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On the derived category of $\overline{M}_{0,n}$

Using Keel's representation and Orlov's theorem, we give an inductive description of the derived category of moduli spaces of n -pointed stable curves of genus zero and some full exceptional collections in it. The detailed calculations are given for $\overline{M}_{0,6}$.

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Dedicated to I.R. Shafarevich on the occasion of his 90th birthday

Introduction

0.1. Summary. Let $f: X \rightarrow Y$ be a monoidal transformation of smooth projective schemes with smooth centre $Z \subset Y$. Then $D^b(\text{Coh } X)$ can in a definite sense be described in terms of $D^b(\text{Coh } Y)$ and $D^b(\text{Coh } Z)$; cf. [1].

We apply this strategy to the calculation of the derived categories of moduli spaces $\overline{M}_{0,n}$. Since there are several methods of representing $\overline{M}_{0,n}$ as the result of a sequence of blow-ups with controlled bases and centres (see, for example, [2], [3]), one may later ask about the interaction of the results obtained, and so on.

In this note we focus on the representation discovered and used by Keel in [2], a remarkable feature of which is that $\overline{M}_{0,n+1}$ is obtained from $\overline{M}_{0,n} \times \overline{M}_{0,4}$ by iterated blow-ups of manifolds isomorphic to $\overline{M}_{0,p+1} \times \overline{M}_{0,q+1}$, $p+q=n$. This produces a straightforward inductive description if one complements the tools used in the computation by a version of the 'Künneth formula' for the derived categories. The simplest useful form is furnished by Proposition 2.1.18 in [4], in which exceptional collections on $V \times W$ are constructed from those on V and W . A much more sophisticated version is contained in [5]; cf. also [6].

This approach might eventually furnish a concise and functorial description of quivers useful for computations of (and in) $D^b(\text{Coh } \overline{M}_{0,n})$, but we have not yet achieved this goal.

We foresee two directions of future work on this subject.

(A) We might expect that replacing motives of $\overline{M}_{0,S}$ by their derived (or enhanced derived) categories could yield a better understanding of their quantum cohomology; cf. [7].

(B) The moduli spaces $\overline{M}_{0,S}$ are rigid, being objects of commutative geometry. Ample experience, including Gauss' q -numbers and quantum groups, shows that non-commutative ('quantum') deformations of rigid objects are especially interesting. The beautiful recent paper [8] shows how to represent $\overline{M}_{0,S}$ as the projective spectrum of a Koszul graded ring of sections of its log-canonical bundle. It would

be interesting to study the deformations of the derived category of $\overline{M}_{0,S}$ in terms of the deformations of this coordinate ring, as well as in terms of Bondal quivers [9].

See [10] for a detailed discussion of the case of del Pezzo surfaces, in particular, $\overline{M}_{0,5}$.

Non-commutative deformations of general $\overline{M}_{0,S}$, together with structural correspondences between them, that we envision, might become a sophisticated counterpart of the theory of quantum groups.

0.2. Plan of the paper. The first section summarizes the main theorem due to Orlov [1], which enables one to reconstruct the derived category of a blow-up. We supplement it by some computations in the derived category, stressing, in particular, the codimension-two blow-ups that will appear in the applications to $\overline{M}_{0,S}$. The second section describes Keel's tower; see [2]. In the third section the inductive construction based on the results of Orlov and Keel is elaborated. It is complemented by a new result (Proposition 3.5 and Corollary 3.6) allowing us in certain circumstances to get full exceptional collections consisting of locally free sheaves. Finally, § 4 gives an application of this construction to $\overline{M}_{0,6}$.

We are very grateful to D. Orlov for detailed and stimulating answers to the questions of Yu. Manin. Our thanks are due to C. Böhning for a discussion of the 'Künneth formula' for derived categories.

§ 1. Orlov's theorem

1.1. Exceptional collections. Let K be a ground field, D a K -linear triangulated category with shift functor T , and $E, F \in \text{Ob } D$. Following [9], we consider the graded complex of K -vector spaces with trivial differential:

$$\text{Hom}_D^\bullet(E, F) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_D^k(E, F)[-k], \quad \text{Hom}_D^k(E, F) := \text{Hom}_D(E, T^k F).$$

Recall the following definitions.

