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SZYSZKOWSKI I.*

WEAK CONVERGENCE OF STOCHASTIC INTEGRALS

Пусть \( W_n = \{W_n(t): t \geq 0\} \), \( n \geq 0 \) есть последовательность согласованных сдайлы случайных процессов таких, что для каждого \( n \geq 0 \) процесс \( W_n \) является семимартингалом и имеет место слабая сходимость в пространстве Скороды \( D(R_+; R^2) \): \( (W_n, [W_n]) \to (W_0, V_0) \) при \( n \to \infty \), где \( V_0 \) — некоторый почти наверное неубывающий процесс. Мы показываем, что \( Z_n(-) = \int_0^\infty f(W_n(t-)) \, dW_n(t) \) слабо сходится в \( D(R_+, R) \) к процессу \( \int_0^\infty f(W_0(t-)) \, dW_0(t) + 2^{-1} \int_0^\infty f'(W_0(t-)) \, dW_0(t) \) для любой аналитической функции \( f \) на \( R \). (Здесь \([W_n]\) обозначает квадратичную вариацию семимартингала \( W_n \).)

Ключевые слова и фразы: слабая сходимость, стохастический интеграл, принцип инвариантности, семимартингал.

1. The main result. For each \( n \geq 0 \) let \( W_n = \{(W_n(t), (F_t^n)): t \geq 0\} \) be an \( R \)-valued сдайлы stochastic process adapted to a right-continuous complete filtration \( (F_t^n)_{t \geq 0} \) on a complete probability space \( (\Omega, \mathcal{F}, P) \). For simplicity of consideration we will always assume that \( W_n(0) = 0 \) a.s., \( n \geq 0 \).

The symbol \( D^k[0, \infty) \) will denote the topological space of \( R^k \)-valued functions on \([0, \infty)\) that are right-continuous and have left-hand limits, endowed with \( J_1 \)-Skorokhod topology (cf. [1], [2]).

By \( D^n(R^k) \) (abbreviated \( D^n \) or \( D \)) we will denote the set of all stochastic processes with paths in the space \( D^k[0, \infty) \), adapted to the filtration \( (F_t^n)_{t \geq 0} \). On \( D \) we put the topology of uniform convergence on compact sets in probability, abbreviated ucp. Note that this topology is metrizable, and the metric

\[
d(X, Y) = \sum_{n=1}^\infty 2^{-n} E \left( \max_{0 \leq t \leq n} \left\| X(t) - Y(t) \right\| \right), \quad X, Y \in D,
\]

(\( \| \cdot \| \) denotes the Euclidean norm in \( R \)) makes \( D \) into a complete metric space compatible with ucp.

The following limit theorem will be proved in Section 2.

*Department of Mathematical Statistics, Rhodes University, 6140 Grahamstown, South Africa.

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Theorem. Let \( \{W_n, \ n \geq 0\} \) be a sequence of semimartingales such that for each \( n \geq 0, W_n \in \mathcal{D}^0(\mathbb{R}) \). If

\[
(W_n, [W_n]) \rightarrow (W_0, V_0) \quad \text{weakly, in } \mathcal{D}^2[0, \infty) \text{ as } n \to \infty;
\]

for some a.s. non-decreasing process \( V_0 \in \mathcal{D}^0(\mathbb{R}), V_0 = 0 \) a.s., then

\[
\int_0^t f(W_n(t-))dW_n(t) \longrightarrow \int_0^t f(W_0(t-))dW_0(t)
\]

\[+ 2^{-1} \int_0^t f'(W_0(t-))d[W_0](t) - 2^{-1} \int_0^t f'(W_0(t-))dV_0(t)\]

weakly, in \( \mathcal{D}^1[0, \infty) \) as \( n \to \infty \), for any analytic function \( f \) ([\( X \)] stands for the quadratic variation process of \( X \)).

Theorem given above generalizes or extends the latest results presented by Strasser [3], Janson and Wichura [4], Yoshihara [5], Jakubowski, Memin and Pages [6], Kurtz and Protter [7], Szyszkowski [8]-[10] (where the case \( V_0 = [W_0] \) was discussed), and Takahata [11].

Note that, in general, \( V_0 \) can be different from \( [W_0] \).

However for some specific sequences \( \{W_n, \ n \geq 0\} \) of semimartingales condition (1) implies \( V_0 = [W_0] \) (see e.g. [12]).

2. The proof (sketch). Note that by utilising [13, Theorems 7.12 and 8.2] and [14, Theorem 4.2], we can reduce our main Theorem to the following “discrete version”.

