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RUIN PROBABILITIES FOR LÉVY PROCESSES WITH
MIXED-EXPONENTIAL NEGATIVE JUMPS

To José Luis Massera (1915–2002)
In Memoriam

Найден явный вид вероятности разорения для процессов Леви (воз­
можно, «убиваемых» с постоянной скоростью), отрицательные скачки
которых имеют смешанное экспоненциальное распределение без ограни­
чений на положительные скачки.

Ключевые слова и фразы: вероятность разорения, явный вид, процесс
Леви, смешанное экспоненциальное распределение.

1. Introduction.
1.1. Let $\{X_t\}_{t \geq 0}$ be a real-valued stochastic process defined on a stochastic
basis $(\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ that satisfies the usual conditions. Assume that $X$ is càdlàg,
adapted, $X_0 = 0$, and for $0 \leq s < t$ the random variable $X_t - X_s$ is independent of the
σ-field $\mathcal{F}_s$ with a distribution that only depends on the difference $t - s$. The stochastic
process $X$ is a process with stationary independent increments (PIIS) or a Lévy process.
For $q \in \mathbb{R}$, $\psi(q)$ denotes the characteristic exponent of $X$ given by the Lévy–Khinchine
formula
\[
\psi(q) = \frac{1}{t} \ln \mathbb{E} e^{iqX_t} = ibq - \frac{1}{2} \sigma^2 q^2 + \int_{\mathbb{R}} \left( e^{iyq} - 1 - i y q 1_{\{|y|<1\}} \right) \Pi(dy),
\]
where $b$ and $\sigma > 0$ are real constants and $\Pi$ is a positive measure on $\mathbb{R} - \{0\}$ such that
\[
\int (1 + y^2) \Pi(dy) < \infty,
\]
called the Lévy measure. The function $\psi(q)$, $q \in \mathbb{R}$, completely
determines the law of the process. For general reference on the subject see [4], [11], or [3].
1.2. Consider now a Lévy process with measure $\Pi$ given by
\[
\Pi(dy) = \begin{cases} 
\pi(dy) & \text{if } y > 0, \\
\lambda \sum_{k=1}^{n} a_k e^{\alpha_k y} dy & \text{if } y < 0,
\end{cases}
\]
(1)
where $\pi$ is an arbitrary Lévy measure concentrated on $(0, \infty)$, $0 < \alpha_1 < \ldots < \alpha_n$, $a_k > 0$,
for $k = 1, \ldots, n$ and $\sum_{k=1}^{n} a_k = 1$. The magnitude of the negative jumps of $X$ is mixed
exponentially distributed, with parameter $\alpha_k$ chosen with probability $a_k$, and they occur
at Poissonian times with rate $\lambda$. As the process considered has a finite number of negative
jumps on $[0, t]$, we consider a truncation function
\[
h(y) = y 1_{\{0 < y < 1\}}.
\]

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Simple computations give

$$\psi(q) = iaq - \frac{1}{2} q^2 + \int_0^\infty (e^{iqy} - 1 - iqb(y)) \pi(dy) - \lambda \sum_{k=1}^n a_k \frac{iq}{\alpha_k + iq}$$  \hspace{1cm} (3)

with $a = b + \lambda \sum_{k=1}^n (a_k/\alpha_k) \{1 - (1 + \alpha_k)/e^{\alpha_k}\}$. We introduce now the Laplace exponent of $X$. For $p \in \mathbb{R}$, $p \leq 0$, $p \neq -\alpha_k$, for $k = 1, \ldots, n$, denote

$$\kappa(p) = ap + \frac{1}{2} \sigma^2 p^2 + \int_0^\infty (e^{py} - 1 - ph(y)) \pi(dy) - \lambda \sum_{k=1}^n a_k \frac{p}{\alpha_k + p}. \hspace{1cm} (4)$$

Considered as a function with complex domain, the characteristic function $iq \mapsto \psi(q)$ in (3) can be extended analytically to the complex strip $\{z = p + iq: p \in (-\alpha_1, 0]\}$ and, for $-\alpha_1 < p \leq 0$, we have $\kappa(p) = \psi(p)$. As the Laplace transform of $X_t$ when $-\alpha_1 < p \leq 0$ is given by $\mathbb{E}e^{pX_t} = e^{t\kappa(p)}$, following [3] we call

$$\kappa(p) = t^{-1} \ln \mathbb{E}e^{pX_t}$$  \hspace{1cm} (5)

the Laplace exponent of $X$.