(i) An object E is said to be *exceptional* if $\text{Hom}_D^\bullet(E, E)$ consists of $K \cdot \text{id}_E$ put in degree zero.

(ii) A family (E_α) of exceptional objects indexed by a totally ordered set of subscripts α is called *an exceptional collection* (or an exceptional sequence), if

$$\text{Hom}_D^\bullet(E_\alpha, E_\beta) = 0 \quad \text{for } \alpha > \beta. \quad (1.1)$$

It is called *full* if it generates D .

(iii) An exceptional sequence (E_α) as above is said to be *strong* if for any $\alpha < \beta$, the complex $\text{Hom}_D^\bullet(E_\alpha, E_\beta)$ has at most one non-vanishing component which is then of degree zero.

1.2. Blow-ups. Let X be a smooth projective variety over K , $Y \subset X$ a smooth closed subvariety and $j: Y \rightarrow X$ the corresponding embedding. We consider the diagram describing the monoidal transformation of X with centre Y :

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{i} & \tilde{X} \\ \downarrow \pi & & \downarrow q \\ Y & \xrightarrow{j} & X \end{array} \quad (1.2)$$

Let $\mathcal{N}$ be the sheaf of sections of the normal bundle to Y in X . The exceptional divisor $\tilde{Y}$ in $\tilde{X}$ is canonically isomorphic to the relative projective spectrum of the symmetric algebra $S_{\mathcal{O}_Y}(\mathcal{N}^t)$ of the dual sheaf of sections of N . The rank $c \geq 2$ of N is equal to the codimension of Y ; the fibres of π are $\mathbf{P}^{c-1}$. This projective bundle carries the standard invertible relative sheaves $\mathcal{O}_\pi(l)$.

From general definitions, we obtain the canonical exact sequences

$$0 \rightarrow \mathcal{N}^t \rightarrow j^*(\Omega_X^1) \rightarrow \Omega_Y^1 \rightarrow 0 \quad (1.3)$$

and

$$0 \rightarrow \pi^*(\Omega_Y^1) \rightarrow \Omega_{\tilde{Y}}^1 \rightarrow \Omega_{\tilde{Y}/Y}^1 \rightarrow 0. \quad (1.4)$$

The dual exact sequence to that in [11, p. 252] reads

$$0 \rightarrow \Omega_{\tilde{Y}/Y}^1(1) \rightarrow \pi^*(\mathcal{N}^t) \rightarrow \mathcal{O}_\pi(1) \rightarrow 0. \quad (1.5)$$

1.3. From exceptional collections on X and Y to those on $\tilde{Y}$ and $\tilde{X}$.

Generally, for a variety Z we denote by $D(Z)$ the bounded derived category $D^b(\text{Coh } Z)$.

Let (E_α) , $\alpha \in A$, be an exceptional collection in $D(X)$ and (F_β) , $\beta \in B$, an exceptional sequence in $D(Y)$. Consider the following sequence of objects in $D(\tilde{X})$ that is indexed by the set of pairs (β, l) for $\beta \in B$, $-c + 1 \leq l \leq -1$ and pairs $(\alpha, l = 0)$ for $\alpha \in A$:

$$(Ri_*(L\pi^*(F_\beta) \otimes \mathcal{O}_\pi(-c + 1)), \dots, Ri_*(L\pi^*(F_\beta) \otimes \mathcal{O}_\pi(-1)), Lq^*(E_\alpha)). \quad (1.6)$$

It is ordered as suggested in (1.6): in each group with fixed l according to the order of A or B , and groups are ordered by increasing l .

1.4. Proposition. (i) *The sequence (1.6) is exceptional.*

(ii) *If (E_α) is full in $D(X)$ and (F_β) is full in $D(Y)$, then (1.6) is full in $D(\tilde{X})$.*

This is Corollary 4.4 in [1] (notice that our notation for the blow-up diagram (1.2) differs from Orlov's and is closer to that of Huybrechts in [11, Ch. 11]).

In the remainder of this section we produce some formulae for morphism spaces between remaining pairs of objects in (1.6).

