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LOCAL AND TRUE MARTINGALES IN DISCRETE TIME\footnote{The second author gratefully acknowledges the financial support received from Austrian Science Fund (FWF) grant F 19456 and from Hungarian Science Foundation (OTKA) grant F 049094. While the research of this paper was conducted the second author was also affiliated with Vienna University of Technology, Financial and Actuarial Mathematics Research Unit.}

Доказывается, что для процесса с дискретным временем и с бесконечным временем горизонтом множество эквивалентных $L^p$-мартингальных мер плотно в множестве эквивалентных локально-мартингальных мер относительно нормы, определяемой полной вариацией.

Ключевые слова и фразы: эквивалентная мартингальная мера; локальный мартингал; норма, определяемая полной вариацией.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. All $\sigma$-algebras we consider are assumed to be $\mathbb{P}$-complete sub-$\sigma$-algebras of $\mathcal{F}$. Let $(\mathcal{G}_n)_{n \geq 0}$ be a discrete-time filtration and $(S_n)_{n \geq 0}$ an $\mathbb{R}^d$-valued local martingale with respect to $(\mathcal{G}_n)_{n \geq 0}$.

The following theorem was formulated in [3].
Theorem 1.1. There is a probability measure \( Q \) on \( \mathcal{F} \) such that \( P \sim Q \) and \( S \) is a \( Q \)-martingale.

In fact, it is a byproduct of results in [6] on arbitrage theory: if \( S \) is a local martingale, then it is known to satisfy the so-called NFLBR property and the main theorem of [6] implies that \( S \) admits an equivalent martingale measure. Though the direct proof given in [3] is much shorter, its arguments, as well as those of [6], are still based on the same functional analytic approach (countably normed spaces, infinite dimensional separation, Komlós' theorem, and [1]). In the present paper we give a proof for the stronger result below, using less heavy artillery (namely [1] and [4] which rely on measurable selection techniques).

Theorem 1.2. For each \( \varepsilon > 0 \), there is a probability measure \( Q \) on \( \mathcal{F} \) such that \( P \sim Q \), \( \|P - Q\| \leq \varepsilon \), and \( S \) is a \( Q \)-martingale.

Here \( \|\cdot\| \) denotes the total variation norm, i.e.,

\[
\|P - Q\| = \sup \left\{ \int X \, d(P - Q) : X \text{ random variable, } |X| \leq 1 \right\}.
\]

Recall that if \( \lambda \) is a common dominating measure, then the total variation distance can be expressed with the densities:

\[
\|P - Q\| = \int_0^\infty |dP - dQ| \, d\lambda.
\]

Remark 1.1. After having read the first version of this paper, Yuri Kabanov has kindly pointed out to us that Theorem 1.2 is a direct consequence of Theorem 1.1 and the main result of [2]. The latter paper uses advanced semimartingale theory, we manage to obtain Theorem 1.2 using only [4] which is the (simpler) discrete-time finite horizon translation of the proof in [2].

Actually, our proof gives much more than Theorem 1.2.

Theorem 1.3. Let \((Y_n)_{n\geq 0}\) be an adapted \( \mathbb{R} \)-valued process. If \((S_n)_{n\geq 0}\) is an \( \mathbb{R}^d \)-valued \( P \)-local martingale, then for each \( \varepsilon > 0 \) there is a probability measure \( Q \) such that \( P \sim Q \), \( \|P - Q\| \leq \varepsilon \), \((S_n)_{n\geq 0}\) is a \( Q \)-martingale, and each \( Y_n \) is \( Q \)-integrable.

Theorem 1.3 clearly implies Theorem 1.2. Let \( 1 \leq p < \infty \). Applying Theorem 1.3 with the choice \( Y_n := |S_n|^p \) we get the following theorem.

Theorem 1.4. For each \( 1 \leq p < \infty \) and each \( \varepsilon > 0 \) there is a probability measure \( Q \) on \( \mathcal{F} \) such that \( P \sim Q \), \( \|P - Q\| \leq \varepsilon \), and \( S \) is a \( Q \)-martingale in \( L^p(Q) \).

