{a} \otimes \tilde{b}))$$

$$= G((a(1 \otimes b(1)) \tilde{a}(1) \otimes b(2) \tilde{b}(2))$$

$$= G(\sigma(1 \otimes \tilde{b}(1)) \otimes b(3) \tilde{b}(2)) G(1 \otimes \tilde{b}(1)) G(1 \otimes \tilde{b}(2) S b(4)) \otimes b(5) \tilde{b}(3))$$

$$= G(\sigma(1 \otimes \tilde{b}(1)) e(b(2)) G(\tilde{a}(1) \otimes \tilde{b}(1)) G(1 \otimes \tilde{b}(2) S b(4)) G(1 \otimes b(3) \tilde{b}(3))$$

$$= G((a \otimes b) \tilde{G}(\tilde{a} \otimes \tilde{b}).$$

Here the first equality holds by definition, the second follows because $G$ is a morphism of $A$-modules, and the third is a consequence of (2.31).

Now we prove that $F$ is a morphism of coalgebras:

$$\delta^* (F(c)) = [p_A F(c(1))]_{(1)}^\tau (\pi c(2)) i \otimes \rho(\pi c(3)) i$$

$$\otimes [p_A F(c(1))]_{(2)}^\tau (\pi c(4)) i \otimes \pi c(4)$$

$$= [p_A F(c(1))]_{(1)} [S p_A F(c(2))]_{(1)} p_A F(c(3)) \otimes \pi c(6)$$

$$\otimes [p_A F(c(1))]_{(2)} [S p_A F(c(2))]_{(2)} p_A F(c(4)) S p_A F(c(5)) p_A F(c(7)) \otimes \pi c(8)$$

$$= p_A F(c(1)) \otimes \pi c(2) \otimes p_A F(c(3) \pi c(4)$$

$$= (F \otimes F) \Delta (c),$$

taking into account the relation $F(c) = p_A F(c(1)) \otimes \pi c(2)$. •
§3. Isomorphisms of extensions

In this section, we study whether two extensions built from compatible data (cf. Definition 2.26) are isomorphic, in terms of the data. We obtain a complete answer in the case of cleft extensions, see 3.2. Our methods are mostly an extension of those in [D]; S. Montgomery communicated us that Blattner and she independently obtained some of the results in [D] (unpublished).

3.1. Rudimentary non-abelian cohomology. Let $A, B$ be two Hopf algebras and let $\text{Reg}(B, A)$ be the group of linear morphisms from $B$ to $A$ which are invertible with respect to the convolution product [Sw]. Set also

\[
\text{Reg}_1(B, A) = \{ \phi \in \text{Reg}(B, A) : \phi(1) = 1 \},
\]
\[
\text{Reg}_e(B, A) = \{ \phi \in \text{Reg}(B, A) : \varepsilon \phi = \varepsilon \},
\]
\[
\text{Reg}_{1,e}(B, A) = \text{Reg}_1(B, A) \cap \text{Reg}_e(B, A);
\]

these are subgroups of $\text{Reg}(B, A)$.

Let $\text{Weak}(B \otimes A, A)$ (\text{Coweak}(B, B \otimes A)) be the set of all weak actions of the Hopf algebra $B$ on the algebra $A$, cf. Definition 2.0 (weak coactions of the Hopf algebra $A$ on the coalgebra $B$, cf. Definition 2.10).

Lemma 3.1.1. (i) Let $\phi \in \text{Weak}(B \otimes A, A)$, $\sigma \in \text{om}(B \otimes B, A)$, $\phi \in \text{Reg}(B, A)$. The formulas

\[
\phi^\sigma(b) = \phi(b) b(1) b(2) \twoheadrightarrow a \phi^{-1}(b(3)),
\]
\[
\phi^\sigma(b \otimes d) = \phi(b) (b(2) \twoheadrightarrow \phi(d(1))) \sigma(b(3) \otimes d(2)) \phi^{-1}(b(4) d(3))
\]

provide a left action of $\text{Reg}_1(B, A)$ on $\text{Weak}(B \otimes A, A) \times \text{om}(B \otimes B, A)$, i.e., $\phi(\cdot, \sigma) = (\phi^\sigma, \sigma)$. (Here $A$ needs to be only an algebra.)

In addition, if $\sigma$ is invertible, then $\phi^\sigma$ also is; in fact

\[
(\phi^\sigma)^{-1}(b \otimes d) = \phi(b) d(1) \sigma^{-1}(b(2) \otimes d(2)) (b(3) \twoheadrightarrow \phi^{-1}(d(3))) \phi^{-1}(b(4)).
\]

(ii) In a similar way, let $\rho \in \text{Coweak}(B, B \otimes A)$, $\tau \in \text{hom}(B, A \otimes A)$. The formulas

\[
\rho^\phi(b) = (1 \otimes \phi^{-1}(b(1))) \rho(b(2)) (1 \otimes \phi(b(3)));
\]
\[
\tau^\phi(b) = \Delta \phi^{-1}(b(1)) \tau(b(2)) (\phi \otimes \text{id}) \rho(b(3)) (1 \otimes \phi(b(4)))
\]

provide a right action of $\text{Reg}_e(B, A)$ on $\text{Coweak}(B, B \otimes A) \times \text{hom}(B, A \otimes A)$, i.e., $(\rho, \tau) \phi = (\rho^\phi, \tau^\phi)$. (Here $B$ needs to be only a coalgebra).

Moreover, if $\tau$ is invertible, then $\tau^\phi$ also is, and the inverse is

\[
(\tau^\phi)^{-1}(b) = (1 \otimes \phi^{-1}(b(1))) (\phi^{-1} \otimes \text{id}) \rho(b(2)) \tau^{-1}(b(3)) \Delta \phi(b(4)).
\]

Proof. (i) is essentially proved in [D], so we prove only the dual statement (ii). First let us show that $\rho^\phi$ is a weak coaction. Condition (2.12) in Definition 2.10 is obvious and
condition (2.13) follows from $\varepsilon \phi = \varepsilon$. Let us proceed with condition (2.11). On the one hand,

$$(\Delta \otimes \text{id}) \rho^\phi(b) = (1 \otimes \phi^{-1}(b_{(1)})) (\Delta \otimes \text{id}) \rho(b_{(2)}) (1 \otimes \phi(b_{(3)}));$$

on the other hand,

$$m^{24}(\rho^\phi \otimes \rho^\phi)(b_{(1)} \otimes b_{(2)})$$

$$= \rho(b_{(2)}) \circ \rho(b_{(5)}) \otimes \phi^{-1}(b_{(1)}) \rho(b_{(2)}) \phi(b_{(3)}) \phi^{-1}(b_{(4)}) \rho(b_{(5)}) \phi(b_{(6)}),$$

and condition (2.11) for $\rho^\phi$ holds.

Let us prove now the group action axioms. Let also $\psi \in \text{Reg}_e(B, A)$. Then

$$(\rho^\phi)^\psi(b) = (1 \otimes \psi^{-1}(b_{(1)})) \rho^\phi(b_{(2)}) (1 \otimes \psi(b_{(3)}))$$

$$= (1 \otimes \psi^{-1}(b_{(1)})) (1 \otimes \phi^{-1}(b_{(2)})) \rho(b_{(3)}) (1 \otimes \phi(b_{(4)})) (1 \otimes \psi(b_{(5)}))$$

$$= \rho(b)^{\phi \ast \psi}.$$ 

On the other hand,

$$(\tau^\phi)^\psi(b) = \Delta \psi^{-1}(b_{(1)}) \tau^\phi(b_{(2)})(\psi \circ \text{id}) \rho(b_{(3)}) (1 \otimes \psi(b_{(4)}))$$

$$= \Delta(\phi \ast \psi)^{-1}(b_{(1)}) \tau(b_{(2)})(\phi \circ \text{id}) \rho(b_{(3)}) (1 \otimes \phi(b_{(4)})) (1 \otimes \phi^{-1}(b_{(5)}))$$

$$= (\psi \circ \text{id}) \rho(b_{(6)}) (1 \otimes \phi(b_{(7)})) (1 \otimes \psi(b_{(8)}))$$

$$= \tau^{\phi \ast \psi};$$

here one uses the axiom (2.11). •

The following simple Lemma is important for what follows.