1.5. Auxiliary results. (a) *Adjunction/duality formula.* For a morphism $f: U \rightarrow V$ of smooth varieties, define $\dim f := \dim U - \dim V$ and put

$$\omega_f := \omega_U \otimes f^*(\omega_V^{-1}). \quad (1.7)$$

Then for $F \in D(U)$, $E \in D(V)$ we have functorial isomorphisms [11, p. 87]

$$\text{Hom}_{D(V)}(Rf_*(F), E) \cong \text{Hom}_{D(U)}(F, Lf^*(E) \otimes \omega_f[\dim f]) \quad (1.8)$$

(tensor products by an invertible or more general locally free sheaf here and below need not be derived).

(b) *The sheaf $\omega_{\tilde{Y}}$.* We will prove that

$$\omega_{\tilde{Y}} \cong \pi^* \circ j^*(\omega_X) \otimes \mathcal{O}_\pi(-c). \quad (1.9)$$

In fact, from (1.4) and (1.3) we get

$$\omega_{\tilde{Y}} \cong \pi^*(j^*(\omega_X) \otimes \det \mathcal{N}) \otimes \omega_{\tilde{Y}/Y}. \quad (1.10)$$

From (1.5) we obtain

$$\pi^*(\det \mathcal{N}) \cong \omega_{\tilde{Y}/Y}^{-1} \otimes \mathcal{O}_\pi(-c). \quad (1.11)$$

Substituting (1.11) to (1.10), we get (1.9).

(c) *The sheaf ω_i .* We wish to prove that

$$\omega_i \cong \mathcal{O}_\pi(-1), \quad \omega_i[\dim i] \cong \mathcal{O}_\pi(-1)[-1]. \quad (1.12)$$

In fact, in the notation of diagram (1.2), we have

$$\omega_{\tilde{X}} \cong q^*(\omega_X) \otimes \mathcal{O}_{\tilde{X}}((c-1)\tilde{Y}), \quad i^*(\mathcal{O}_{\tilde{X}}(\tilde{Y})) \cong \mathcal{O}_\pi(-1), \quad (1.13)$$

where we write $\tilde{Y}$ in place of the exceptional divisor $i(\tilde{Y})$; cf. [11, p. 252].

From (1.7) and (1.13) we find that

$$\omega_i = \omega_{\tilde{Y}} \otimes i^*(\omega_{\tilde{X}}^{-1}) \cong \omega_{\tilde{Y}} \otimes i^* \circ q^*(\omega_X^{-1}) \otimes \mathcal{O}_\pi(c-1). \quad (1.14)$$

Since $i^* \circ q^* = \pi^* \circ j^*$, substituting (1.9) into (1.14) yields (1.12).

(d) *The sheaf ω_π .* From (1.11) we can calculate $\omega_\pi := \omega_{\tilde{Y}} \otimes \pi^*(\omega_Y^{-1}) \cong \omega_{\tilde{Y}/Y}$:

$$\omega_\pi \cong \det \pi^*(\mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c)$$

and

$$\omega_\pi[\dim \pi] \cong \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c)[c-1]. \quad (1.15)$$

(e) *The sheaf ω_j .* We have

$$\omega_j \cong \det \mathcal{N}, \quad \omega_j[\dim j] \cong \det \mathcal{N}[-c]. \quad (1.16)$$

This follows directly from (1.3).

(f) *The sheaf ω_q .* Finally, from (1.7) and (1.13) we get

$$\omega_q \cong \mathcal{O}_{\tilde{X}}((c-1)\tilde{Y}) \cong \omega_q[\dim q].$$

1.6. Calculations of Hom's. Below we will need some formulae for the morphism spaces between objects of the sequence (1.6) that do not follow directly from the exceptionality of this sequence, that is, $\text{Hom}_{D(\tilde{X})}^\bullet(G, H)$, where G, H are objects of (1.6), indexed respectively by (β_1, l_1) and (β_2, l_2) with $l_1 < l_2$, or else by (β, l) , $l < 0$, and $(\alpha, 0)$. In fact, as Orlov has proved, for other pairs the respective Hom's come from Y , X or vanish.

Moreover, in our applications to Keel's tower, all centres of consecutive blow-ups have codimension $c = 2$. In this case the only relevant value of l is $l = -1$, so that we only need to consider the second option.

Here we take two arbitrary objects $E \in D(X)$ and $F \in D(Y)$.