Now let \( (S_t)_{t \geq 0} \) be an arbitrary process on \((\Omega, \mathcal{F}, \mathbb{P})\) adapted to \((\mathcal{F}_t)_{t \geq 0}\). Let \( \mathcal{M}_p^\mathbb{P} \) denote the set of \( Q \sim P \) such that \( S \) is a \( Q \)-martingale in \( L^p(Q) \) (respectively, \( Q \)-local martingale) with respect to the given filtration. The following result is an obvious consequence of Theorem 1.4.

Corollary 1.1. \( \mathcal{M}_p^\mathbb{P} \) is dense in \( \mathcal{M}_\mathbb{P}^\omega \) with respect to \( \|\cdot\| \).

In the usual setting of mathematical finance \( S \) represents the price process of an asset. The sets \( \mathcal{M}_p^\mathbb{P}, \mathcal{M}_\mathbb{P}^\omega \) correspond to pricing functionals for derivative securities and play a prominent role in the arbitrage theory of financial markets.

2. Proof. We recall that, for arbitrary nonnegative random variable \( Y \) and \( \sigma \)-algebra \( \mathcal{G} \), one can define \( \mathbb{E}(Y \mid \mathcal{G}) \) (finite or infinite). If both the positive and negative parts \( X_+, X_- \) of a random variable \( X \) satisfy \( \mathbb{E}(X_+ \mid \mathcal{G}) < \infty \), \( \mathbb{E}(X_- \mid \mathcal{G}) < \infty \), then \( \mathbb{E}(X \mid \mathcal{G}) := \mathbb{E}(X_+ \mid \mathcal{G}) - \mathbb{E}(X_- \mid \mathcal{G}) \) can be defined. Multidimensional extension of this notion is straightforward.

Definition 2.1. An \( \mathbb{R}^d \)-valued process \( (S_t)_{t \geq 0} \) adapted to \((\mathcal{F}_t)_{t \geq 0}\) is called a generalized martingale if, for all \( n \geq 0 \), \( \mathbb{E}(S_{n+1} \mid \mathcal{F}_n) = S_n \) in the above generalized sense.

The following result goes back to Meyer, see [7, Chap. VII, Theorem 1].
Proposition 2.1. On a discrete-time filtration $\mathcal{F}_k$ every local martingale is a generalized martingale.

We note that if $\mathcal{G}_0$ is trivial, then the reverse implication also holds. Our arguments will be based on the following lemma.

Lemma 2.1. Let $X: \Omega \to \mathbb{R}^h$, $Y: \Omega \to \mathbb{R}^n$ be random variables and $\mathcal{G} \subset \mathcal{F}$ a $\sigma$-algebra. Assume that $E(X \mid \mathcal{G})$ exists and $E(X \mid \mathcal{G}) = 0$. Then for any $\varepsilon > 0$ there is a bounded random variable $Z > 0$ such that

(i) $E[Z - 1] < \varepsilon$;

(ii) $E(ZX \mid \mathcal{G})$ exists and $E(ZX \mid \mathcal{G}) = 0$;

(iii) $ZY$ is bounded.

Proof. Define $\mathcal{H} := \mathcal{G} \vee \sigma(X) \vee \sigma(Y)$. We recall one direction of Theorem 2.4 in [1].

Theorem 2.1. Let $\mathcal{G} \subset \mathcal{H}$ be (complete) sub-$\sigma$-algebras of $\mathcal{F}$, and let $W: \Omega \to \mathbb{R}^h$ be an $\mathcal{H}$-measurable random variable satisfying

$$ (W, R) \geq 0 \text{ a.s.} \Rightarrow (W, R) = 0 \text{ a.s.} \tag{1} $$

for each $\mathcal{G}$-measurable $h$-dimensional random variable $R$; here $\langle \cdot, \cdot \rangle$ denotes scalar product. Then there exists an $\mathcal{H}$-measurable scalar random variable $L$ such that $0 < L \leq 1$ a.s., $E(W \mid L) < \infty$ and $E(WL \mid \mathcal{G}) = 0$ a.s.

We claim that (1) holds for $W := X$. Indeed, if $(X, R) \geq 0$ a.s., then

$$ E\langle X, R \rangle = E E\langle (X, R) \mid \mathcal{G} \rangle = E(E(X \mid \mathcal{G}), R) = 0, $$

showing (1). Define now a new probability $P_1 \sim P$ by

$$ \frac{dP_1}{dP} = \frac{(1 + Y)^{-1}}{E(1 + Y)^{-1}}. $$

Note that (1) continues to hold under $P_1$ hence applying Theorem 2.1 (under $P_1$) we may set

$$ \frac{dP_2}{dP_1} := \frac{L}{E(L)} $$

and get that $X$ is $P_2$-integrable with $E_2(X \mid \mathcal{G}) = 0$. Here and in what follows we use $E_2$ for the expectation with respect to $P_2$. Now we restate Proposition 2.3 of [4].