Lemma 3.1.6. (i) Let $(-, \sigma) \in \text{Weak}(B \otimes A, A) \times \text{hom}(B \otimes B, A), \phi \in \text{Reg}_1(B, A)$. Then the (not necessarily associative) algebras $A \#_\sigma B$ and $A \#_\sigma B$ are isomorphic.

(ii) Let $(\rho, \tau) \in \text{Cowek}(B, B \otimes A) \times \text{hom}(B, A \otimes A), \psi \in \text{Reg}_e(B, A)$. Then the (not necessarily coassociative) coalgebras $A^\ast \# B$ and $A^\ast \# B$ are isomorphic.

Proof. (i) Let $F = F_\phi: A \#_\sigma B \rightarrow A \#_\sigma B$ be given by

$$F(a \# b) = a \phi^{-1}(b_{(1)}) \# b_{(2)}.$$ 

Clearly, $F_\phi F_\psi = F_{\phi \ast \psi}$. Then, on the one hand,

$$F((a \# b)(c \# d)) = F(a(b_{(1)} \rightarrow c) \sigma(b_{(2)} \otimes d_{(1)}) \# b_{(3)} d_{(2)})$$

$$= a(b_{(1)} \rightarrow c) \sigma(b_{(2)} \otimes d_{(1)}) \phi^{-1}(b_{(3)} d_{(2)}) \# b_{(4)} d_{(3)};$$

on the other hand,

$$F(a \# b) F(c \# d)$$

$$= (a \phi^{-1}(b_{(1)}) \# b_{(2)}) (c \phi^{-1}(d_{(1)}) \# d_{(2)})$$

$$= a \phi^{-1}(b_{(1)}) (b_{(2)} \phi \rightarrow c \phi^{-1}(d_{(1)})) \phi(b_{(3)} \otimes d_{(2)}) \# b_{(4)} d_{(3)}$$

$$= a(b_{(1)} \rightarrow c \phi^{-1}(d_{(1)}))(b_{(2)} \rightarrow \phi(d_{(2)})) \sigma(b_{(3)} \otimes d_{(3)}) \phi^{-1}(b_{(4)} d_{(4)}) \# b_{(5)} d_{(5)};$$
we have only used the definitions and (2.1).

(ii) Let \( \mathcal{G} : A \#^r B \to A \#^\varphi B \) be the mapping

\[
\mathcal{G}(a \# b) = a\psi(b(1)) \otimes b(2).
\]

Then

\[
(G \otimes G)\Delta(a \# b)
= (G \otimes G)(a(1)\tau(b(1))_j \otimes \rho(b(2))_i \otimes a(2)\tau(b(2))_j \otimes \rho(b(3))_i \otimes b(3))
= a(1)\psi((b(1))_j \otimes \rho((b(2)))_i \otimes a(2)\psi((b(1))_j \otimes \rho((b(2)))_i \otimes b(2)) \otimes b(3)
\]

and this is equal to

\[
\Delta G(a \# b) = \Delta(a\psi(b(1)) \otimes b(2))
= a(1)\psi((b(1))_j \otimes \rho((b(2)))_i \otimes a(2)\psi((b(1))_j \otimes \rho((b(2)))_i \otimes b(2)) \otimes b(3)
\]

by (3.1.4), (3.1.5).

Observe that, if \( A \#^r B \) denotes the "bialgebra" obtained from \((-\sigma, \tau) \in \text{Weak}(B \otimes A, A) \times \text{hom}(B \otimes B, A)\) (without associativity, coassociativity, or compatibility between the multiplication and comultiplication), and \( \phi \in \text{Reg}_l \beta(B, A) \), then \( A \#^\varphi B \) and \( A \#^r B \) are isomorphic, as follows from the proof of the preceding Lemma.

Let us now introduce the following notation:

\[
Z^{1,0}(B, A) = \{(-\sigma, \tau) \in \text{Weak}(B \otimes A, A) \times \text{hom}(B \otimes B, A) : \sigma(h, 1) = \epsilon(h)1, (-\sigma, \tau) \text{ satisfies the cocycle condition (2.8)} \}
\text{and the T.M.C. (2.9)},
\]

\[
Z^{0,1}(B, A) = \{(\rho, \tau) \in \text{Coweak}(B, B \otimes A) \times \text{hom}(B, A \otimes A) : (\epsilon \otimes \text{id})\tau = \epsilon, (\rho, \tau) \text{ satisfies the co-cocycle condition (2.18)} \}
\text{and the T.C-M.C. (2.19)},
\]

\[
Z^1(B, A) = \{D = (-\sigma, \rho, \tau) \in Z^{1,0}(B, A) \times Z^{0,1}(B, A) : D \text{ is a compatible data}\},
\]

\[
Z^1(B, A) = \{D = (-\sigma, \rho, \tau) \in Z^{1,0}(B, A) \times Z^{0,1}(B, A) : D \text{ is a Hopf data}\}.
\]

We shall consider the left action of \( \text{Reg}_l \beta(B, A) \) on the set of data \( \text{Weak}(B \otimes A, A) \times \text{hom}(B \otimes B, A) \times \text{Cowork}\times \text{hom}(B, A \otimes A) \) given by \( \phi D = \phi D \), where if \( D = (-\sigma, \rho, \tau) \) then \( \phi D = (\phi - \sigma, \rho^{\phi^{-1}}, \tau^{\phi^{-1}}) \).
Proposition 3.1.7. (i) $Z^1,0$ is stable by the action of $\text{Reg}_1(B,A)$ defined in Lemma 3.1.1 (i). Let $H^{1,0}(B,A)$ be the quotient of $Z^{1,0}$ by the action of $\text{Reg}_1(B,A)$. Then $(-,\sigma) \mapsto A\#_\sigma B$ induces a mapping from $H^{1,0}(B,A)$ to the set of isomorphism classes of $B$-extensions of algebras of $A$ (see 3.2 below).

(ii) $Z^{0,1}$ is stable by the action of $\text{Reg}_e(B,A)$ defined in Lemma 3.1 (ii). Let $H^{0,1}(B,A)$ be the quotient of $Z^{0,1}$ by the action of $\text{Reg}_e(B,A)$. Then $(\rho,\tau) \mapsto A^\tau\#_\rho B$ gives rise to a mapping from $H^{0,1}(B,A)$ to the set of isomorphism classes of $A$-extensions of coalgebras of $B$.

(iii) $Z^1$ is stable by the action of $\text{Reg}_{1,e}(B,A)$ defined above. Let $H^1(B,A)$ be the quotient of $Z^1(B,A)$ by the action of $\text{Reg}_{1,e}(B,A)$. Then $(-,\sigma,\rho,\tau) \mapsto A^\tau\#_\rho B$ gives rise to a mapping from $H^1(B,A)$ to the set of isomorphism classes of $B$-extensions of bialgebras of $A$.

(iv) The statement obtained from (iii) by replacing “compatible” by “Hopf” is still true.

Proof. (i) follows from Lemma 3.1.6 (i) and Proposition 2.6; in turn, (ii) follows from Lemma 3.1.6 (ii) and Proposition 2.16. (iii) follows from the preceding and 2.20, and (iv) from (iii) and [Sw, 4.0.4]. •

3.2. Cleft extensions. In this subsection, we first recall some facts about cleft extensions of algebras from [D] and then state their dual analogues.

