On equivalence of some subcategories of modules in Morita contexts

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ABSTRACT. A Morita context \((R, R V S, S W R, S)\) defines the isomorphism \(L_0(R) \cong L_0(S)\) of lattices of torsions \(r \geq r_I\) of \(R\)\(-\)Mod and torsions \(s \geq r_J\) of \(S\)\(-\)Mod, where \(I\) and \(J\) are the trace ideals of the given context. For every pair \((r, s)\) of corresponding torsions the modifications of functors \(T^W = W \otimes_{R^-} -\) and \(T^V = V \otimes_{S^-} -\) are considered:

\[
\begin{align*}
R\text{-Mod} & \supseteq \mathcal{P}(r) \quad \xrightarrow{T^W = (1/s) \cdot T^W} \mathcal{P}(s) \subseteq S\text{-Mod}, \\
& \xleftarrow{T^V = (1/r) \cdot T^V}
\end{align*}
\]

where \(\mathcal{P}(r)\) and \(\mathcal{P}(s)\) are the classes of torsion free modules. It is proved that these functors define the equivalence

\[
\mathcal{P}(r) \cap J_I \approx \mathcal{P}(s) \cap J_J,
\]

where \(\mathcal{P}(r) = \{R M \mid r(M) = 0\}\) and \(J_I = \{R M \mid IM = M\}\).

Let \((R, R V S, S W R, S)\) be an arbitrary Morita context with the bimodule morphisms

\[
(,): V \otimes_S W \longrightarrow R, \quad [,]: W \otimes_R V \longrightarrow S,
\]

satisfying the conditions of associativity:

\[
(v, w)v_1 = v[w, v_1], \quad [w, v]w_1 = w(v, w_1) \tag{1}
\]

for \(v, v_1 \in V\) and \(w, w_1 \in W\). We denote by \(I = (V, W)\) and \(J = [W, V]\) the trace ideals of this context, where \(I\) is ideal of \(R\) and \(J\) is ideal of \(S\).

2000 Mathematics Subject Classification: 16S90, 16D90.

Key words and phrases: torsion (torsion theory), Morita context, torsion free module, accessible module, equivalence.
They define the torsions $r_I$ in $R$-$\text{Mod}$ and $r_J$ in $S$-$\text{Mod}$ such that the classes of torsion free modules are:

\[
\mathcal{P}(r_I) = \{RM \mid I m = 0, m \in M \implies m = 0\},
\]
\[
\mathcal{P}(r_J) = \{SN \mid J n = 0, n \in N \implies n = 0\},
\]
i.e. $r_I$ and $r_J$ are determined by the smallest Gabriel filters, containing $I$ and $J$, respectively [7].

In the lattices $\mathcal{L}(R)$ and $\mathcal{L}(S)$ of all torsions of $R$-$\text{Mod}$ and $S$-$\text{Mod}$, respectively, we distinguish the following sublattices:

\[
\mathcal{L}_0(R) = \{r \in \mathcal{L}(R) \mid r \geq r_I\},
\]
\[
\mathcal{L}_0(S) = \{s \in \mathcal{L}(S) \mid s \geq r_J\}.
\]

The following result is well known ([1], [4], [5], [7]).

**Theorem 1.** There exists a preserving order bijection between the torsions of $R$-$\text{Mod}$ containing $r_I$ and torsions of $S$-$\text{Mod}$ containing $r_J$, i.e. $\mathcal{L}_0(R) \cong \mathcal{L}_0(S)$. □

This bijection is obtained with the help of the functors:

\[
R$-$\text{Mod} \xleftrightarrow{H^V = \text{Hom}_R(V, -)} S$-$\text{Mod},
\]
\[
H^W = \text{Hom}_S(W, -)
\]
acting by $H^V$ and $H^W$ to the injective cogenerators of torsions [4]. From the definitions it follows