Lemma 2.2. Let $\sigma$-algebras $\mathcal{G}$ and $\mathcal{H}$ satisfy $\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}$. Let $W: \Omega \to \mathbb{R}^d$, $F: \Omega \to \mathbb{R}^d \setminus \{0\}$ be $\mathcal{H}$-measurable random variables with

$$ E|W| < \infty, \quad E(W \mid \mathcal{G}) = 0, \quad E(F \mid \mathcal{G}) = 1, \quad E(WF \mid \mathcal{G}) = 0. $$

Then there is a sequence of strictly positive $\mathcal{H}$-measurable bounded random variables $F_n$ such that

$$ E(F_n \mid \mathcal{G}) = 1, \quad E(WF_n \mid \mathcal{G}) = 0 \tag{2} $$

and we also have

$$ F_n \to F \text{ a.s.} \tag{3} $$

Use the above result under $P_2$ with the choice $W := X$ and

$$ F := \frac{\zeta}{E(\zeta \mid \mathcal{G})}, \quad \text{where} \quad \zeta = \frac{dP}{dP_2}. $$

Note that $1/\zeta = dP/dP_2$ is bounded and so is $Y/\zeta$ by the choice of $P_1$. Define

$$ Z_n := \frac{E_2(\zeta \wedge n \mid \mathcal{G})}{E_2(\zeta \wedge n)}, \quad \text{where} \quad \zeta = \frac{dP}{dP_2}. $$

Since $F_n, 1/\zeta, \zeta \wedge n, \text{and } Y/\zeta$ are all bounded, so are $Z_n$ and $Z_n Y$. By (2),

$$ E_2(XF_n \mid \mathcal{G}) = 0, $$

and hence $Z_n Y$ is bounded as well.
consequently,

$$E(XZ_n | \mathcal{F}) = 0. \quad (4)$$

By construction,

$$EZ_n = E_2 \left( E_2(F_n | \mathcal{F}) \frac{E_2(\zeta \land n | \mathcal{F})}{E_2(\zeta \land n)} \right) = 1,$$

and

$$Z_n \to F \frac{E_2(\zeta | \mathcal{F})}{E_2(\zeta)} = 1 \quad \text{a.s.}$$

by (3) and by dominated convergence theorem, thus Scheffé’s theorem shows that choosing \( n \) large enough \( Z_n \) will satisfy all the statements (i), (ii), (iii) above. Lemma 2.1 is proved.

Remark 2.1. If \( h = 1 \), then there is an elementary proof of Lemma 2.1. Unfortunately, this does not provide a proof of Theorems 1.2 and 1.4 in the one-dimensional case. The reason is that our arguments (see the proof of Theorem 1.3 below) require Lemma 2.1 for \( d \) = \( d \) + 1-dimensional random variables. We also note that the rest of this section relies on elementary measure-theoretic techniques only. In particular, we use the following observation, which is a useful corollary of the monotone convergence theorem.

Proposition 2.2. Let \( X_n \) be a sequence of integrable random variables such that

$$\sum_n E|X_n - X_{n+1}| < \infty.$$

Then \( X_n \) is convergent both in \( L^1 \) and a.s.

Lemma 2.1 extends easily to finite sequences of generalized martingale differences. Under this we mean a sequence \( \{(X_k, \mathcal{G}_k): 0 \leq k \leq n\} \) such that

(i) \( X_k \) is a \( \mathcal{G}_k \)-measurable \( R^d \)-valued random variable for \( k = 0, \ldots, n \),

(ii) \( E(X_k | \mathcal{G}_{k-1}) \) exists and is a.s. zero, for \( k = 1, \ldots, n \).