Let $B$ be a Hopf algebra. A $B$-comodule algebra is an algebra $C$, which is simultaneously a $B$-comodule whose structural morphism $\gamma: C \rightarrow C \otimes B$ is a morphism of algebras. The subalgebra of invariants is

$$\{c \in C : \gamma(c) = c \otimes 1\}.$$ 

Let $A$ be the subalgebra of invariants: then one says that $C$ is a $B$-extension of $A$ (more precisely, an extension of algebras), and denotes $C/A$. A morphism of extensions preserves, by definition, the algebra and the comodule structures, and induces the identity on the algebra of invariants. An extension is cleft if there exists a $\chi \in \text{Reg}_1(B,C)$ such that

$$\gamma\chi = (\chi \otimes \text{id})\Delta;$$

such $\chi$ is called a section. Note that [D]

$$\gamma\chi^{-1}(b) = \chi^{-1}(b_{(2)}) \otimes S(b_{(1)}).$$

Assume that $B$ acts weakly on an algebra $A$ and let $\sigma$ be a cocycle satisfying (2.3), (2.4), (2.5). Then $C = A\#_\sigma B$ is a $B$-extension of $A$, up to identifying the last with the subalgebra of invariants via the map $a \mapsto a \# 1$. Here, the comodule map is

$$a \# b \mapsto a \# b_{(1)} \otimes b_{(2)}.$$ 

Moreover, the morphism $\chi: B \rightarrow C$, $\chi(b) = 1 \# b$ is a section if $\sigma$ belongs to $\text{Reg}(B \otimes B, A)$; the only non-trivial part follows from [BM, Proposition 1.8]. Let us recall from loc. cit. that

$$\chi^{-1}(b) = \sigma^{-1}(Sb_{(2)} \otimes b_{(3)}) \otimes Sb_{(1)}.$$ 

Thus, in such case, $A\#_\sigma B$ is a cleft $B$-extension of $A$. Conversely, one has the following important fact:
Theorem 3.2.1 ([DT, Theorem 11]). Let $C$ be a cleft $B$-extension of $A$, $\chi: B \to C$ a section. Define $\rightarrow: B \otimes A \to A$, $\sigma: B \otimes B \to A$, by

$$b \rightarrow a = \chi(b(1))a\chi^{-1}(b(2)), \quad (3.2.2)$$

$$\sigma(b \otimes b) = \chi(b(1))\chi(b(1))\chi^{-1}(b(2))\chi^{-1}(b(2)). \quad (3.2.3)$$

Then $\rightarrow$ is a weak action, $\sigma$ is invertible, the algebra $A \#_\sigma B$ is associative, and the cleft extensions $C/A$ and $A \#_\sigma B/A$ are isomorphic via

$$a \# b \mapsto a\chi(b), \quad c \mapsto c(0)\chi^{-1}(c(1)) \# c(2).$$

For obvious reasons, now we change our notation. Let $A$ be a Hopf algebra. An $A$-module coalgebra is a coalgebra $C$, which is simultaneously an $A$-module whose structural morphism $\mu: A \otimes C \to C$ is a morphism of coalgebras. The coalgebra of covariants is $C/A^+C$. If $B$ is the coalgebra of covariants, then one says that $C$ is an $A$-extension of $B$ (of coalgebras) and denotes $C \backslash B$. We shall denote by $\pi$ the canonical morphism $C \to B$. Morphisms of extensions of coalgebras preserve both the module and the coalgebra structures and induce the identity on the coalgebra of covariants. An extension is cleft if there exists a $\xi \in \text{Reg}_e(C, A)$ such that

$$\xi(ac) = a\xi(c), \quad \forall a \in A, \ c \in C; \quad (3.2.4)$$

then $\xi$ is called a retraction. Note that

$$\xi^{-1}(ac) = \xi^{-1}(c)S(a).$$

Now assume that $B$ is a coalgebra provided with a weak coaction $\rho: B \to B \otimes A$. Let us also fix a co-cocycle $\tau: B \to A \otimes A$ satisfying $(e \otimes \text{id})\tau = e$, (2.13), (2.14). Then $C = A \#^\tau B$ is an $A$-extension of $B$, up to identifying the last with the coalgebra of covariants via the map $a \# b \mapsto e(a)b$. $A$ acts by $a.(a \# b) = a\alpha \# b$. Moreover, the morphism $\xi: C \to A$, $\xi(a \# b) = ae(b)$ is a retraction if $\tau$ belongs to $\text{Reg}(B, A \otimes A)$. This follows from the following Lemma, whose proof runs as that of [BM, Proposition 1.8].

Lemma 3.2.5. $\xi$ is invertible if and only if $\tau$ is.

Proof. Assume that $\xi$ is invertible. One easily proves that

$$\tau(\pi c) = \Delta(\xi^{-1}(c_1))\xi(c_2) \otimes \xi(c_3);$$

from this one finds the expression

$$\tau^{-1}(\pi c) = \xi^{-1}(c_2) \otimes \xi^{-1}(c_1)\Delta(\xi(c_3)).$$

Conversely, assume that $\tau$ is invertible. Let

$$\eta(a \# b) = \tau^{-1}(b)_kS(\alpha^{-1}(b^k)).$$
One easily proves that $\xi \ast \eta$ is the trivial morphism. The other multiplication is the only nontrivial point of the Lemma. One has

$$\eta \ast \xi(a \# b) = \varepsilon(a)\tau^{-1}(\rho(b^{(2)}))_k S \left(\tau(b^{(1)})_j \tau^{-1}(\rho(b^{(2)}))_k\right)^i \tau(b^{(1)})^j \rho(b^{(2)})^i. \quad (3.2.6)$$

Multiplying the co-cocycle condition on $b^{(2)}$ by $(\Delta \otimes \text{id})\tau^{-1}(b^{(1)})$, one gets

$$(\tau \otimes \text{id})\rho(b)$$

$$= [\tau^{-1}(b^{(1)})^{(1)}]_j \tau(b^{(2)}) j$$

$$\otimes [\tau^{-1}(b^{(1)})^{(2)}]_k [\tau(b^{(2)})]^{(1)}_l \rho(b^{(3)})^k \otimes [\tau^{-1}(b^{(1)})]_m [\tau(b^{(2)})]_l \tau(b^{(3)})^k.$$ \quad (3.2.7)

On the other hand, since $\rho$ is a weak coaction, the inverse of the mapping $b \mapsto (\tau \otimes \text{id})\rho(b)$ is $b \mapsto (\tau^{-1} \otimes \text{id})\rho(b)$. Then one deduces from (3.2.7) that

$$(\tau^{-1} \otimes \text{id})\rho(b)$$

$$= \tau^{-1}(b^{(2)}) j [\tau(b^{(3)})]^{(1)}$$

$$\otimes \tau^{-1}(b^{(1)}) h [\tau^{-1}(b^{(2)})]^{(1)}_l [\tau(b^{(3)})]^{(2)}_m \otimes \tau^{-1}(b^{(1)})^k [\tau^{-1}(b^{(2)})]^{(2)}_l \tau(b^{(3)})^k.$$ \quad (3.2.8)

Now one can proceed with (3.2.6), thanks to (3.2.8):

$$\tau^{-1}(\rho(b^{(2)}))_k S \left(\tau(b^{(1)})_j \tau^{-1}(\rho(b^{(2)}))_k\right)^i \tau(b^{(1)})^j \rho(b^{(2)})^i$$

$$= \tau^{-1}(b^{(3)})_h [\tau(b^{(4)})_l]^{(1)} S \left(\tau(b^{(1)})_j \tau^{-1}(b^{(2)})_h [\tau^{-1}(b^{(3)})^k]^{(1)} \tau(b^{(4)})]^{(2)}_l \tau(b^{(3)})^k\right)$$

$$\tau(b^{(1)})^j \tau^{-1}(b^{(2)})^k [\tau^{-1}(b^{(3)})^k]^{(1)} \tau(b^{(4)})^g \tau^{-1}(b^{(2)})^g$$

$$= \tau^{-1}(b)_h \varepsilon(\tau^{-1}(b)_h)$$

$$= (\text{id} \otimes \varepsilon)\tau^{-1}(b)$$

$$= \varepsilon(b).$$

This proves that $\eta \ast \xi$ is the trivial morphism and hence $\xi$ is invertible. •

Now we prove the dual version of Theorem 3.2.1.