1.7. Proposition. *There are isomorphisms functorial in E, F :*

$$\mathrm{Hom}_{D(\tilde{X})}(Ri_*(L\pi^*(F) \otimes \mathcal{O}_\pi(l)), Lq^*(E)) \quad (1.17)$$

$$\cong \mathrm{Hom}_{D(Y)}(F \otimes S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N})[1], Lj^*(E)) \quad (1.18)$$

$$\cong \mathrm{Hom}_{D(X)}(Rj_*(F \otimes S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N}) \otimes \det \mathcal{N})[1-c], E). \quad (1.19)$$

PROOF. Applying (1.8) to $i: \tilde{Y} \rightarrow \tilde{X}$ in place of f , we see that (1.17) is isomorphic to

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), Li^* \circ Lq^*(E) \otimes \omega_i[-1]). \quad (1.20)$$

Replacing here ω_i by (1.12) and $Li^* \circ Lq^*$ by $L\pi^* \circ Lj^*$, we rewrite (1.20) as

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), L\pi^* \circ Lj^*(E) \otimes \mathcal{O}_\pi(-1)[-1]). \quad (1.21)$$

Multiply both arguments of (1.21) by the same invertible sheaf $\pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(1-c)$ and then apply to them the same shift $[c]$ without changing Hom . Using (1.15) to rewrite the new second argument, we see that the result will be

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(l+1-c)[c], L\pi^* \circ Lj^*(E) \otimes \omega_\pi[\dim \pi]). \quad (1.22)$$

We can rewrite (1.22) using the adjunction formula (1.8) for π :

$$\mathrm{Hom}_{D(Y)}(R\pi_*[L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)][c], Lj^*(E)). \quad (1.23)$$

Now apply the projection formula for π (cf. [11, (3.11), p. 83]) to the first argument:

$$\begin{aligned} R\pi_*[L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)][c] \\ \cong F \otimes (\det \mathcal{N})^{-1} \otimes^L R\pi_*(\mathcal{O}_\pi(l+1-c))[c], \end{aligned}$$

and remark that the complex $R\pi_*(\mathcal{O}_\pi(l+1-c))$ for $1-c \leq l \leq -1$ is quasi-isomorphic to $R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c))[1-c]$. Hence (1.23) will become

$$\mathrm{Hom}_{D(Y)}(F \otimes (\det \mathcal{N})^{-1} \otimes R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c))[1], Lj^*(E)). \quad (1.24)$$

Finally, relative Serre duality and (1.5) imply that, in the range of l under consideration, we have

$$R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c)) \cong S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N}) \otimes \det \mathcal{N}.$$

Thus, we have proved (1.18).

Multiplying both arguments by $\det \mathcal{N}$, shifting by $-c$ and applying (1.16), we get (1.19).

This finishes the proof.

REMARK 1.1. For $l = -1$, $c = 2$, (1.18) becomes simply

$$\mathrm{Hom}_{D(Y)}(F[1], Lj^*(E)). \quad (1.25)$$

This is the only case that must be considered when $c = 2$. We will use this formula in §§ 3, 4.

REMARK 1.2. We could have started a proof of Proposition 1.7 by applying formula (1.8) to q rather than to i , thus following the diagram (1.8) clockwise rather than anti-clockwise.

1.8. Reconstruction of the derived category of a blow-up. For the sake of simplicity, we continue with the case of codimension 2 blow-up. The main result of [1] in this case establishes the canonical semiorthogonal decomposition of $D(X)$, whose components are admissible subcategories: $(\tilde{D}(Y)_{-1}, D(X)_0)$, where $D(X)_0 := Lq^*(D(X))$ and $\tilde{D}(Y)_{-1}$ consists of objects $Ri_*(L\pi^*(F) \otimes \mathcal{O}_\pi(-1))$, where $F \in \text{Ob } D(Y)$.

Together with the formula (1.17) $\cong$ (1.19) for $c = 2$, $l = -1$, this semiorthogonal decomposition allows us to reconstruct $D(\tilde{X})$, not directly however but only through the mediation of DG-categories, say, of the canonical enhancements of all relevant derived categories (cf. [12] and [13]).