Lemma 2.3. Let \( \{(X_k, \mathcal{G}_k): 0 \leq k \leq n\} \) be a finite sequence of generalized martingale differences, and let \( Y \) be a \( \mathcal{G}_n \)-measurable nonnegative random variable, and \( \varepsilon > 0 \). Then there is a \( \mathcal{G}_n \)-measurable positive random variable \( D \) such that

(i) \( EY = 1, E|D - 1| < \varepsilon \),

(ii) \( D \) and \( YD \) are bounded,

(iii) \( E(X_kD | \mathcal{G}_{k-1}) = 0 \) for \( k = 1, \ldots, n \).

Proof. Let \( \mathcal{G}_* = \{\emptyset, \Omega\} \). Fix \( \eta > 0 \). First we apply Lemma 2.1 with

$$X = X_n, \quad Y = Y, \quad \mathcal{G} = \mathcal{G}_{n-1}, \quad \varepsilon = \eta. \quad (5)$$

This gives a bounded \( Z_n \) such that \( Z_nY \) is also bounded and \( E|Z_n - 1| < \eta \).

If \( Z_n, \ldots, Z_{k+1} \) (0 \( \leq k < n \)) are defined, then we apply Lemma 2.1 with

$$X = \begin{cases} X_k & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases} \quad Y = \frac{1}{E(Z_{k+1} | \mathcal{G}_k)}, \quad \mathcal{G} = \mathcal{G}_{k-1}, \quad \varepsilon = \eta$$

to obtain a bounded \( Z_k \) such that \( Z_k/E(Z_{k+1} | \mathcal{G}_k) \) is also bounded and \( E|Z_k - 1| < \eta \).

With this choice the following variables are all bounded:

$$Z_n, \quad Z_{n-1}, \quad \ldots, \quad Z_1, \quad Z_0,$$

$$Y = \begin{cases} Z_{n-1} & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases} \quad \mathcal{G} = \mathcal{G}_{n-1}, \quad \varepsilon = \eta$$

Put

$$D = \frac{Z_n}{E(Z_n | \mathcal{G}_{n-1})} \ldots \frac{Z_1}{E(Z_1 | \mathcal{G}_0)} \frac{Z_0}{E(Z_0 | \mathcal{G}_0)}.$$
Then $D, DY$ are bounded, and for $1 \leq k \leq n$,

$$E(X_k D | \mathcal{G}_{k-1}) = \frac{Z_{k-1}}{Z_0} E(Z_k | \mathcal{G}_k) \cdot \cdots \cdot Z_1 E(Z_1 | \mathcal{G}_1) \cdot E(Z_0) E(Z_k X_k | \mathcal{G}_{k-1}) = 0.$$ 

We can carry out the above construction for all $\eta > 0$ and in this way we get a sequence of random variables with the obvious notation

$$D_{\eta} = \frac{Z_n(\eta) \cdots Z_0(\eta)}{E(Z_n(\eta) | \mathcal{G}_{n-1}) \cdots E(Z_0(\eta))}.$$ 

Now let $\eta_m > 0$ be such that $\sum_{m=0}^{\infty} \eta_m < \infty$. Let us apply Proposition 2.2 to $X_n := Z_n$ noting that $E|Z_{n+1} - Z_n| \leq \eta_{n+1} + \eta_m$. We get that $\lim_{m \to \infty} Z_k(\eta_m) = 1$ and $\lim_{m \to \infty} E(Z_k(\eta_m) | \mathcal{G}_{k-1}) = 1$ both in $L_1$ and a.s. for $0 \leq k \leq n$. But this implies that $D_{\eta_m} \to 1$ a.s. as $m \to \infty$.

Since $E.D_{\eta_m} = 1$, Scheffé’s theorem gives that for large $m$, $D = D_{\eta_m}$ satisfies $E|D - 1| \leq \varepsilon$ also. Lemma 2.3 is proved.

We can finally prove Theorem 1.3.

Proof. Let $X_0 = S_0, X_n = S_n - S_{n-1}$ for $n > 0$. Take $\varepsilon_n > 0$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \min(\varepsilon, 1)/3$. We may and will suppose that $Y_n \geq |X_n|$ for each $n$.

Using Lemma 2.3 (recall Proposition 2.1) we define a sequence $(D_n)$ in a recursive way. Besides $(D_n)$ a sequence of positive numbers $(c_n)$ is to be defined. These sequences will satisfy the following properties for each $n$:

(i) $0 < D_n(1 + Y_n) \leq c_n$,

(ii) $E|D_n - 1| \leq \varepsilon_n/(1 + \prod_{k<n} c_k)$,

(iii) for $k \leq n$, $E(D_0 \cdots D_n X_k | \mathcal{G}_{k-1}) = 0$,

(iv) $D_n$ is $\mathcal{G}_n$-measurable.