**Lemma 2.** ([4], Lemma 4). If $(r, s)$ is a pair of corresponding torsions in the sense of Theorem 1 (i.e. $H^V(r) = s$ and $H^W(s) = r$), then $H^V(\mathcal{P}(r)) \subseteq \mathcal{P}(s)$ and $H^W(\mathcal{P}(s)) \subseteq \mathcal{P}(r)$, where $\mathcal{P}(r)$ and $P(s)$ are (P) the classes of torsion free modules. □

Now we consider the following functors accompanying the given Morita context:

\[
R$-$\text{Mod} \xleftrightarrow{T^W = W \otimes_R -} S$-$\text{Mod}
\]
with the natural transformations

\[
\eta: T^V T^W \longrightarrow 1_{R$-$\text{Mod}}, \quad \rho: T^W T^V \longrightarrow 1_{S$-$\text{Mod}},
\]
defined by the rules:

\[
\eta_M(v \otimes w \otimes m) = (v, w)m, \quad \rho_S(w \otimes v \otimes n) = [w, v]n,
\]

(5)
for \( v \otimes w \otimes m \in T^V T^W(M), \, M \in R-Mod \) and \( w \otimes v \otimes n \in T^W T^V(N), \, N \in S-Mod \). By definitions it follows:

\[
\text{Im } \eta_M = IM, \quad \text{Im } \rho_N = JN.
\]

It is easy to verify the following relations:

\[
T^W(\eta_M) = \rho_{T^W(M)}, \quad (6)
\]

\[
T^V(\rho_N) = \eta_{T^V(N)}, \quad (7)
\]

for every \( M \in R-Mod \) and \( N \in S-Mod \) (i.e. \((T^W, T^V)\) and \((\eta, \rho)\) define a wide Morita context in the sense of [3]).

For an arbitrary class of modules \( K \subseteq R-Mod \) we denote:

\[
K^\uparrow = \{ X \in R-Mod \mid \text{Hom}_R(X, Y) = 0 \quad \forall Y \in K \},
\]

\[
K^\downarrow = \{ Y \in R-Mod \mid \text{Hom}_R(X, Y) = 0 \quad \forall X \in K \}.
\]

If \( r \) is a torsion of \( R-Mod \), \( \mathcal{R}(r) = \{ M \in R-Mod \mid r(M) = M \} \) and \( \mathcal{P}(r) = \{ M \in R-Mod \mid r(M) = 0 \} \), then \( \mathcal{R}(r) = \mathcal{P}(r)^\uparrow \) and \( \mathcal{P}(r) = \mathcal{R}(r)^\downarrow \) ([5], [7], [8]).

The following statement is known ([6], lemma 3), but for convenience we give the proof.

**Lemma 3.** If \((r, s)\) is a pair of corresponding torsions in the sense of Theorem 1, then \( T^W(\mathcal{R}(r)) \subseteq \mathcal{R}(s) \) and \( T^V(\mathcal{R}(s)) \subseteq \mathcal{R}(r) \).

**Proof.** Let \( sN \in \mathcal{R}(s) = \mathcal{P}(s)^\uparrow \), i.e. \( \text{Hom}_S(N, Y) = 0 \) for every \( Y \in \mathcal{P}(s) \). If \( M \in \mathcal{P}(r) \), then by Lemma 2 \( sH^V(M) = \text{Hom}_R(V, M) \in \mathcal{P}(s) \).