Proposition. Let \( \{X_{nk}, \ k \geq 1, \ n \geq 1\} \) be a double array of random variables defined on a common complete probability space \( (\Omega, \mathcal{F}, P) \) and let for each \( n \geq 1, (k_n(t))_{t \geq 0} \) be a right-continuous, non-decreasing deterministic integer-valued process taking all the values between zero and \( k_n(\infty) \). Define

\[ W_n(t) = S_n(k_n(t)), \quad t \geq 0, \]

where \( S_n,k = \sum_{i=1}^k X_{ni}, k \geq 1, n \geq 1, \mathcal{F}_n^t = \sigma(X_{ni}: i = 1, 2, \ldots, k_n(t)) \). Assume (1) holds for some semimartingales \( W_0 \) and \( V_0 \). Then (2) holds for any analytic function \( f \) on \( \mathbb{R} \).

Remark 1. By (1) it follows that, for each \( T \geq 0 \),

\[
\lim_{C \to \infty} \limsup_{n \to \infty} \mathbf{P}\left\{ \sup_{0 \leq t \leq T} \left| W_n(t) \right| > C \right\} = 0.
\]

Hence to prove our Proposition it is enough to show that (2) holds when \( \sup_{0 \leq t \leq T} |W_n(t)| \) is a.s. bounded, for each \( n \geq 1 \) and \( T > 0 \).

Proof of the Proposition. Observe that it is enough to prove our Proposition for the case \( T = 1 \) only (cf. [1]). Thus in the sequel we restrict our considerations to the interval \([0, 1]\), using notation and some basic facts contained in the well-known Billingsley’s book [14, Ch. 2 and 3].

Let \( \Gamma_T \) denote the set of all finite non-random partitions \( \gamma: 0 = t_0 < t_1 < \cdots < t_k = T \) of the interval \([0, T]\) and

\[ |\gamma| = \max_{1 \leq i \leq k} (t_i - t_{i-1}). \]

For \( \gamma = (t_i) \in \Gamma_1 \) and \( X \in \mathcal{D} \), we adopt the following notation:

\[
\int_0^t g^\gamma(X) dX = \sum_{t_i \in \gamma} g\left(X(t_i)\right) \left(X(t_{i+1} \wedge t) - X(t_i \wedge t)\right),
\]

\[ [X]^\gamma(t) = [X]^\gamma_t = \sum_{t_i \in \gamma} \left(X(t_i \wedge t) - X(t_{i-1} \wedge t)\right)^2. \]
For each $\gamma \in \Gamma_1$ and $t \in [0, 1]$, the stochastic integral

$$Z_n(t) = \int_0^t f(W_n) \, dW_n = \sum_{j=0}^{k_n(t)-1} f(S_{nj}) X_{nj+1}$$

\hspace{1cm}
can be expressed as follows:

$$Z_n(t) = \int_0^t f^*(W_n) \, dW_n + \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \left[ f(S_{nj}) - f(W_n(t_i)) \right] X_{nj+1},$$

where $B(t_i, n, t) = \{j: [k_n(t_i \land t)] \leq j < [k_n(t_{i+1} \land t)]\}$.

Moreover, we have

$$\sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \left[ f(S_{nj}) - f(W_n(t_i)) \right] X_{nj+1}$$

$$= \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \sum_{k=1}^{\infty} \frac{d^k}{dx^k} (W_n(t_i))(S_{nj} - W_n(t_i))^k \frac{1}{k!} X_{nj+1}$$

$$= \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \sum_{k=1}^{\infty} \frac{d^k}{dx^k} (W_n(t_i))(S_{nj} - W_n(t_i))^k \frac{1}{k!} X_{nj+1}$$

$$= \sum_{k=1}^{\infty} \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \frac{d^k}{dx^k} (W_n(t_i))(S_{nj} - W_n(t_i))^k \frac{1}{k!} X_{nj+1},$$

where the interchanges of the summation signs are justified because, on each event of the form $\{\sup_{0 \leq s \leq 1} |W_n(t)| \leq C\}$, $C > 0$, the series under consideration is absolutely convergent (see also Remark 1).