This will prove the statement by the following reasoning. Put $H_n = \prod_{k \leq n} D_k$. Then

$$E|H_{n+1} - H_n| = E|H_n(D_{n+1} - 1)| \leq \prod_{k \leq n} c_k E|D_{n+1} - 1| \leq \varepsilon_{n+1}.$$ 

Since $\varepsilon_n$ is summable, we can use Proposition 2.2 to obtain that $H_n$ is convergent in $L_1$ and a.s., the limit is denoted by $H$. It is also clear that

$$E|H - 1| \leq \sum_{n=0}^{\infty} \varepsilon_n < \frac{\min(\varepsilon, 1)}{3}.$$ 

This implies that $dQ = (H/E.H) dP$ defines a probability measure $Q \ll P$ on $\mathcal{F}$. Elementary calculation gives that

$$\|P - Q\| = E\left|\frac{H}{E.H} - 1\right| \leq \varepsilon.$$ 

To prove that $Q$ is actually equivalent we need that $H > 0$ a.s. This easily follows from

$$\sum_{n=0}^{\infty} |D_n - 1| < \infty \quad \text{a.s.}$$

which is true, since by monotone convergence

$$E\left(\sum_{n=0}^{\infty} |D_n - 1|\right) = \sum_{n=0}^{\infty} E|D_n - 1| \leq \sum_{n=0}^{\infty} \varepsilon_n < \infty.$$ 

In a similar way we can prove that $S$ is a $Q$-martingale and each $Y_k$ is $Q$-integrable. To show this we need that for each $k, |X_k| \leq Y_k$ is in $L^1(Q)$ and $E_Q(X_k | \mathcal{G}_{k-1}) = 0$. In other words, we need that

$$E(HY_k) < \infty \quad \text{and} \quad E(HX_k | \mathcal{G}_{k-1}) = 0.$$
Both follows if we show that $H_n Y_n$, $n \geq 0$, is a Cauchy sequence in $L^1(\mathcal{P})$, since it implies that $H_n X_k \rightarrow H X_k$ not only a.s. but also in $L^1(\mathcal{P})$ and we can use property (iii).

Now for $n \geq k$

$$E |H_{n+1} Y_k - H_n Y_k| = E (|D_{n+1} - 1| D_n \cdots D_{k+1} (D_k Y_k) D_{k-1} \cdots D_0) \leq E (|D_{n+1} - 1|) \prod_{m \leq n} \varepsilon_m \leq \varepsilon_{n+1}.$$ 

So the theorem is proved provided that we can carry out the recursive definition of $(D_n)$ and $(c_n)$ with the properties listed above.

In the recursive definition of the sequence $(D_n)$ we actually use the following equivalent form of property (iii):

$$E (E (H_n | \mathcal{G}_k) S_k | \mathcal{G}_{k-1}) = E (H_n | \mathcal{G}_{k-1}) S_{k-1}, \quad n \geq 0, \quad k > 0, \quad (6)$$

where $H_n = D_0 \cdots D_n$. It is obviously true for $k > n$ by the $\mathcal{G}_n$-measurability of $H_n$, while it follows from property (iii) for $k \leq n$. It is also clear that (6) implies property (iii) by the tower rule.

First we define $D_0, c_0$. Let $Z_n = 1/(1 + Y_0/n)$, and let $n$ be so large that $D_0 = Z_n / E Z_n$ satisfies $E (D_0 - 1) \leq \varepsilon_0$. Clearly $D_0$ and $D_0 Y_0$ are bounded and we can define $c_0$ as the sum of their essential suprema.

Assume that $D_k, c_k, 0 \leq k < n$, are already defined, with the properties listed above. We can apply Lemma 2.3 to $Y = Y_n$, $\varepsilon$ chosen according to (ii), and the differences of the generalized martingale $X_k := S'_k - S'_{k-1}, 1 \leq k < n, X_0 := S'_0$, where

$$S'_k = E (H_{n-1} | \mathcal{G}_k) \left( \frac{S_k}{1} \right), \quad k = 0, \ldots, n.$$ 

The fact that $S'_k, k = 0, \ldots, n$, is a generalized martingale is obvious for the last coordinate, while for the rest of the coordinates it follows from property (iii) of $H_{n-1}$, cf. (6). Note that $S'_k$ has one extra coordinate $E (H_{n-1} | \mathcal{G}_k)$.