Now from \( N \in \mathcal{R}(s) \) it follows that \( \text{Hom}_S(N, \text{Hom}_R(V, M)) = 0 \). By adjunction

\[
\text{Hom}_R(V \otimes_S N, M) \cong \text{Hom}_S(N, \text{Hom}_R(V, M)) = 0
\]

for every \( M \in \mathcal{P}(r) \), therefore \( V \otimes_S N \in \mathcal{P}(r)^\uparrow = \mathcal{R}(r) \), i.e. \( T^V(\mathcal{R}(s)) \subseteq \mathcal{R}(r) \). By symmetry the relation \( T^W(\mathcal{R}(r)) \subseteq \mathcal{R}(s)(\mathcal{R}) \) is true. \( \Box \)

In continuation we mention some facts about the classes of modules determined by trace ideals \( I \triangleleft R \) and \( J \triangleleft S \) in the categories \( R-Mod \) and \( S-Mod \), respectively. The ideal \( I \triangleleft R \) defines in \( R-Mod \) the following classes of modules:

\[
\mathcal{A}(I) = \{ M \in R-Mod \mid IM = 0 \},
\]

\[
\mathcal{J}_I = \{ M \in R-Mod \mid IM = M \},
\]

\[
\mathcal{F}_I = \{ M \in R-Mod \mid Im = 0, \, m \in M \implies m = 0 \} = \mathcal{P}(r_I).
\]
The modules of \( J_I \) are called \( I \)-accessible and

\[
J_I = \{ M \in R\text{-Mod} \mid \text{Im } \eta_M = M \}.
\]

The following relations are known ([7], [8]):

\[
J_I = A(I)^\uparrow, \quad F_I = A(I)^\perp. \quad (8)
\]

Similarly we define the classes \( A(J), J_J \) and \( F_J \) in \( S\text{-Mod} \) with the relations

\[
J_J = A(J)^\uparrow \quad \text{and} \quad F_J = A(J)^\perp, \quad \text{where} \quad F_J = P(r_J).
\]

**Lemma 4.** Let \((r, s)\) be a pair of corresponding torsions (Theorem 1). Then \( A(I) \subseteq \mathcal{R}(r) \) and \( A(J) \subseteq \mathcal{R}(s) \).

**Proof.** From \( r \geq r_I \) it follows \( \mathcal{P}(r) \subseteq \mathcal{P}(r_I) = \mathcal{F}_I \) and by (8) we obtain

\[
\mathcal{R}(r) = \mathcal{P}(r)^\uparrow \supseteq \mathcal{P}(r_I)^\uparrow = \mathcal{F}_I^\uparrow = A(I)^{\perp \uparrow} \supseteq A(I).
\]

Similarly, \( \mathcal{R}(s) \supseteq A(J) \). \( \Box \)

From now on we fix an arbitrary pair \((r, s)\) of corresponding torsions, i.e. \( r \geq r_I, \ s \geq r_J, \ s = H^V(r) \) and \( r = H^W(s) \) (Theorem 1). We consider the following modifications of the functors \( T^W \) and \( T^V \):

\[
\begin{array}{ccc}
R\text{-Mod} & \xrightarrow{T^W} & S\text{-Mod} \\
\downarrow & & \downarrow \\
1/r & & 1/s \\
\xrightleftharpoons{T^V} & & \xrightleftharpoons{T^V} \quad S\text{-Mod},
\end{array}
\]

where \((1/r)(M) = M/r(M), (1/s)(N) = N/s(N), \ T^W = (1/s) \cdot T^W \) and \( T^V = (1/r) \cdot T^V \). So, by definition:

\[
\tilde{T}^W(rM) = (W \otimes_R M)/s(W \otimes_R M), \quad \tilde{T}^V(sN) = (V \otimes_S N)/r(V \otimes_S N) \quad (9)
\]

for \( M \in R\text{-Mod} \) and \( N \in S\text{-Mod} \). Denote by \( \alpha \) and \( \beta \) the natural transformations:

\[
\alpha : T^W \rightarrow \tilde{T}^W, \quad \beta : T^V \rightarrow \tilde{T}^V,
\]

where

\[
\alpha_M : T^W(M) \rightarrow T^W(M)/s(T^W(M))
\]