**Proposition 3.2.9.** Let $C$ be a clef $A$-extension of a coalgebra $B$ and let $\xi \in \text{Reg}_B(C, A)$ be a retraction. Define $\overline{\rho}: C \rightarrow B \otimes A$, $\overline{\tau}: C \rightarrow A \otimes A$ by

$$\overline{\rho}(c) = \pi(c^{(2)}) \otimes \xi^{-1}(c^{(1)})_l \xi(c^{(3)}) \quad \text{and} \quad \overline{\tau}(c) = \Delta \xi^{-1}(c^{(1)})_l \xi(c^{(2)}) \otimes \xi(c^{(3)}). \quad (3.2.10)$$

$$\overline{\tau}(c) = \Delta \xi^{-1}(c^{(1)})_l \xi(c^{(2)}) \otimes \xi(c^{(3)}). \quad (3.2.11)$$
Then \( \overline{\rho}, \overline{\tau} \) give rise to \( \rho : B \rightarrow B \otimes A, \tau : B \rightarrow A \otimes A; \rho \) is a weak coaction; \( (\rho, \tau) \) belongs to \( Z^{0,1}(B, A) \), and hence \( A \ast \# B \) is a cleft \( A \)-extension of \( B \). Moreover, the mapping \( C \rightarrow A \ast \# B \) given by
\[ c \mapsto \xi(c(1)) \otimes \pi(c(2)) \]
is an isomorphism of \( A \)-extensions of \( B \), whose inverse is induced by the map \( A \otimes C \rightarrow C \),
\[ a \otimes c \mapsto a \xi^{-1}(c(1))c(2). \]

**Proof.** It is straightforward; for example one proves that both sides of the co-cocycle condition are equal to \( (\Delta \otimes \text{id})\Delta \xi^{-1}(c(1))(\xi^{-\otimes 2} \Delta(c(2))) \); and both sides of the twisted comodule condition are equal to \( \pi(c(3)) \otimes \Delta \xi^{-1}(c(1))(\xi^{-\otimes 2} \Delta(c(2))) \). We leave the details to the reader. •

Let us now consider
\[
Z^{{1,0}}(B, A) = \{ (-, \sigma) \in Z^{{1,0}}(B, A) : \sigma \text{ is invertible} \},
\]
\[
Z^{{0,1}}(B, A) = \{ (\rho, \tau) \in Z^{{0,1}}(B, A) : \tau \text{ is invertible} \}.
\]

Let \( H_{*}^{{1,0}} \) (resp., \( H_{*}^{{0,1}} \)) be the quotient of \( Z^{{1,0}} \) (resp., \( Z^{{0,1}} \)) by the action of \( \text{Reg}_{*} \) (resp., \( \text{Reg}_{*} \)).

**Proposition 3.2.12.** The mappings defined in Proposition 3.1.7 give rise to bijections
\[
H_{*}^{{1,0}}(B, A) \simeq \{ \text{isomorphism classes of cleft extensions of algebras } C/A \},
\]
\[
H_{*}^{{0,1}}(B, A) \simeq \{ \text{isomorphism classes of cleft extensions of coalgebras } C\backslash B \}.
\]

**Proof.** The first is proved in [D]. As for the second, surjectivity follows from Proposition 3.2.9. Let us prove injectivity. Let \( (\rho, \tau), (\rho_1, \tau_1) \in Z_{*}^{0,1}(B, A) \) and let \( \eta : C = A \ast \# B \rightarrow A \ast \# B = C_1 \) be an isomorphism of extensions. Let \( \xi : C_{1} \rightarrow A, \xi_1 : C_1 \rightarrow A \) be the rejections defined above and let \( \bar{\xi} = \xi_1 \eta, \bar{\tau} = \xi_1^{-1} \ast \bar{\xi} \). Then \( \bar{\tau} \) factorizes through \( \nu \in \text{Reg}_{*}(B, A) \); indeed, \( \bar{\nu}(ac) = \xi_1^{-1}(a_1)\xi(a_2)c(1) = \xi_1^{-1}(c_1)S(a_1)a_2\bar{\xi}(c(2)) = \varepsilon(a)\bar{\nu}(c) \).

\[
\rho''(\pi c) = (1 \otimes \nu^{-1}(c(1)))\rho(\pi c(2))(1 \otimes \nu\pi(c(3)))
\]
\[
= \left(1 \otimes (\bar{\xi})^{-1}(c(1))\xi(c(2))\right)\left(\pi c(4) \otimes \xi^{-1}(c(3))\xi(c(5))\right)\left(1 \otimes \xi^{-1}(c(6))\bar{\xi}(c(7))\right)
\]
\[
= \pi c(2) \otimes (\xi_1)^{-1}(\eta c(1))\xi_1(\eta c(3))
\]
\[
= \rho_1(\pi c).
\]

Here one uses all the requirements to a morphism of extensions. Similarly, \( \tau'' = \tau_1 \), and injectivity follows. •
**Definition 3.2.13.** Let

\[ 0 \to A \xrightarrow{i} C \xrightarrow{\pi} B \to 0 \]  

be an exact sequence of Hopf algebras. We shall say that \((C)\) is cleft, or else that \(C\) is a cleft extension of the Hopf algebra \(A\) by the Hopf algebra \(B\), if there exist a section \(\chi \in \text{Reg}_1(B, C)\) of the algebra extension and a retraction \(\xi \in \text{Reg}_e(C, A)\) of the coalgebra extension, satisfying the following equivalent conditions:

1. \(\chi^{-1}(\pi c) = S(c_{(1)})\xi(c_{(2)})\),
2. \(\chi(\pi c) = \xi^{-1}(c_{(1)})c_{(2)}\),
3. \(\xi^{-1}(c) = \chi(\pi c_{(1)})S(c_{(2)})\),
4. \(\xi(c) = c_{(1)}\chi^{-1}(\pi c_{(2)})\),
5. \(\xi \chi = e_{\pi A}\).

**Proof of the equivalence.** (1) \(\iff\) … \(\iff\) (4) \(\iff\) (5) is easy. (5) \(\implies\) (1): Let \(\eta(\pi c) = S(c_{(1)})\xi(c_{(2)})\). Since \(\chi\) is a section, one easily shows that \(\chi(\pi c_{(1)})S(c_{(2)}) \in A\). Then \(\chi \eta(\pi c) = \chi(\pi c_{(1)})S(c_{(2)})\xi(c_{(3)}) = \xi(\chi(\pi c_{(1)})S(c_{(2)})c_{(3)}) = \varepsilon(c)\), by the hypothesis. Since \(\chi\) is invertible, this implies that \(\eta = \chi^{-1}\). 

Now we are ready to present the main result of this section. Let

\[ Z^1_{\pi}(B, A) = \{(-\sigma, \rho, \tau) \in Z^1(B, A) : \sigma \text{ and } \tau \text{ are invertible}\}. \]

We shall see (cf. Lemma 3.2.17 below) that the bialgebra \(A \#_{\pi} B\) is actually a Hopf algebra if \((\sigma, \rho, \tau) \in Z^1_{\pi}(B, A)\). Let \(H^1_{\pi}\) be the quotient of \(Z^1_{\pi}\) by the action of \(\text{Reg}_1, \varepsilon\) defined before Proposition 3.1.7.

**Theorem 3.2.14.** \(H^1_{\pi}(B, A)\) classifies the cleft extensions \(0 \to A \xrightarrow{i} C \xrightarrow{\pi} B \to 0\) up to isomorphism.