In this way we get $D_k; c_k$ is defined as in (i). We have to show property (iii). By the $\mathcal{G}_{n-1}$-measurability of $H_{n-1}$ we get

$$E (H_n | \mathcal{G}_{n-1}) = E (D_n | \mathcal{G}_{n-1}) E (H_{n-1} | \mathcal{G}_{n-1}).$$

The last coordinate of $D_n S'$ satisfies, by the choice of $D_n$,

$$E \left( D_n (E (H_{n-1} | \mathcal{G}_k) - E (H_{n-1} | \mathcal{G}_{k-1})) | \mathcal{G}_{k-1} \right) = 0, \quad 1 < k \leq n.$$ 

This implies

$$E (D_n | \mathcal{G}_k) E (H_{n-1} | \mathcal{G}_k) | \mathcal{G}_{k-1}) = E (D_n | \mathcal{G}_{k-1}) E (H_{n-1} | \mathcal{G}_{k-1}),$$

for $1 \leq k \leq n$. Applying it for $k = n - 1$ gives that

$$E (H_n | \mathcal{G}_{n-2}) = E (E (H_n | \mathcal{G}_{n-1}) | \mathcal{G}_{n-2})$$

$$= E (E (D_n | \mathcal{G}_{n-1}) E (H_{n-1} | \mathcal{G}_{n-1}) | \mathcal{G}_{n-2})$$

$$= E (D_n | \mathcal{G}_{n-2}) E (H_{n-1} | \mathcal{G}_{n-2}).$$

Repeating this we get that

$$E (H_n | \mathcal{G}_k) = E (D_n | \mathcal{G}_k) E (H_{n-1} | \mathcal{G}_k), \quad 0 \leq k \leq n.$$ 

We can apply this and $E (D_n (S'_k - S'_{k-1}) | \mathcal{G}_{k-1}) = 0$ to calculate

$$E (H_n S_k | \mathcal{G}_{k-1}) = E (E (H_n | \mathcal{G}_k) S_k | \mathcal{G}_{k-1})$$

$$= E (E (D_n | \mathcal{G}_k) E (H_{n-1} | \mathcal{G}_k) S_k | \mathcal{G}_{k-1})$$

$$= E (D_n | \mathcal{G}_{k-1}) E (H_{n-1} | \mathcal{G}_{k-1}) S_k = E (H_n | \mathcal{G}_{k-1}) S_k,$$

which is just the equivalent form (6) of property (iii). So the recursion can be continued and Theorem 1.3 is proved.
3. Examples. The continuous-time counterpart of Theorem 1.1 is not true in general. For completeness we give here two simple counterexamples.

Example 3.1. Let $X$ be the three-dimensional squared Bessel process starting from some positive value. Then $X$ satisfies the following equation:

$$dX_t = 2\sqrt{X_t} \, dW_t + 3 \, dt, \quad X_0 > 0,$$

with a Wiener process $W$. It is well known (see, e.g., [5, Chap. XI]) that the solution of the above equation is strong, so for deterministic $X_0$, the process $X_t$ is adapted to the filtration of $W$ and $(Y_t = 1/\sqrt{X_t}, \mathcal{F}_t)$ is a local martingale, where $\mathcal{G}_t = \sigma(W_s, s \leq t)$. This is a well-known example of a local martingale not being a martingale.

Assume that there is $Q \ll P$ such that, under $Q$, $Y$ is a martingale. Put

$$D = \frac{dQ}{dP} \bigg|_{\mathcal{G}_\infty}, \quad D_t = E(D \mid \mathcal{G}_t).$$