Let us examine the roots of the equation $\kappa(p) = 0$, for $p \leq 0$. Since the integrand in (4) is convex in $p$ for fixed $y$, we obtain that $\kappa(p)$ is convex on $(-\alpha_1, 0]$. From this it follows when $\sigma > 0$, that under the condition

$$\kappa'(0-) = \lim_{p \to 0^-} \frac{1}{p} \kappa(p) = a + \int_1^\infty y \pi(dy) - \lambda \sum_{k=1}^n \frac{a_k}{\alpha_k} > 0,$$ \hspace{1cm} (6)

(where the integral can take the value $\infty$) the equation $\kappa(p) = 0$ has $n + 1$ negative roots $-p_j$, $j = 1, \ldots, n + 1$, that satisfy

$$0 < p_1 < \alpha_1 < p_2 < \cdots < p_n < \alpha_n < p_{n+1}. \hspace{1cm} (7)$$

Furthermore, observe that when $\gamma > 0$, equation $\kappa(p) = \gamma$ has always $n + 1$ negative roots $-p_j$ satisfying (7).

Denote by $\tau(\gamma)$ an exponential random variable with parameter $\gamma > 0$, independent of $X$, and for $\gamma = 0$, put $\tau(0) = \infty$. We are interested in the random variables

$$M = \sup_{0 \leq t < \tau(\gamma)} X_t \text{ and } I = \inf_{0 \leq t < \tau(\gamma)} X_t$$ \hspace{1cm} (8)

called the supremum and infimum of the process, killed at rate $\gamma$ if $\gamma > 0$ (see [3]).

The following result gives a closed formula for the ruin probability,

$$R(x) = \mathbb{P}\{\exists t \in [0, \tau(\gamma)): x + X_t \leq 0\} = \mathbb{P}\{I \leq -x\}, \hspace{1cm} x \geq 0,$$ \hspace{1cm} (9)

for the process with characteristic exponent given by (3).

**Theorem 1.1.** Let $X$ be a Lévy process with characteristic exponent given by (3) and $\sigma > 0$.

(a) Assume that condition (6) holds. Then $\mathbb{P}\{\lim_{t \to \infty} X_t = \infty\} = 1$, i.e., the process drifts to infinity.

(b) Let $\gamma > 0$. Assume that condition (6) holds when $\gamma = 0$. Then the ruin probability (9) is given by

$$R(x) = \sum_{j=1}^{n+1} A_j e^{-p_j x}, \hspace{1cm} x \geq 0,$$ \hspace{1cm} (10)

where $-p_1, \ldots, -p_{n+1}$ are the negative roots of the equation $\kappa(p) = \gamma$ and the constants $A_1, \ldots, A_{n+1}$ are given by

$$A_j = \frac{\prod_{k=1}^n (1 - p_j/\alpha_k)}{\prod_{k=1, k \neq j}^{n+1} (1 - p_j/p_k)}, \hspace{1cm} j = 1, \ldots, n + 1. \hspace{1cm} (11)$$

Remark. As noticed by the referee (and can be confirmed reading the proof), the existence of a real negative root in the interval $(-\infty, -\alpha_n)$ ensures the result (10) also in the case $\sigma = 0$. 

1.3. Example. In the following example, the distributions of the supremum and infimum are obtained simultaneously, when jumps are completely specified. Let the process \( X = \{X_t\}_{t \geq 0} \) be given by

\[
X_t = at + \sigma B_t + \sum_{k=1}^{N_t} Y_k,
\]

where \( B = \{B_t\}_{t \geq 0} \) is a standard Brownian motion, \( N = \{N_t\}_{t \geq 0} \) is a Poisson process with parameter \( \lambda \), \( Y = \{Y_k\}_{k \in \mathbb{N}} \) is a sequence of independent and identically distributed random variables with density