**Proof.** By [BM, Prop. 1.8], Lemma 3.2.5, and condition (5) in Definition 3.2.13, the mapping considered in Proposition 3.1.7 gives rise to a map from \(H^1_{\pi}(B, A)\) to the set of isomorphism classes of cleft extensions of Hopf algebras. Let us prove that it is surjective. Let \((C)\) be a cleft exact sequence of Hopf algebras, with a section \(\chi\) and a retraction \(\xi\) as in Definition 3.2.13. Let \(-\sigma, \rho, \tau\) be defined by (3.2.2), (3.2.3), (3.2.10), (3.2.11). Let \(F\) be defined by \(F(a \# b) = \alpha \chi(b); F\) is an isomorphism of extensions of algebras (Theorem 3.2.1). But \(F(a \# \pi c) = a \xi^{-1}(c_{(1)})c_{(2)}\) (by (2) in 3.2.13), and hence \(F\) is also an isomorphism of extensions of coalgebras (Proposition 3.2.9). This implies the surjectivity.

Now let us proceed with the injectivity. Let \((\sigma, \rho, \tau), (\sigma', \rho', \tau') \in Z^1_{\pi}(B, A)\). Let

\[
\begin{array}{c}
0 \to A \xrightarrow{i} A \#_{\pi} B \xrightarrow{\pi} B \to 0 \\
\text{id} \downarrow \quad \varepsilon \downarrow \quad \text{id} \downarrow \\
0 \to A \xrightarrow{\sigma} A \#_{\sigma_1} B \xrightarrow{\pi_1} B \to 0
\end{array}
\]

be an isomorphism of (cleft) exact sequences of Hopf algebras (with inverse \(\Xi\)) and let \(\chi, \xi, \chi_1, \xi_1\) be the corresponding sections and retractions. Let \(\nu \in \text{Reg}_{\pi}(B, A)\) be such
that \( \nu \pi = \xi^{-1} \ast (\xi_1 \Theta) \) and let \( \mu \in \text{Reg}_1(B, A) \) such that \( \nu \mu = (\Theta \chi_1) \ast \chi^{-1} \). We know that 
\((\nu, \sigma_1) = (\mu, \ast, \sigma) \) and \((\rho_1, \tau_1) = (\rho, \tau, \nu) \) ([D, Lemma 2.1] and Proposition 3.2.12).

So we only need to prove that
\[
\mu \overset{?}{=} \nu^{-1}. \tag{3.2.15}
\]

Let \( c = a \# b \in A \# \sigma B \). Then
\[
\nu(\pi c) = \xi^{-1} \left( a_{(1)} \tau (b_{(1)} j \# \rho(b_{(2)}))_i \right) \xi_1 \Theta \left( a_{(2)} \tau (b_{(1)})^j \rho(b_{(2)})^i \# b_{(3)} \right)
\]
\[
= \tau^{-1} \left( \rho(b_{(2)})_i \right) S \left( a_{(1)} \tau (b_{(1)})^j \tau^{-1} \left( \rho(b_{(2)})_i \right)^k \right) a_{(2)} \tau (b_{(1)})^j \rho(b_{(2)})^i \xi_1 \Theta (1 \# b_{(3)})
\]
\[
= e(a) \xi_1 \Theta (1 \# b)
\]
\[
= \xi_1 \Theta (1 \# \pi c).
\]

Here the first equality is by definition, the second uses the formula in Lemma 3.2.4 and the fact that both \( \xi_1 \) and \( \Theta \) are morphisms of \( A \)-modules; the third one follows because (3.2.6) is the trivial morphism. On the other hand, from \( c = \xi(c_{(1)}) \otimes \pi(c_{(2)}) \) it follows that
\[
\Upsilon(1 \# b) = \xi \Upsilon(1 \# b_{(1)}) \# b_{(2)} \tag{3.2.16}
\]
and hence
\[
\nu(\pi c) = \Upsilon(1 \# b_{(1)}) \sigma^{-1}(Sb_{(3)} \otimes b_{(4)}) \# S(b_{(2)})
\]
\[
= \xi \Upsilon(1 \# b_{(1)})(b_{(2)} \rightarrow \sigma^{-1}(Sb_{(7)} \otimes b_{(9)})) \sigma(b_{(3)} \otimes Sb_{(6)}) \# b_{(4)} Sb_{(5)}
\]
\[
= \xi \Upsilon(1 \# b) \# 1,
\]
where we have used the formulas in the proof of [BM, Prop. 1.8]. Now applying \( \Theta \) to (3.2.16) we obtain
\[
1 \# b = \xi \Upsilon(1 \# b_{(1)}) \Theta (1 \# b_{(2)}) = \xi \Upsilon(1 \# b_{(1)}) \xi_1 \Theta (1 \# b_{(2)}) \# b_{(3)}
\]
and hence \( \mu \ast \nu \) is the trivial morphism, i.e., (3.2.15) holds. 

**Lemma 3.2.17.** Let \( (\cdot, \sigma, \rho, \tau) \in Z_4^4(B, A) \). Then the bialgebra \( A \# \sigma B \) is a Hopf algebra and its antipode is defined by
\[
S(a \# b) = \left[ (\sigma^{-1}(S\rho(b_{(2)})_h \otimes \rho(b_{(3)}))_i) \otimes S\rho(b_{(1)})_j \right]
\]
\[
[r^{-1}(b_{(4)})_k S(a\rho(b_{(1)})^i \rho(b_{(2)})^j \rho(b_{(3)})^l \tau^{-1}(b_{(4)})^k) \otimes 1] \tag{3.2.18}
\]

**Proof.** Let \( \xi, \chi \) be the "canonical" retraction and section of \( A \# \sigma B \). Then the equality \( c = \xi(c_{(1)}) \otimes \pi(c_{(2)}) \) can be rephrased as \( \text{id}_C = (\xi \ast (\chi \pi)) \) in the algebra \( \text{End}(C) \). But we know that \( \xi \) and \( \chi \) are invertible ([BM, Prop. 1.8] and Lemma 3.2.5). It follows that \( \text{id}_C \) is invertible and \( S_C = \text{id}_C^{-1} = (\chi^{-1} \pi) \ast (\xi^{-1}) \). The expressions for \( \chi^{-1} \) and \( \xi^{-1} \) imply (3.2.18). 


Lemma 3.2.19. Let us consider a morphism of exact sequences of Hopf algebras

\[ 0 \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \longrightarrow 0 \]

\[ \text{id} \downarrow \quad \Theta \downarrow \quad \text{id} \downarrow \]

\[ 0 \longrightarrow A \xrightarrow{\iota_1} C_1 \xrightarrow{\pi_1} B \longrightarrow 0 \]

where the top exact sequence is cleft. Then the bottom one is also cleft and \( \Theta \) is an isomorphism.

Proof. Let \( \chi, \xi \) be as in Definition 3.2.13. Let \( \chi_1 = \Theta \chi \); clearly, \( \chi_1 \) is a retraction of \( C_1 \). Let \( d \in C_1 \); one has \( d = \iota_1 \iota(\pi(c)) = \iota_1(\pi(c_1)\chi_1^{-1}) \). But \( \iota_1 \iota(d) \in A \subset \text{Im} \Theta \) and hence \( \Theta \) is surjective. Next, we claim that \( \chi_1(d) = \chi(c) \) if \( \Theta(c) = d \) is well defined. For, if \( \Theta(c) = 0 \), then \( \chi(c) = c_1 \chi_1^{-1}(\pi(c_2)) = \Theta(c_1)\chi_1^{-1} \pi_1 \Theta(c_2) = 0 \). Moreover, \( \chi_1 \) is a section and the bottom exact sequence is cleft. Now assume again that \( \Theta(c) = 0 \).

Then \( 0 = (\chi_1 \otimes \pi_1)(\Theta \otimes \Theta)(\pi \otimes \pi)(c) = (\chi \otimes \pi)(\pi_1 \otimes \pi)(c) \) and hence \( c = c_1 \chi_1^{-1}(\pi(c_2)) \chi(\pi(c_2)) = \chi(c_1)\chi(\pi(c_2)) = 0 \). •

Appendix. Bicovariant bimodules

(N. Andruskiewitsch)

The notion of a bicovariant bimodule was introduced in [W], see also [W3]. A crucial feature is that each bicovariant bimodule comes equipped with a solution of the braided (or Yang–Baxter) equation. According to [PW, p. 411], Connes conjectured in 1986 that “bicovariant bimodules over the algebra of smooth functions on a quantum group are (in natural way) labeled by representations of another quantum group”. This was solved (affirmatively) in [PW] by introducing the quantum double, in a dual way to \([D1]\).