The relevant construction on the level of DG-categories is described in [14, Definition 3.57], under the name ‘upper triangular DG-categories’. Here we briefly describe its relation to the (two-term) semiorthogonal decompositions of triangulated categories, as was explained to us in a private letter from D. Orlov to the first author.

Generally, let $\mathbf{A}$ be a linear pretriangulated DG-category and $\mathcal{A} := H^0(\mathbf{A})$ its homotopy category with its natural triangulated structure. Assume that $\mathcal{A}$ admits a semiorthogonal decomposition, $(\mathcal{E}, \mathcal{F})$ as above, in the case of blow-up. Lift $\mathcal{F}, \mathcal{E}$ to the subcategories $\mathbf{F}, \mathbf{E}$ of $\mathbf{A}$ and consider a pair of objects F, E of the respective categories. The bifunctor given on objects by $(F, E) \mapsto \text{Hom}_{\mathbf{A}}(F, E)$ with values in complexes of linear spaces can be regarded as a DG-functor $\Phi: \mathbf{E} \rightarrow \mathbf{Mod}\text{-}\mathbf{F}$, ‘the upper right corner of the upper triangular DG-category’ in the sense of [14].

Conversely, given a triple $(\mathbf{F}, \mathbf{E}, \Phi)$ as above, one can explicitly construct a new DG-category $\mathbf{B}$. Its objects are triples (F, E, q) , where $q \in \Phi(E)(F)^1$. Moreover, the morphisms constitute the complex

$$\text{Hom}_{\mathbf{B}}((F, E, q), (F', E', q')) := \text{Hom}_{\mathbf{F}}(F, F') \oplus \text{Hom}_{\mathbf{E}}(E, E') \oplus \Phi(E', F)$$

with fairly obvious composition rule.

Now the principle result is that $\mathbf{B}$ is quasi-equivalent to $\mathbf{A}$.

This fact, together with the uniqueness of enhancements and explicit description (1.25) of the bifunctor Φ in the blow-up case, furnishes the reconstruction of the triangulated category.

§ 2. Keel’s tower

2.1. Notation: combinatorics of marks. Let S be a finite set, $\text{card } S = n \geq 3$. We will call the choice of a three-element subset $P \subset S$ an *inductive structure* on S . We will sometimes write $\Sigma := S \setminus P$, so that $S = \Sigma \sqcup P$.

Recall that the boundary strata of $\overline{M}_{0,S}$ are bijectively numbered by (isomorphism classes of) stable S -marked trees. Such a tree describes the dual combinatorial type of the curve parametrized by the generic point of the respective stratum. The number of edges of such a tree equals the codimension of the stratum.

In particular, boundary divisors, that is, trees with one edge, are determined by *unordered 2-partitions* $S = S_1 \sqcup S_2$, which are stable in the sense that $|S_i| \geq 2$. Furthermore, codimension-two strata are determined by *3-partitions* $S = S_1 \sqcup S_2 \sqcup S_3$ in which the middle term S_2 is uniquely defined, whereas S_1 and S_3 can be interchanged. The stability condition here means that $|S_1|, |S_3| \geq 2$, $|S_2| \geq 1$.

Whenever an inductive structure P is chosen on S and $|S| \geq 4$, we may and will order each stable 2-partition by the condition $|S_1 \cap P| \leq 1$, and for $|S| \geq 4$ we will order each stable 3-partition by the condition $|(S_1 \cup S_2) \cap P| \leq 1$.

2.2. Keel's blow-down. Now, assuming $n := |S| \geq 4$ and given an inductive structure on S , consider a one-point set $\{\bullet\}$ disjoint from S and the diagram of two forgetful morphisms, forgetting respectively sections marked by Σ and the one marked by $\bullet$:

$$\begin{array}{ccc} \overline{M}_{0,S \sqcup \{\bullet\}} & \xrightarrow{f_\Sigma} & \overline{M}_{0,P \sqcup \{\bullet\}} \\ \downarrow f_{\{\bullet\}} & & \\ \overline{M}_{0,\Sigma \sqcup P} & & \end{array}$$

Notice that $\overline{M}_{0,P \sqcup \{\bullet\}}$ is $\mathbf{P}^1$ endowed with three boundary points. They are canonically marked by unordered partitions of $P \sqcup \{\bullet\}$ into two parts of cardinality 2. Such a partition, is in turn determined by an element p of P (one part is $\{\bullet, p\}$) or a two-element subset of P (the part not containing $\bullet$).