and

\[
\beta_N : T^V(N) \rightarrow T^V(N)/r(T^V(N))
\]
are the natural epimorphisms. Since the functors $T^W$ and $T^V$ are right exact, it is clear that the functors $\bar{T}^W$ and $\bar{T}^V$ preserve epimorphisms. By definitions of $\bar{T}^W$ and $\bar{T}^V$ it follows that $\bar{T}^W(M) \in \mathcal{P}(s)$ and $\bar{T}^V(N) \in \mathcal{P}(r)$ for every $M \in R\text{-Mod}$ and $N \in S\text{-Mod}$, therefore we can consider the restrictions of these functors on the subcategories $\mathcal{P}(r)$ and $\mathcal{P}(s)$:

$$\mathcal{P}(r) \xrightarrow{\bar{T}^W} \mathcal{P}(s). \quad (10)$$

In the situation (10) there exist the modifications of natural transformations $\eta$ and $\rho$:

$$\bar{\eta} : \bar{T}^V \bar{T}^W \rightarrow 1_{\mathcal{P}(r)}, \quad \bar{\rho} : \bar{T}^W \bar{T}^V \rightarrow 1_{\mathcal{P}(s)},$$

which are defined (see [3]) as follows. For every $M \in \mathcal{P}(r)$ applying $T^V$ to the exact sequence

$$0 \rightarrow s(T^W(M)) \xrightarrow{i_M} T^W(M) \xrightarrow{\alpha_M} T^W(M)/s(T^W(M)) \rightarrow 0, \quad (11)$$

we obtain the diagram:

$$\begin{array}{c}
0 \\
\downarrow i \\
\downarrow \eta_M \\
\downarrow \eta'_M \\
\downarrow \bar{\eta}_M \\
M
\end{array}$$

Since $s(T^W(M)) \in \mathcal{R}(s)$, by Lemma 3 $T^V(s(T^W(M))) \in \mathcal{R}(r)$, so from $M \in \mathcal{P}(r)$ it follows $\text{Hom}_R(T^V(s(T^W(M))), M) = 0$, therefore $\eta_M \cdot T^V(i_M) = 0$. Since $\text{Im} T^V(i_M) = \text{Ker} T^V(\alpha_M) \subseteq \text{Ker} \eta_M$ and $T^V(\alpha_M)$ is an epimorphism, there exists an unique morphism $\eta'_M$ such that $\eta'_M \cdot T^V(\alpha_M) = \eta_M$. The following step: from $M \in \mathcal{P}(r)$ and $r(T^V \bar{T}^W(M)) \in \mathcal{R}(r)$ it follows $\eta'_M \cdot i = 0$ and there exists an unique morphism $\bar{\eta}_M$ such that $\bar{\eta}_M \cdot \beta_{\bar{T}^W(M)} = \eta'_M$. So, by definitions we have:

$$\eta_M = \bar{\eta}_M \cdot \beta_{\bar{T}^W(M)} \cdot T^V(\alpha_M). \quad (13)$$

In such a way it is obtained a natural transformations $\bar{\eta}$ ([3]) and symmetrically $\bar{\rho}$ is defined. From these definitions follows immediately
Lemma 5. \( a \) If the module \( M \in \mathcal{P}(r) \) is \( I \)-accessible (i.e. \( \eta_M \) is epi), then \( \bar{\eta}_M \) is an epimorphism.

\( b \) If the module \( N \in \mathcal{P}(s) \) is \( J \)-accessible, then \( \bar{\rho}_N \) is an epimorphism.

Now we consider in \( \mathcal{P}(r) \) and \( \mathcal{P}(s) \) the following subcategories of torsion free and accessible modules:

\[
\mathcal{A} = \mathcal{P}(r) \cap \mathcal{J}_I \subseteq R\text{-Mod}, \quad \mathcal{B} = \mathcal{P}(s) \cap \mathcal{J}_J \subseteq S\text{-Mod}.
\]

Lemma 6. The functors \( \bar{T}^W \) and \( \bar{T}^V \) transfer subcategories \( \mathcal{A} \) and \( \mathcal{B} \) each one in another, i.e. \( \bar{T}^W(\mathcal{A}) \subseteq \mathcal{B} \) and \( \bar{T}^V(\mathcal{B}) \subseteq \mathcal{A} \).