In [T2], an alternative description of the quantum double was given. It turns out [T2] that the representations of the quantum double are exactly the crossed bimodules for the original algebra. (This was also previously observed in [M] under a finiteness hypothesis.) Crossed bimodules were introduced in [Y] and it was proved there that their category is braided.

In this appendix, we briefly review these facts and complete this circle of ideas. We show that the space of “left invariants” is in fact a crossed bimodule and that there is a one-to-one correspondence between crossed bimodules and bicovariant bimodules. (This is merely a translation of some facts in [W] into a coordinate-free language.) Moreover, the category of such bimodules is quasitensorial [D2], hence braided, and the corresponding solution of the quantum Yang–Baxter equation (found in [Y]) is that of [W]. (I noticed recently that these facts were also remarked in [Sc]).

Interesting examples of crossed bimodules are the right adjoint corepresentation with the right multiplication (dually) the right adjoint representation with the right comultiplication. They give rise to the solutions of the QYBE, which were first presented in [W2], by a direct computation. (See Corollary A.3) Moreover, by means of these solutions, any Hopf algebra is generalized commutative and generalized cocommutative, in the sense of [GRR, C, Mn].
A.1. Left covariant bimodules. We preserve the notation of the paper. The structural morphism of a comodule will usually be denoted by $c$; when one space carries two different comodule structures, we shall write $c_r$ ($c_l$ or simply $c$) for the right (left) one.

A left covariant bimodule $M$ has, by definition, a bimodule structure and a left comodule structure, over $A$ such that the comodule structural morphism $c: M \rightarrow A \otimes M$ is a morphism of bimodules; here as always we use the comultiplication to endow $A \otimes M$ with a bimodule action of $A$.

Let $N$ be a right $A$-module. Then $M := A \otimes N$ is a left covariant bimodule via the following formulae:

$$a(b \otimes n) = ab \otimes n; \quad (b \otimes n)a = ba(1) \otimes na(2); \quad c(b \otimes n) = \Delta(b)(1 \otimes 1 \otimes n). \quad (A.1)$$

Moreover, any left covariant bimodule arises in this way. Indeed, let $M$ be a left covariant bimodule and let

$$M_{\text{inv}} = \{ m \in M : c(m) = 1 \otimes m \}.$$

Consider also $P: M \rightarrow M$, $P(m) = S(m(1))m(2)$.

Proposition A.1. $P$ is a projector whose image is $M_{\text{inv}}$, and the latter acquires a right module structure by $n.a = P(na)$; let us denote it by $N$. Then $M$ is isomorphic, as a left covariant bimodule, to $A \otimes N$ with the structure explained above.

Sketch of Proof. (See [W, Lemma 2.2, Theorem 2.1].) First, one shows easily that $c(P(m)) = 1 \otimes P(m)$ and then, that $\text{Im} P = M_{\text{inv}}$, $P^2 = P$. It is also obvious that $P(am) = c(a)P(m)$ and $m = m(1)P(m(2))$. In particular, the restriction of the multiplication is an epimorphism $\varphi: A \otimes N \rightarrow M$. Suppose that there exist $a_i \in A$, $m_i \in N$ such that $0 = \sum a_i m_i$. Applying $(\text{id} \otimes P)c$ we get $0 = \sum a_i \otimes m_i$, and hence $\varphi$ is a linear isomorphism. $N$ is a right module with the action defined above, as follows immediately from the formula $P(na) = S(a(1))na(2)$, $n \in N$. Consider $N$ as a left covariant bimodule via (A.1); obviously, $\varphi$ preserves the left action of $A$ and the comodule structure. It is also easy to show that $\varphi$ preserves the right action: $\varphi((1 \otimes n)b) = b(1)P(nb(2)) = b(1)S(b(2))nb(3) = nb$. •

Remark. Compare the preceding with [Sw, 4.1].

A.2. Bicovariant bimodules. The notion of "left covariant bimodule" has an immediate translation to "right covariant bimodule". Now let us recall the definition of a bicovariant bimodule. It is a bimodule $M$, which is in addition left covariant, with structural morphism $c_r: M \rightarrow M \otimes A$, and right covariant, via $c_r: M \rightarrow A \otimes M$; moreover, the following diagram must commute:

$$
\begin{array}{ccc}
M & \xrightarrow{c_r} & A \otimes M \\
\downarrow c_r & & \downarrow \text{id} \otimes c_r \\
M \otimes A & \xrightarrow{c_r \otimes \text{id}} & A \otimes M \otimes A.
\end{array}
$$

Let now $N$ be a right $A$-module and a right $A$-comodule, such that

$$(1 \otimes a(1))c_r(na(2)) = c_r(n)\Delta(a), \quad n \in N, \ a \in A. \quad (A.2)$$
Let \( M = A \otimes N \) be provided with the left covariant bimodule structure explained in A.1 and extend \( c_r \) to \( M \) in the following way:

\[
c_r(a \otimes n) = \Delta(a)c_r(n) = a(1) \otimes n(0) \otimes a(2)n(1). \tag{A.3}
\]

It is easy to see that in this way \( M \) becomes a bicovariant bimodule. In fact, (A.2) guarantees that \( c_r \) is a morphism of right modules. Moreover, any bicovariant bimodule is obtained in this guise. Finally, it is obvious that a morphism of right modules and right comodules \( f: N \to N' \) gives rise to a morphism of bicovariant bimodules \( \text{id} \otimes f: M \to M' \), and that any such morphism has this form.

**Example A.1.** Consider \( A \) as a right module via the right multiplication and as a right comodule via the adjoint; recall that \( \text{ad}: A \to A \otimes A \) is defined by \( \text{ad}(a) = b(2) \otimes \epsilon(b(1))b(3) \).

We claim that the preceding data satisfies (A.2). Indeed,

\[
(1 \otimes a(1))\text{ad}(ba(2)) = (ba(2))(1) \otimes a(1)S((ba(2))(1))(ba(2))(3) = b(2)a(3) \otimes a(1)S(a(2))S(b(1))b(3)a(4) = \text{ad}(b)\Delta(a).
\]

Observe also that \( \ker \epsilon \) is a right subcomodule for the adjoint. In fact, one has \( (\epsilon \otimes \text{id})\text{ad} = \epsilon \), and also \( (\text{id} \otimes \epsilon)\text{ad} = \text{id} \).

**Example A.2.** Now consider \( A \) as a right comodule via the comultiplication and as a right module via the Adjoint, that is, via the following formula: \( u \text{Ad}(v) = S(v(1))uv(2) \).

Again, (A.2) is fulfilled:

\[
(1 \otimes a(1))\Delta(b \text{Ad}(a(2))) = (S(a(2))ba(3))(1) \otimes a(1)(S(a(2))ba(3))(2) = S(a(2))b(1)a(4) \otimes a(1)S(a(2))b(2)a(5) = S(a(1))b(1)a(2) \otimes b(2)a(3).
\]

**Remark.** Now let us assume that \( U \) is a Hopf algebra dual to \( A \). The left \( U \)-module structures on \( A \) provided respectively by \( (u.a,v) = (a,v \text{Ad}(u)) \) and (A.16) below applied to \( (A,\text{ad}) \) coincide. In fact, one has \( (v \otimes u,\text{ad}a) = (v \text{Ad}(u),a) \).

**Example A.3.** Consider a right comodule \( N \) as a trivial \( A \)-module, i.e., \( na = \epsilon(a)n \). Then the compatibility condition (A.2) reads \( n(0) \otimes an(1) = n(0) \otimes n(1)a \), which is fulfilled if \( A \) is commutative.