We summarize below some results of [2], showing, in particular, that the map (inverse to)

$$(f_{\{\bullet\}}, f_\Sigma): \overline{M}_{0,S \sqcup \{\bullet\}} \rightarrow \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}} \quad (2.1)$$

is a composition of blow-ups of smooth codimension-two subvarieties isomorphic to boundary divisors of $\overline{M}_{0,S}$. These blow-ups naturally form a sequence of $n-3$ steps $b_k: B_{k,S} \rightarrow B_{k-1,S}$, $k = 2, \dots, n-2$. At each step, a union of pairwise-disjoint connected smooth submanifolds of codimension two is blown up.

In order to form a bridge between our notation and Keel's, the reader should have in mind the following case:

$$S := \{1, \dots, n\}, \quad \bullet := n+1, \quad P := \{1, 2, 3\}, \quad \Sigma := \{4, \dots, n\}. \quad (2.2)$$

2.3. Exceptional divisors of $(f_{\{\bullet\}}, f_\Sigma)$. In our notation, Lemma 1 of [2] establishes that the exceptional divisors of the morphism $(f_{\{\bullet\}}, f_\Sigma)$ are exactly all boundary divisors of $\overline{M}_{0,S \sqcup \{\bullet\}}$ corresponding to the stable 2-partitions of $S \sqcup \{\bullet\}$ satisfying the following condition:

(*) *the part containing $\bullet$ contains no more than one element of P and has cardinality ≥ 3 .*

As an independent check, the reader can convince himself that the number of such partitions coincides with the difference between the ranks of the Picard groups:

$$\mathrm{rk} \mathrm{Pic} \overline{M}_{0,S \sqcup \{\bullet\}} - \mathrm{rk} \mathrm{Pic}(\overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}}) = 2^{n-1} - n - 1.$$

Let $\bar{\sigma}$ be a partition of $S \sqcup \{\bullet\}$ satisfying (*) above, and let σ be the corresponding partition of S obtained by deleting $\bullet$. Obviously, we have

$$f_\bullet(D_{\bar{\sigma}}) = D_\sigma. \quad (2.3)$$

We will call the cardinality of the second part of $\bar{\sigma}$ (and of σ) *the height* of $D_{\bar{\sigma}}$.

2.4. Keel's tower. The main result of [2, § 1] can now be stated in the following way.

The morphism (2.1) can be represented as a product of blowing-downs,

$$\overline{M}_{0,S \sqcup \{\bullet\}} =: B_{n-2,S} \rightarrow B_{n-3,S} \rightarrow \dots \rightarrow B_{1,S} := \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}}, \quad (2.4)$$

satisfying the following conditions:

(i) The image of any exceptional divisor $D_{\bar{\sigma}}$ of height h remains a divisor in $B_{h,S}$, but becomes a closed subscheme of codimension 2 in $B_{h-1,S}, \dots, B_{1,S}$. The product of the subsequent arrows followed by the projection $B_{1,S} \rightarrow \overline{M}_{0,S}$ identifies this subscheme with D_{σ} .

(ii) Each morphism $B_{h+1,S} \rightarrow B_{h,S}$ is the blow-up of the disjoint union of those subschemes in $B_{h,S}$ that are images of exceptional divisors of height $h+1$. Hence, the connected components of the centre of the respective blow-up are isomorphic to $\overline{M}_{0,p+1} \times \overline{M}_{0,q+1}$, $p, q \leq n-2$.

§ 3. Inductive construction of semiorthogonal decompositions and full exceptional collections

3.1. The inductive step I: functoriality in S . In order to calculate $D^b(\overline{M}_{0,S \sqcup \{\bullet\}})$ assuming that the derived categories of the respective moduli spaces of smaller dimension are already known, we will apply to Keel's tower the results of Orlov summarized in § 1.

More precisely, Keel's tower depends on the choice of an *inductive structure* on S in the sense of § 2.1. In order to be able to induce a given inductive structure on the subsets of S , we will adopt the following convention, essentially returning us to Keel's choice (2.2).