Next, we have

$$\sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \frac{df}{dx} (W_n(t_i))(S_{nj} - W_n(t_i)) X_{nj+1}$$

$$= \sum_{t_i \in \gamma} \frac{df}{dx} (W_n(t_i)) \sum_{j \in B(t_i, n, t)} (S_{nj} - W_n(t_i)) X_{nj+1}$$

$$= 2^{-1} \left( \int_0^t \frac{df^\gamma}{dx} (W_n) \, d[W_n]^\gamma - \int_0^t \frac{df^\gamma}{dx} (W_n) \, dW_n \right)$$

$$= \int_0^t \frac{df^\gamma}{dx} (W_n) \, d\{2^{-1}([W_n]^\gamma - [W_n])\}$$

$$= \int_0^t \frac{df^\gamma}{dx} (W_n) \, dD_n^\gamma \ (\text{say}), \quad n \geq 1.$$

Note that

$$\sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \frac{d^k}{dx^k} (W_n(t_i))(S_{nj} - W_n(t_i))^{k-1} X_{nj+1}$$

$$= \int_0^t \Delta_k^\gamma (W_n) \, dD_n^\gamma, \quad t \in [0, 1], \quad n \geq 1,$$

where

$$\Delta_k^\gamma (W_n)(s) = \frac{d^k}{dx^k} (W_n(t_i)) \frac{(W_n(s) - W_n(t_i))^{k-1}}{k!},$$

for $t_i \leq s < t_{i+1}$, $k \geq 1$. 

Now, combining the above results, we obtain for each \( \gamma \in \Gamma_1 \):

\[
Z_n(\cdot) = \int_0^\gamma f^\gamma(W_n)\,dW_n + \sum_{k=1}^\infty \int_0^\gamma \Delta_k^\gamma(W_n)\,dD_n^\gamma = I_n^\gamma(\cdot) + I_n^\gamma(\cdot) \quad \text{(say)},
\]

where

\[
D_n^\gamma(s) = 2^{-1}\left([W_n]^\gamma(s) - [W_n](s)\right).
\]

Without loss of generality we may and do assume that all the partitions \( \gamma \) under consideration consist of continuity points of the limiting process \( (W_0, V_0) \).

It is easy to show that, for each \( \gamma \in \Gamma_1 \), the following weak limit theorem holds in the Skorokhod space \( D^2[0,1] \):

\[
\lim_{n \to \infty} \left(I_n^\gamma, J_n^\gamma\right) = \left(\int_0^\gamma f^\gamma(W_0(t))\,dW_0(t), \sum_{k=1}^\infty \int_0^\gamma \Delta_k^\gamma(W_0)\,dD_0^\gamma\right).
\]

Hence, for each \( \gamma \in \Gamma_1 \),

\[
\lim_{n \to \infty} Z_n(\cdot) = \int_0^\gamma f^\gamma(W_0(t))\,dW_0(t) + \sum_{k=1}^\infty \int_0^\gamma \Delta_k^\gamma(W_0)\,dD_0^\gamma,
\]

weakly in \( D[0,1] \). But

\[
\lim_{|\gamma| \to 0} \int_0^\gamma f^\gamma(W_0(t))\,dW_0(t) = \int_0^\gamma f(W_0(t-))\,dW_0(t),
\]

uniformly on \([0,1]\), in probability.

Moreover, the following limit theorems hold with probability one in the space \( D[0,1] \):

\[
\lim_{|\gamma| \to 0} D_0^\gamma(\cdot) = D_0(\cdot) = 2^{-1}\left\{[W_0](\cdot) - V_0(\cdot)\right\},
\]

\[
\lim_{|\gamma| \to 0} \int_0^\gamma \Delta_k^\gamma(W_0)\,dD_0^\gamma = 2^{-1}\left\{\int_0^\gamma f'(W_0(t-))\,d[W_0](t) - \int_0^\gamma f'(W_0(t-))\,dV_0(t)\right\}
\]

and, for each \( k \geq 2 \),

\[
\lim_{|\gamma| \to 0} \int_0^\gamma \Delta_k^\gamma(W_0)\,dD_0^\gamma \equiv 0.
\]

Finally

\[
\lim_{n \to \infty} Z_n(\cdot) = \lim_{|\gamma| \to 0} \int_0^\gamma f^\gamma(W_0(t))\,dW_0(t) + \lim_{|\gamma| \to 0} \sum_{k=1}^\infty \int_0^\gamma \Delta_k^\gamma(W_0)\,dD_0^\gamma
\]

\[
= \int_0^\gamma f(W_0(t-))\,dW_0(t)
\]

\[
+ 2^{-1}\left\{\int_0^\gamma f'(W_0(t-))\,d[W_0](t) - \int_0^\gamma f'(W_0(t-))\,dV_0(t)\right\}
\]

weakly in \( D[0,1] \), what completes the proof of our Proposition.

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