\[
d(y) = \begin{cases} 
\sum_{k=1}^{m} b_k \beta_k e^{-\beta_k y} & \text{if } y > 0, \\
\sum_{k=1}^{n} a_k \alpha_k e^{\alpha_k y} & \text{if } y < 0,
\end{cases}
\]

with \( 0 < \alpha_1 < \cdots < \alpha_n \), \( 0 < \beta_1 < \cdots < \beta_m \), \( a_k \) and \( b_k \) strictly positive for all \( k \), and \( \sum_{k=1}^{m} b_k + \sum_{k=1}^{n} a_k = 1 \). The processes \( B, N, \) and \( Y \) are independent. The Laplace exponent \( \kappa(p) \) is given by

\[
\kappa(p) = ap + \frac{1}{2} \sigma^2 p^2 + \lambda \sum_{k=1}^{m} b_k \frac{p}{\beta_k - p} - \lambda \sum_{k=1}^{n} a_k \frac{p}{\alpha_k + p}.
\]

A simple application of Theorem 1.1 to \( X \) and \( -X \) gives the following corollary.

**Corollary 1.1.** Let \( X \) be given in (12).

(a) For \( \gamma > 0 \), the infimum \( I \) in (8) has a density given by

\[
f_I(x) = \sum_{k=1}^{n+1} A_k p^k e^{p x}, \quad x \leq 0,
\]

where \( -p_1, \ldots, -p_{n+1} \) are the negative roots of the equation \( \kappa(p) = \gamma \) and \( A_1, \ldots, A_{n+1} \) are given by (11).

(b) For \( \gamma > 0 \), the supremum \( M \) in (8) has density given by

\[
f_M(x) = \sum_{k=1}^{m+1} B_k r^k e^{-r x}, \quad x \geq 0,
\]

where \( r_1, \ldots, r_{m+1} \) are the positive roots of the equation \( \kappa(p) = \gamma \) and the coefficients are given by

\[
B_j = \frac{\prod_{k=1}^{m} (1 - r_j / \beta_k)}{\prod_{k=1}^{n+1} (1 - r_j / \alpha_k)}, \quad j = 1, \ldots, m+1.
\]

(c) If \( \kappa'(0) > 0 \) (or \( \kappa'(0) < 0 \)), then the process drifts to \( \infty \) (or to \( -\infty \)), and the density of \( I \) (of \( M \)) in (8) with \( \gamma = 0 \) is given by (14) (or (15)).

**Remark.** (a) and (b) in Corollary 1 can also be obtained from [10] by fractional expansion. In case of Theorem 1.1, in order to use this factorization methods, complex variable results must be considered in order to count and locate the roots of \( \psi(z) = \gamma \).

1.4. Related results to Theorem 1.1 and applications can be found in [1]. The distribution of the infimum of a process with no negative jumps was found by Zolotarev [13], see also [8] or [3] and the references therein. Results giving double Laplace transform of finite time ruin can be found in [2], based on combinatorial methods and complex analysis. Rogozin [10] gives a factorization concerning the ruin of a killed process, extending combinatorial identities for random walks by Spitzer [12]. Theorem 1.1 generalizes previous work in the case of a Wiener process with negative exponential jumps ([5]) and of a Lévy process with negative exponential jumps ([6]). Within the possible applications, the results are used to give closed solutions to optimal stopping problems for Lévy processes ([7]).

The method of proof is direct, along the lines of the barrier problem for simple random walks, consisting in finding a martingale with value 1 in the stopping region that vanishes if this region is not attained.
2. Proofs. First introduce some notation and necessary facts. The infinitesimal generator of the process $X$ with exponent in (3) is

$$
(L^X f)(x) = af'(x) + \frac{1}{2} \sigma^2 f''(x) + \int_0^\infty \int_{-\infty}^\infty \left( f(x+y) - f(x) - f'(x) h(y) \right) \pi(dy)
$$

$$
+ \lambda \sum_{k=1}^n a_k \int_{-\infty}^0 \left( f(x+y) - f(x) \right) \alpha_k e^{\alpha_k y} dy
$$

(16)

with $h$ in (2). The jump measure of $X$ is

$$
\mu^X = \mu^X(\omega, dt, dy) = \sum_s 1_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(dt, dy).
$$