**A.3. The quantum Yang–Baxter equation.** Let \( \text{Bicov} \) (\( \text{Bicov}_A \) if necessary) be the category of all right modules and right comodules satisfying the compatibility condition (A.2); of course the morphisms must preserve both structures. Let \( N, N' \) be objects of \( \text{Bicov} \), and let \( R_{N,N'} \), be defined by the commutativity of the following diagram:

\[
\begin{array}{ccc}
N \otimes N' & \xrightarrow{R_{N,N'}} & N \otimes N' \\
\text{id} \otimes \epsilon & & \mu \otimes \text{id} \\
N \otimes N' \otimes A & \xrightarrow{T_{28}} & N \otimes A \otimes N';
\end{array}
\tag{A.4a}
\]
let $S_{N,N'}: N \otimes N' \to N' \otimes N$ be defined by

$$S_{N,N'} = TR_{N,N'}.$$  \hfill (A.4b)

(We shall omit the subscripts whenever no danger of confusion is present.)

**Proposition A.2.** (i) $Bicov$ is a quasitensor category (cf. [D2]), whose associativity constraint is the usual one and whose "commutativity constraint" is $S_{N,N'}$.

(ii) $S$ satisfies the quantum Yang–Baxter equation (QYBE for short); that is, if $N$, $N'$, $N''$ are objects of $Bicov$, then

$$(S_{N',N''} \otimes \text{id})(\text{id} \otimes S_{N,N''})(S_{N,N'} \otimes \text{id})(\text{id} \otimes S_{N',N''}) = (\text{id} \otimes S_{N,N'})(S_{N,N''} \otimes \text{id})(\text{id} \otimes S_{N',N''}).$$  \hfill (A.5)

(iii) $S$ is invertible and its inverse is given by

$$N \otimes N' \xrightarrow{S^{-1}} N' \otimes N,$$

$$c \otimes \text{id} \downarrow \uparrow \tau$$

$$N \otimes A \otimes N' \xrightarrow{\text{id} \otimes \mu_{S^{-1}}} N \otimes N'.$$

Here $\mu_{S^{-1}}: A \otimes N' \to N'$ is the left module structure given by $\mu_{S^{-1}}(a \otimes n') = n'S^{-1}(a)$ (we use the fact that $S^{-1}$ is an antimhomorphism of algebras).

In other words, $S_{N,N''}$ is a morphism in $Bicov$, in fact a natural transformation, and the following diagrams must commute:

$$
\begin{array}{c}
(N_1 \otimes N_2) \otimes N_3 \xrightarrow{S} N_3 \otimes (N_1 \otimes N_2) \xrightarrow{\sim} (N_3 \otimes N_1) \otimes N_2 \\
\downarrow \quad \uparrow S \otimes \text{id} \\
N_1 \otimes (N_2 \otimes N_3) \xrightarrow{\text{id} \otimes S} N_1 \otimes (N_3 \otimes N_2) \xrightarrow{\sim} (N_1 \otimes N_3) \otimes N_2,
\end{array}
$$

$$
\begin{array}{c}
N_1 \otimes (N_2 \otimes N_3) \xrightarrow{S} (N_2 \otimes N_3) \otimes N_1 \xrightarrow{\sim} N_2 \otimes (N_3 \otimes N_1) \\
\downarrow \quad \uparrow \text{id} \otimes S \\
(N_1 \otimes N_2) \otimes N_3 \xrightarrow{\text{S} \otimes \text{id}} (N_2 \otimes N_1) \otimes N_3 \xrightarrow{\sim} N_2 \otimes (N_1 \otimes N_3).
\end{array}
$$

**Proof.** (i) First let us prove that $S_{N,N''}$ is a morphism of $A$-modules. Let $a \in A$, $n \in N$, $n' \in N'$. Then

$$S((a \otimes n')a) = S(na(1) \otimes n'a(2)) = (n'a(2))(0) \otimes na(1)(n'a(2))(1)$$

$$= n'_0 a(1) \otimes nn'(1)a(2)$$

$$= S(n \otimes n')a,$$
thanks to (A.2).

Now let us show that it is a morphism of comodules. Using the compatibility condition (A.2) and the first axiom of comodules, we have:

\[(S \otimes \text{id})c(n \otimes n') = (S \otimes \text{id}) \left( n(0) \otimes n'(0) \otimes n(1)n'(1) \right) = n'(0) \otimes n(0)n'(1) \otimes n(1)n'(2) \]
\[= n'(0) \otimes (nn'(2))(0) \otimes n'(1)(nn(2))(1) \]
\[= c(S(n \otimes n')). \]

The commutativity of (A.7 a) (resp., (A.7 b)) follows from the coassociativity in the definition of comodule (resp., the definition of tensor product of comodules).

If \( f : N \to P \) (resp., \( g : N' \to P' \)) is a morphism of modules (resp., of comodules) then \( S_{P,P'}(f \otimes g) = (f \otimes g)S_{N,N'} \). In particular, \( S \) is a natural transformation.

(ii) follows from (i), see [D2, Remark 4 before Prop. 3.1]. (A direct proof is straightforward.)

(iii): use that \( S^{-1} \) is the antipode for the opposite comultiplication and the same multiplication. •

Remark. Proposition A.2 is a generalization of [W, Prop. 3.1]. Indeed, our formula (A.4), in the case \( N = N' = N'' \), coincides with [W, (3.5)]; this follows from [W, (3.15), (2.35) and (2.13)]. On the other hand, assertions (ii) and (iii) of Proposition A.2 were first proved in [Y].

Corollary A.3. (i) Let \( S_0 : A \otimes A \to A \otimes A \) be defined by

\[ S_0(a \otimes b) = b(2) \otimes aS(b(1))b(3). \] (A.8)

Then \( S_0 \) satisfies the quantum Yang–Baxter equation. Moreover, \( S_0(a \otimes b \Delta(c)) = S_0(a \otimes b)\Delta(c) \) and \( (S_0 \otimes \text{id})\text{ad}^{\otimes 2} = \text{ad}^{\otimes 2} S_0 \).

(ii) Let \( S_1 : A \otimes A \to A \otimes A \) be defined by

\[ S_1(a \otimes b) = b(1) \otimes S(b(2))ab(3). \] (A.9)

Then \( S_1 \) is also a solution of the QYBE.

Proof. Apply the Proposition to \( A \) within the setting of Example A.1 (resp. A.2) above. Note that if \( U \) is as in the Remark following Example A.2, then

\[ \langle S_1(u \otimes v), a \otimes b \rangle = \langle u \otimes v, S_0(a \otimes b) \rangle, \quad a, b \in A, \quad u, v \in U. \]

Now let us consider the quantum double \( E \) of \( A \), as defined in [T2]: as a vector space, \( E = \text{End}(A) \); the multiplication is given by \( (T_1T_2)(a) = T_1(a(1))T_2(a(2)) \); the comultiplication, by

\[ \tilde{\Delta}(T)(x \otimes y) = (1 \otimes x(1))\Delta(T(yx(2)))(1 \otimes S(x(3))); \]

cf loc cit for the remaining definitions.
Proposition A.4. (i) Let $V$ be a right $\mathcal{E}$-comodule, with structural morphism $c_\mathcal{E}$. Then $V$ is an object of $\text{Bicov}_A$, with action and coaction defined by

$$v.a = v(1)(\varepsilon, v(2)(a)), \quad c(v) = v(1) \otimes v(2)(1).$$

Conversely, any object $V$ of $\text{Bicov}_A$ is an $\mathcal{E}$-comodule via the mapping $c_\mathcal{E}: V \to V \otimes \mathcal{E} \cong \text{Hom}(A, V \otimes A)$ defined by $c_\mathcal{E}(v)(a) = c(va)$. These assignments are inverse to each other and hence the categories $\text{Bicov}_A$ and $\text{Comod}_\mathcal{E}$ are equivalent.