It is easy to see that the martingale property under $Q$ would mean

$$E_Q(Y_t \mid \mathcal{G}_t) = E(D_t Y_t \mid \mathcal{G}_t) = Y_t,$$

in other words, $(DY, \mathcal{G})$ is a martingale under $P$. Since $\mathcal{G}$ is a Brownian filtration, $D_t = D_0 + \int_0^t H_s \, dW_s$ with some predictable $H$. This gives that

$$d(DY)_t = D_t \, dY_t + Y_t \, dD_t + d(Y, D)_t.$$

Here the first two terms are local martingales, so if $DY$ were a local martingale also, then $(Y, D)$ would be identically zero. However

$$d(Y, D)_t = -H Y_t^2 \, dt,$$

since $dY_t = -Y_t^2 \, dW_t$. As $Y_t > 0$ a.s. for all $t$, $(Y, D) = 0$ would imply $H = 0$ and $D = D_0$. Since $D$ is $\mathcal{G}_0$-measurable and $\mathcal{G}_0$ is trivial, $D = 1$ and $Q \big|_{\mathcal{G}_\infty} = P \big|_{\mathcal{G}_\infty}$ is a contradiction.

Example 3.2. This example is due to Christophe Stricker. Let $X_t := \exp \left( W_t - \frac{t}{2} \right)$, $t \geq 0$, where $W$ is Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ is its natural filtration. Define the (a.s. finite) stopping time

$$\tau := \inf \left\{ t: X_t = \frac{1}{2} \right\},$$

and set

$$S_t := X_{t \wedge \tau}, \quad 0 \leq t < \frac{\pi}{2}, \quad S_t = \frac{1}{2}, \quad t \geq \frac{\pi}{2}.$$

We can unify the notation, by using the convention $\tau t = \infty$, for $t \geq \pi/2$ in this example, since $\tau$ is finite a.s. Then $S_t = X_{t \wedge \tau}$ and we can define $\mathcal{G}_t := \mathcal{F}_{t \wedge \tau}$. We find that $(S_t)$ is a $(\mathcal{G}_t)$-adapted continuous process. It cannot be a martingale under any measure $P' \sim P$, since $S_{\pi/2} = 1/2$ and $S_t$ is not constant for $t < \pi/2$.

It is, however, a $(\mathcal{G}_t)_{t \geq 0}$-local martingale. Take

$$\sigma_n := \inf \{ t: X_{t \wedge \tau} = n \},$$

then $\tau_n := (\arctg \sigma_n) I_{\{ \sigma_n < \infty \}} + \infty I_{\{ \sigma_n = \infty \}}$, $n \geq 0$, is a sequence of $(\mathcal{G}_t)_{t \geq 0}$-stopping times tending a.s. to infinity.

Since $X_t$ is an $(\mathcal{F}_t)_{t \geq 0}$-martingale, the stopped process $X_{t \wedge \tau \wedge \sigma_n}$ is a (bounded) $(\mathcal{F}_t)_{t \geq 0}$-martingale. Then $X_{t \wedge \tau \wedge \sigma_n} = X_{t \wedge (t \wedge \tau) \wedge \tau} = S_{t \wedge \tau}$ is a $(\mathcal{G}_t)_{t \geq 0}$-martingale, showing that $S$ is indeed a local martingale.
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ALMOST SURE LIMIT THEOREMS FOR GAUSSIAN SEQUENCES

Пусть \{X_n, n \geq 1\} — гауссовые случайные величины с нулевым средним, единичной дисперсией и корреляциями \(r_{ij} = \mathbb{E} X_i X_j\). Предположим, что существует последовательность \(0 \leq \rho_n < 1\), \(n \geq 1\), такая, что \(|r_{ij}| \leq \rho_{|j-i|}\) для \(i \neq j\) и \(\rho_n \ln n(\ln \ln n)^{1+\varepsilon} = O(1)\) при \(n \to \infty\). Для последовательности уровней \(\{u_{nk}, 1 \leq k \leq n, n \geq 1\}\), положим \(\lambda_n = \min_{1 \leq k \leq n} u_{nk}\), и пусть последовательность \(n(1 - \Phi(\lambda_n))\) ограничена. Мы выводим центральные предельные теоремы типа "почти наверное" для \((\ln n)^{-1} \sum_{k=1}^n k^{-1} I(X_1 \leq u_{k1}, \ldots, X_k \leq u_{kk})\) и \((\ln n)^{-1} \times \sum_{k=1}^n k^{-1} I(\max_{1 \leq i \leq k} X_i \leq \lambda_k)\).

Ключевые слова и фразы: центральная предельная теорема типа "почти наверное", гауссовская последовательность, логарифмическое среднее.

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