Its compensator is a deterministic measure given by $\nu = \nu(dt, dy) = dt \Pi(dy)$ with $\Pi$ in (1). The process $M = \{M_t\}_{t \geq 0}$ given by

$$
M_t = \sigma B_t + \int_{[0,t] \times \mathbb{R}} h(y)(\mu^X - \nu),
$$

where $B = \{B_t\}_{t \geq 0}$ is a standard Brownian motion, is a martingale with no negative jumps, and $X$ has the decompositions

$$
X_t = M_t + at + \int_{[0,t] \times \mathbb{R}} (y - h(y)) \mu^X
$$

$$
= M_t + at + \sum_{0 < s \leq t} \Delta X_s 1_{\{\Delta X_s \geq 1\}} - \sum_{k=1}^N Y_k,
$$

(17)

(18)

where $N = \{N_t\}_{t \geq 0}$ is a Poisson process with parameter $\lambda$, $Y = \{Y_k\}_{k \in \mathbb{N}}$ is a sequence of positive identically distributed random variables with density $d(y) = \sum_{i=1}^n a_k \alpha_k e^{-\alpha_k y}$, and $B$, $N$ and $Y$ are independent.

The following result gives (20), the integro-differential equation that the ruin probability must satisfy.

Lemma 2.1. (a) Let $0 < p_1 < \alpha_1 < p_2 < \cdots < p_n < \alpha_n < p_{n+1}$, and define the constants

$$
A_j = \frac{\prod_{k=1}^n (1 - p_j/\alpha_k)}{\prod_{k=1, k \neq j}^{n+1} (1 - p_j/p_k)}, \quad j = 1, \ldots, n+1.
$$

Then

$$
\sum_{j=1}^{n+1} A_j = 1,
$$

$$
\sum_{j=1}^{n+1} A_j \frac{p_j}{p_j - \alpha_k} = 0, \quad k = 1, \ldots, n.
$$

(19)

(b) For the infinitesimal generator $L^X$ in (16), the function $R(x)$ in (10), where now $-p_1, \ldots, -p_{n+1}$ are the negative roots of the equation $\kappa(p) = \gamma$ (such that (7) holds), and $x > 0$, we have

$$
(L^X R)(x) = \gamma R(x).
$$

(20)

with $R(x) = 1$ if $x < 0$.

Proof (a) By the theorem on simple fractional expansion, there exist unique constants $A_1, \ldots, A_{n+1}$ such that

$$
\prod_{j=1}^{n+1} (1 + p/\alpha_j) = \sum_{j=1}^{n+1} A_j \frac{p_j}{p_j + p}.
$$

(21)

Each equation of the linear system (19) is equation (21) evaluated at $p = 0$ and $p = -\alpha_k$ for $k = 1, \ldots, n$, respectively. On the other hand, the relation for $A_j$ ($j = 1, \ldots, n+1$) is obtained by standard methods, i.e., multiply (21) by $(p_j + p)/p_j$ and substitute $p = p_j$.
(b) Denote $f_p(x) = e^{-px^+}$, where $x^+ = \max(x, 0)$. As $\sum_{j=1}^{n+1} A_j = 1$, we can write $R(x) = \sum_{j=1}^{n+1} A_j f_p(x)$. For $x > 0$, \( (L^X f_p)(x) = e^{-px} \left( \frac{1}{2} \sigma^2 p^2 - ap + \int_0^\infty (e^{-py} - 1 + ph(y)) \pi(dy) \right) + \lambda \sum_{k=1}^n a_k \int_{-\infty}^0 (e^{-p(x+y)^+} - e^{-px}) \alpha_k e^{\alpha_k y} dy \)

\( = e^{-px} \left( \frac{1}{2} \sigma^2 p^2 - ap + \int_0^\infty (e^{-py} - 1 + ph(y)) \pi(dy) + \lambda \sum_{k=1}^n a_k \frac{p}{\alpha_k - p} \right) + \lambda \sum_{k=1}^n a_k e^{-\alpha_k x} \frac{p}{p - \alpha_k}. \)

In conclusion, for $x > 0$,

\( L^X R(x) = \sum_{k=1}^{n+1} A_j (L^X f_p)(x) \)

\( = \sum_{j=1}^{n+1} A_j e^{-pY^+} \kappa(-p_j) + \lambda \sum_{k=1}^n a_k e^{-\alpha_k x} \sum_{j=1}^{n+1} A_j \frac{p_j}{p_j - \alpha_k} = \gamma R(x), \)

since $\kappa(-p_j) = \gamma$ for $j = 1, \ldots, n + 1$, and (19).