(ii) Let $B: \mathcal{E} \times \mathcal{E} \to k$ be the bilinear form defined by

$$B(F, G) = \langle \varepsilon, FG(1) \rangle. \quad (A.10)$$

Let $V, V'$ be two $\mathcal{E}$-comodules, let $R_{V, V'}$ be defined by the commutativity of the following diagram:

$$\begin{array}{ccc}
V \otimes V' & \xrightarrow{R_{V, V'}} & V \otimes V' \\
c_\mathcal{E} \otimes c_\mathcal{E} & \downarrow & \uparrow \text{id} \otimes B \\
V \otimes \mathcal{E} \otimes V' \otimes \mathcal{E} & \xrightarrow{\tau \otimes \tau} & V \otimes V' \otimes \mathcal{E} \otimes \mathcal{E};
\end{array} \quad (A.11a)$$

and let $S_{V, V'}$ be given by

$$S_{V, V'} = \tau R_{V, V'}. \quad (A.11b)$$

Then $\text{Comod}_\mathcal{E}$ is a quasitensor category whose commutativity constraint is $S_{V, V'}$ and the equivalence stated in (i) preserves this additional structure.

Proof. (i) See [T2]. (ii) is left to the reader.

Part (ii) of the above Proposition is a special case of the following fact, a slight generalization of [Ly, Theorem 2.3.3]:

Proposition A.5. Let $\rho: A \otimes A \to k$ be a nondegenerate bilinear form satisfying

$$f(1)g(1)\rho(g(2), f(2)) = \rho(g(1), f(1))g(2)f(2),$$

$$\rho(fg, h) = \rho(f, h(1))\rho(g, h(2)), $$

$$\rho(h, gf) = \rho(h(1), f)\rho(h(2), g)$$

for any $f, g, h \in A$. Then the category of right $A$-comodules is quasitensorial and the commutativity constraint $S_{M, N}: M \otimes N \to N \otimes M$ is given by

$$S_{M, N}(m \otimes n) = n(1) \otimes m(1)\rho(m(2), n(2)).$$

The data $(A, \rho)$ is the dual version of a quasitriangular Hopf algebra, see below.
A.4. Quasitriangular Hopf algebras. Let us assume the existence of a Hopf algebra $U$ such that $A \hookrightarrow U^*$ is a Hopf algebra dual to $U$. First let us suppose that $U$ is quasitriangular [D1], i.e., there exists an invertible $R \in U \otimes U$ satisfying

$$\Delta'(u) = R\Delta(u)R^{-1}, \quad u \in U,$$

$$\Delta \otimes \text{id}(R) = R^{13}R^{23}, \quad \text{id} \otimes \Delta(R) = R^{13}R^{12},$$

where $\Delta'$ is the opposite comultiplication. (A.13) implies that the mapping $A \rightarrow U$ given by $a \mapsto \text{id} \otimes \langle a,\cdot\rangle(R)$ (resp., $a \mapsto \langle a,\cdot\rangle \otimes \text{id}(R)$) is an antihomomorphism (resp., a homomorphism) of algebras. Note that:

(A.14) If $U_{\text{op}}$ denotes $U$ as Hopf algebra with the opposite comultiplication, then $(U_{\text{op}}, R^{-1})$ is also a quasitriangular Hopf algebra, as well as $(U_{\text{op}}, R^1)$.

(A.15) If $(V, T)$ is another quasitriangular Hopf algebra, then $(U \otimes V, T^{23}(R \otimes S))$ is also one.

Now let $N$ be a right $A$-comodule, hence a left $U$-module with the action defined by the commutativity of the diagram:

$$
\begin{array}{ccc}
U \otimes N & \longrightarrow & N \\
\tau \downarrow & & \uparrow \text{id} \otimes (\cdot) \\
N \otimes U & \longrightarrow & N \otimes A \otimes U.
\end{array}
$$

(A.16)

We shall consider $N$ as a right $A$-module by composing the preceding with the antihomomorphism $a \mapsto \text{id} \otimes \langle a,\cdot\rangle(R)$. In concrete terms,

$$na = n_{(1)} \langle n_{(2)} \otimes a, R \rangle, \quad n \in N, \ a \in A.$$  

(A.17)

To insure the pertinence of $N$ to $Bicov$, we need to check the compatibility condition (A.2). First note that for any $u \in U$, $a, b \in A$ (A.12) implies

$$\langle a(1) \otimes b(1) \otimes a(2) \otimes b(2), \Delta'(u) \otimes R \rangle = \langle a(1) \otimes b(1) \otimes a(2) \otimes b(2), R \otimes \Delta(u) \rangle$$

and thus

$$\langle a(2) \otimes b(2), R \rangle b(1)a(1) = \langle a(1) \otimes b(1), R \rangle a(2)b(2).$$

(A.18)

Taking into account (A.17), the left hand side of (A.2) is

$$\langle n_{(2)} \otimes a(2), R \rangle n_{(0)} \otimes a(1)n_{(1)};$$

the right one is

$$\langle n_{(1)} \otimes a(1), R \rangle n_{(0)} \otimes n_{(2)}a(2),$$

and the equality follows from (A.18) (that is, from (A.12)).

On the other hand, let $N'$ be another $A$-comodule, $n \in N, n' \in N'$. Then the mapping (A.4) gives in this case, thanks to (A.17),

$$n \otimes n' \mapsto \langle n_{(1)} \otimes n'_{(1)}, R \rangle n_{(0)} \otimes n'_{(0)}$$
which is the same as the action of $R$ on the $U \otimes U$-module $N \otimes N'$. We have therefore proved the following fact, essentially due to Rosso [R]:

**Proposition A.6.** If $U$ is quasitriangular, then any $A$-comodule belongs to $Bicov$ in a "canonical" way; moreover, the solutions of the quantum Yang–Baxter equation provided by (A.4) and $R$ coincide.

Let us consider $A$ as $A \otimes A^{cop}$-comodule via $a \mapsto a_{(2)} \otimes a_{(3)} \otimes a_{(1)}$. It follows from (A.14) and (A.15) that the mapping $S_2: A \otimes A \rightarrow A \otimes A$ given by

$$S_2(a \otimes b) = (a_{(3)} \otimes a_{(1)}, R)(b_{(3)} \otimes b_{(1)}, R^{-1})b_{(2)} \otimes a_{(2)}$$

(A.19)

satisfies the Yang–Baxter equation.

Now let us recall that a *generalized commutative* algebra is a pair $(B, S)$, where $B$ is an algebra with multiplication $m$ and $S: B \otimes B \rightarrow B \otimes B$ is a solution of the QYBE such that

$$mS = m,$$  

(A.20a)

$$S(b \otimes 1) = 1 \otimes b, \quad S(1 \otimes b) = b \otimes 1,$$  

(A.20b)

$$S(m \otimes id) = (id \otimes m)S^{12}S^{23}, \quad S(id \otimes m) = (m \otimes id)S^{23}S^{12}.$$  

(A.20c)

The following fact is well known.

**Proposition A.7.** $(A, S_2)$ is a generalized commutative algebra.

**Proof.** We already showed that $S_2$ satisfies the quantum Yang–Baxter equation. (A.20a and c) are direct consequences of (A.12 and 13), respectively, whereas to prove (A.20b) one uses the following equalities (cf. [D3, Prop. 3.1]):

$$(\varepsilon \otimes id)(R) = 1 = (id \otimes \varepsilon)(R).$$

(A.21)

Now we show that any Hopf algebra is generalized commutative.

**Proposition A.8.** $(A, S_1)$ is a generalized commutative algebra.

**Proof.** It is straightforward. For example,

$$mS_1(a \otimes b) = b_{(1)}S(b_{(2)})ab_{(3)} = ab;$$

$$S(m \otimes id)S^{12}S^{23}(a \otimes b \otimes c) = c_{(1)} \otimes S(c_{(2)}a)c_{(3)}S(c_{(4)})bc_{(5)} = c_{(1)} \otimes S(c_{(2)})abc_{(3)} = S_1(m \otimes id)(a \otimes b \otimes c);$$

and the rest is similar. •

In the same vein, one defines *generalized cocommutative* coalgebras and proves that if $A$ is a Hopf algebra, then $(A, S_0)$ is generalized cocommutative.
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EXTENSIONS OF HOPF ALGEBRAS


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