**Proof.** Let \( M \in \mathcal{A} \). Since \( \bar{T}^W(M) \in \mathcal{P}(s) \), it is sufficient to check that \( \bar{T}^W(M) \in \mathcal{J}_J \). For that we consider the following commutative diagram:

\[
\begin{array}{ccc}
T^W T^V T^W(M) & \overset{\rho_{T^W(M)}}{\longrightarrow} & T^W(M) \\
\downarrow & & \downarrow \alpha_M \\
T^W T^V \bar{T}^W(M) & \overset{\rho_{T^W(M)}}{\longrightarrow} & \bar{T}^W(M)
\end{array}
\tag{14}
\]

Since \( M \in \mathcal{J}_I \), \( \eta_M \) is epi, therefore \( T^W(\eta_M) \) is epi. From (6) \( \rho_{T^W(M)} = T^W(\eta_M) \), so \( \rho_{T^W(M)} \) is epi, therefore \( \alpha_M \cdot \rho_{T^W(M)} \) also is epi. Now diagram (14) shows that \( \rho_{T^W(M)} \) is epimorphism, i.e. \( \bar{T}^W(M) \in \mathcal{J}_J \). This proves that \( \bar{T}^W(\mathcal{A}) \subseteq \mathcal{B} \). By symmetry \( \bar{T}^V(\mathcal{B}) \subseteq \mathcal{A} \). \( \square \)

Another proof of Lemma 6 follows from the remark that

\[
\begin{align*}
T^W(\mathcal{J}_I) & \subseteq \mathcal{J}_J, & T^V(\mathcal{J}_J) & \subseteq \mathcal{J}_I. \tag{15}
\end{align*}
\]

Indeed, if \( M \in \mathcal{J}_I \) then:

\[
\begin{aligned}
J(W \otimes_R M) &= [W, V]W \otimes_R M = W(V, W) \otimes_R M = \\
&= W \otimes_R (V, W)M = W \otimes_R IM = W \otimes_R M,
\end{aligned}
\]

i.e. \( T^W(M) \in \mathcal{J}_J \), and similarly for the second relation.

Now from (15) for every \( M \in \mathcal{J}_I \) we obtain:

\[
\begin{aligned}
J \cdot \bar{T}^W(M) &= J \cdot [(W \otimes_R M)/s(W \otimes_R M)] = \\
&= [J(W \otimes_R M) + s(W \otimes_R M)]/s(W \otimes_R M) \overset{(15)}{=} \\
&= [W \otimes_R M + s(W \otimes_R M)]/s(W \otimes_R M) = \\
&= (W \otimes_R M)/s(W \otimes_R M) = \bar{T}^W(M),
\end{aligned}
\]
therefore $T^W(M) \in \mathcal{J}_J$.

Lemma 6 permits to obtain by restriction the functors:

$$
\mathcal{A} \xrightarrow{T^W} \mathcal{B} \xrightarrow{T^V} \mathcal{A}
$$

(16)

with the natural transformations $\eta$ and $\rho$.

**Lemma 7.** a) For every $M \in \mathcal{P}(r)$, $I \cdot \text{Ker } \eta_M = 0$, i.e. $\text{Ker } \eta_M \in \mathcal{A}(I) \subseteq \mathcal{R}(r)$.

b) For every $N \in \mathcal{P}(s)$, $J \cdot \text{Ker } \rho_N = 0$, i.e. $\text{Ker } \rho_N \in \mathcal{A}(J) \subseteq \mathcal{R}(s)$.