Next lemma summarizes the application of the Meyer–Itô formula to the process $X$ with $\psi$ in (3) and the function $R$ in (10). Denote $Y = \{Y_t = x + X_t\}_{t \geq 0}$.

**Lemma 2.2.** Let $\tau_0 = \inf\{t \geq 0: x + X_t \leq 0\}$. For $R$ in (10)

\( R(Y_{s \wedge \tau_0}) - R(x) = \int_0^{s \wedge \tau_0} (L^X R)(Y_s) \, ds + M(R)_{s \wedge \tau_0} \)

(22) with $L^X$ in (16) and $M(R) = \{M(R)_t\}_{t \geq 0}$ given by

\( M(R)_t = \int_0^t R'(Y_{s-}) \, dM_s + \int_{[0,t] \times \mathbb{R}} W(s, y)(\mu^X - \nu), \)

(23)

is a local martingale, where $W(s, y) = R(Y_{s-} + y) - R(Y_{s-}) - R'(Y_{s-}) h(y)$.

Proof. Since the function $R$ is not $C^2(R)$ we will apply the Meyer–Itô formula ([9, IV.51]) denoting by $m(da)$ the signed measure that is the second derivative of $R$ in the sense of distributions, when restricted to compacts:

\( R(Y_t) - R(x) = \int_0^t R'(Y_{s-}) \, dY_s + \sum_{0 < s \leq t} \left( R(Y_s) - R(Y_{s-}) - R'(Y_{s-}) \Delta Y_s \right) \)

\( + \frac{1}{2} \int_R l^Y(a, t) m(da), \)

(24)

where $l^Y(a, t)$ is the local time of the process $Y$ at level $a$ and time $t$.

First, observe that $m(da) = R''(a) da + R'(0+) \delta_0(da)$ with $R''(a)$ the second derivative of $R$ if $a \neq 0$, $R'(0+)$ the right derivative of $R$ at the point $x = 0$ and $\delta_0(da)$ the point mass at $x = 0$. Applying Corollary 1 to Theorem IV.51 in [9],

\( \int_R l^Y(a, t) m(da) = \int_0^t R''(Y_{s-}) \, d(Y^c, Y^c) \geq + R'(0+) l^Y(0, t). \)

(25)

In view of (17) and (25), as $d(Y^c, Y^c)_s = \sigma^2 ds$, (24) gives

\( R(Y_t) - R(x) = \int_0^t R'(Y_{s-}) \, dM_s + \int_0^t \left( \frac{1}{2} \sigma^2 R''(Y_{s-}) + a R'(Y_{s-}) \right) ds \)

\( + \frac{1}{2} R'(0+) l^Y(0, t) + \sum_{0 < s \leq t} W(s, \Delta X_s). \)

(26)
Now, we compensate the jumps in the last term. As $R$ is bounded, the left-hand side in (26) is a special semimartingale. On the right-hand side, the first term is a local martingale and the second and third are continuous processes, and in consequence, locally integrable. We can apply II.1.28 in [4], obtaining

$$
\sum_{0 < s \leq t} W(s, \Delta X_s) = \int_{[0,t] \times \mathbb{R}} W \mu^X = \int_{[0,t] \times \mathbb{R}} W(\mu^X - \nu) + \int_{[0,t] \times \mathbb{R}} W\nu
$$

$$
= \int_{[0,t] \times \mathbb{R}} W(\mu^X - \nu) + \int_{0}^{t} \left[ \int_{\mathbb{R}} W(s, y) \Pi(dy) \right] ds.
$$

(27)

In order to evaluate (26) at $t_0$, observe that

$$
\lim_{t \to t_0} X_t = 0 \quad \text{because} \quad x + X_t > 0 \quad \text{on the set} \quad [0, t_0] \quad \text{and the local time is continuous in} \quad t \quad \text{(see [9, p. 165]).}
$$

In view of (26) and (27), (22) follows.

Proof of Theorem 1.1. (a) In order to see that

$$
P\{\lim_{t \to \infty} X_t = \infty\} = 1
$$

(28)

consider the decomposition (18). Denote $d = \lambda \sum_{k=1}^{n} \alpha_k / \alpha_k$. By (6) there exists $\varepsilon > 0$ such that

$$
a + \int_{1}^{\infty} y \pi(dy) - (d + \varepsilon) > 0.
$$

(29)

Consider $X^1 = \{X^1_t\}_{t \geq 0}$ and $X^2 = \{X^2_t\}_{t \geq 0}$ with

$$
X^1_t = at + M_t + \sum_{0 < s \leq t} \Delta X_s 1_{\{\Delta X_s \geq 1\}} - (d + \varepsilon) t, \quad X^2_t = (d + \varepsilon) t - \sum_{k=1}^{N_t} Y_k.
$$

On the one hand, as $X^1$ has only positive jumps, by (29) and VII.1.2 in [3] we deduce $P\{\lim_{t \to \infty} X^1_t = \infty\} = 1$. On the other hand, as $E(X^2_t) = \varepsilon t > 0$, we have $P\{\lim_{t \to \infty} X^2_t = \infty\} = 1$ and conclude (28).

Proof of (b). According to Lemma 2.2, $\{R(Y_t \wedge \tau_0)\}$ is a semimartingale. So the quadratic covariation $[R(Y_t \wedge \tau_0), e^{-\gamma t}] = 0$ since the deterministic process $\{e^{-\gamma t}\}$ is continuous and has bounded variation. This gives

$$
e^{-\gamma (t \wedge \tau_0)} R(Y_t \wedge \tau_0) - R(x) = \int_{0}^{\tau_0 \wedge t} e^{-\gamma s} dR(Y_s \wedge \tau_0) - \int_{0}^{\tau_0 \wedge t} R(Y_s \wedge \tau_0) \gamma e^{-\gamma s} ds
$$

$$
= \int_{0}^{\tau_0 \wedge t} e^{-\gamma s} [L^X R(Y_s) - \gamma R(Y_s)] ds + \int_{0}^{\tau_0 \wedge t} e^{-\gamma s} dM(R),
$$

with $M(R)$ in (23), by Lemma 2.1(b), as $Y_s = x + X_s > 0$ on $[0, \tau_0]$ and (20). Since the function $R$ is bounded, taking expectations and limits as $t \to \infty$, it follows that

$$
R(x) = E(e^{-\gamma \tau_0} R(x + X_{\tau_0})) = E e^{-\gamma \tau_0} = P\{\tau_0 < \tau(\gamma)\} = P\{\exists t \in [0, \tau(\gamma)) : x + X_t \leq 0\}
$$

concludes the proof.

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HÖLDER EQUALITY FOR CONDITIONAL EXPECTATIONS WITH APPLICATION TO LINEAR MONOTONE OPERATORS

Let \((\Omega, \mathcal{F}, \mathbf{P})\) be a probability space with a probability measure \(\mathbf{P}(A), A \in \mathcal{F}\), on a \(\sigma\)-algebra \(\mathcal{F}\) of sets \(A \subseteq \Omega\), and let
\[ L_p = L_p(\Omega, \mathcal{F}, \mathbf{P}), \quad 1 \leq p < \infty, \]
be a standard \(L_p\)-space of real random variables \(X = X(\omega), \omega \in \Omega\), with \(\|X\| = (\mathbf{E}|X|^p)^{1/p}\). Here we consider the conditional expectation
\[ x(X) = \mathbf{E}(X f \mid \mathcal{G}), \quad X \in L_p, \]
associated with an appropriate factor \(f = f(\omega), \omega \in \Omega\), and a \(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\). Naturally the factor \(f\) should be at least such that \(\mathbf{E}|X f| < \infty\) for all \(X \in L_p\).

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