**Proof.** From definition of $\eta_M$ (see (12), (13)) it is clear that $\eta_M$ acts as follows:

$$
\eta_M(v \otimes (w \otimes m + s(W \otimes_R M))) = \eta_M(v \otimes w \otimes m) = (v, w)m,
$$

where $(v \otimes (w \otimes m + s(W \otimes_R M))) = \beta_{T^W(M)}T^V(\alpha_M)(v \otimes w \otimes m)$.

If $(v \otimes (w \otimes m + s(W \otimes_R M)) \in \text{Ker } \eta_M$,

then $\eta_M(v \otimes w \otimes m) = (v, m)m = 0$ and for every $(v', w') \in I$ we obtain:

$$(v', w')(v \otimes (w \otimes m + s(W \otimes_R M)) =
(v', w')v \otimes (w \otimes m + s(W \otimes_R M)) =
= (v', w')v[w', v] \otimes (w \otimes m + s(W \otimes_R M)) =
= v' \otimes ([w', v]w \otimes m + s(W \otimes_R M)) =
= v' \otimes (w'(v, w) \otimes m + s(W \otimes_R M)) =
= v' \otimes (w' \otimes (v, w)m + s(W \otimes_R M)) = 0,$$

because $(v, w)m = 0$. From this we can conclude that $I \cdot \text{Ker } \eta_M = 0$ and by Lemma 4 $\text{Ker } \eta_M \in \mathcal{A}(I) \subseteq \mathcal{R}(r)$. The statement (b) follows from symmetry.

**Lemma 8.** a) $\text{Ker } \eta_M = 0$ for every $M \in \mathcal{P}(r)$. b) $\text{Ker } \rho_N = 0$ for every $N \in \mathcal{P}(s)$.

**Proof.** Since $\text{Ker } \eta_M \subseteq \overline{T^V T^W(M)} \in \mathcal{P}(s)$, we have $\text{Ker } \eta_M \in \mathcal{P}(r)$. By Lemma 7 $\text{Ker } \eta_M \in \mathcal{R}(r)$, therefore $\text{Ker } \eta_M \in \mathcal{R}(r) \cap \mathcal{P}(r) = \{0\}$. Similarly $\text{Ker } \rho_N = 0$ for $N \in \mathcal{P}(s)$.

**Theorem 9.** For every pair $(r, s)$ of corresponding torsions (in the sense of Theorem 1) the functors $\overline{T^W}$ and $\overline{T^V}$ (see (10)) with natural transformations $\eta$ and $\rho$ define an equivalence between the subcategories of torsion free and accessible modules $\mathcal{A} = \mathcal{P}(r) \cap \mathcal{J}_I \subseteq \text{R-Mod}$ and $\mathcal{B} = \mathcal{P}(s) \cap \mathcal{J}_J \subseteq \text{S-Mod}$. 


Proof. If $M \in \mathcal{A}$, then by Lemma 5 a) $\bar{\eta}_M$ is epi. Moreover, from $M \in \mathcal{P}(r)$ by Lemma 8 a) we conclude that $\bar{\eta}_M$ is mono, so $\bar{\eta}_M$ is an isomorphism. Symmetrically, for every $N \in \mathcal{B}$ we obtain that $\bar{\rho}_N$ is an isomorphism. Therefore the functors $\bar{T}^W$ and $\bar{T}^V$ with the natural transformations $\bar{\eta}$ and $\bar{\rho}$ establish the equivalence $\mathcal{A} \approx \mathcal{B}$.

The more general situation of wide Morita contexts is studied in [3]. The equivalence of Theorem 9 can be proved by [3, Theorem 2.6], using the preceding lemmas. We exposed the direct proof of this result.

For the particular case of the smallest pair $(r_I, r_J)$ of corresponding torsions we have

Corollary 10. ([2], [3]). The subcategories of torsion free and accessible modules $\mathcal{P}(r_I) \cap J_I$ and $\mathcal{P}(r_J) \cap J_J$ are equivalent.

References


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Received by the editors: 04.06.2003
and final form in 27.10.2003.