S is totally ordered, and for $|S| \geq 3$, $P \subset S$ consists of the first three elements of S .

The inductive structure induced on subsets of S is then defined by the induced order.

Under these conditions, one easily sees that a bijection between two sets of marks compatible with their respective orders lifts to a unique isomorphism of Keel towers.

3.2. The inductive step II: Keel blow-ups. Our 'inductive leap' from S to $S \sqcup \{\bullet\}$ breaks down into a sequence of small inductive steps corresponding to consecutive floors of the Keel tower (2.4). They will allow us to obtain an inductive description of a class of semiorthogonal decompositions of $D^b(\text{Coh } \overline{M}_{0,S})$. Concretely, in view of Orlov's theorem, each floor $p_k: B_{k+1,S} \rightarrow B_{k,S}$ provides a semiorthogonal decomposition of $D^b(\text{Coh } B_{k+1,S})$ of the form

$$D^b(\text{Coh } B_{k+1,S}): (\tilde{D}(Y_{k,S})_{-1}, D(B_{k,S})_0). \quad (3.1)$$

To be more precise, we recapitulate Orlov's results in this context. Consider the following specializations of the blow-up diagram (1.2):

$$\begin{array}{ccc} \widehat{Y}_{k,S} & \xrightarrow{i_k} & B_{k+1,S} \\ \downarrow \pi_k & & \downarrow q_k \\ Y_{k,S} & \xrightarrow{j_k} & B_{k,S} \end{array} \quad (3.2)$$

in which $i_k: \widehat{Y}_{k,S} \rightarrow B_{k+1,S}$ is the embedding of the exceptional divisor of q_k . Since q_k blows up codimension-two submanifolds, $\pi_k: \widehat{Y}_{k,S} \rightarrow Y_{k,S}$ is a fibration with fibre $\mathbf{P}^1$. Let $O(-1)$ be the corresponding relative sheaf on $\widehat{Y}_{k,S}$. Then $D^b(\text{Coh } \widehat{Y}_{k,S})$ has the semiorthogonal decomposition

$$D^b(\text{Coh } \widehat{Y}_{k,S}): (L\pi_k^*(D^b(\text{Coh } Y_{k,S})) \otimes O(-1), L\pi_k^*(D^b(\text{Coh } Y_{k,S}))). \quad (3.3)$$

Now, the subcategory $\tilde{D}(Y_{k,S})_{-1}$ in (3.1) is defined as

$$\tilde{D}(Y_{k,S})_{-1} := Ri_{k*} [L\pi_k^*(D^b(\text{Coh } Y_{k,S})) \otimes \mathcal{O}(-1)] \quad (3.4)$$

whereas $D(B_{k,S})_0$ in (3.1) is

$$D(B_{k,S})_0 := Lq_k^*(D^b(\text{Coh } B_{k,S})). \quad (3.5)$$

Moreover, since q_k blows up a disjoint union of smooth submanifolds Y_σ , $\sigma = (S_1, S_2)$, each of the subcategories $\tilde{D}(Y_{k,S})_{-1}$ admits an *orthogonal* decomposition

$$\tilde{D}(Y_{k,S})_{-1} : (\tilde{D}(Y_\sigma)_{-1} \mid \text{card } S_2 = k + 2). \quad (3.6)$$

Finally, we have a canonical identification

$$Y_\sigma = \overline{M}_{0,S_1 \sqcup \{\bullet_\sigma\}} \times \overline{M}_{0,S_2 \sqcup \{\bullet_\sigma\}}, \quad (3.7)$$

where $\bullet_\sigma$ corresponds to the point of intersection of two components. Therefore $D^b(\text{Coh } Y_\sigma)$ is generated by the external product $\boxtimes$ of any two exceptional collections generating $\overline{M}_{0,S_1 \sqcup \{\bullet_\sigma\}}$ and $\overline{M}_{0,S_2 \sqcup \{\bullet_\sigma\}}$ respectively.

This ‘Künneth formula’ for derived and DG-categories is known on various levels of generality. Böhning [4] establishes it for collections of locally free sheaves; the proof generalizes to complexes of locally free sheaves. The DG case is treated in [5].

The base $B_{1,S}$ has a similar decomposition (cf. (2.4)). Thus, the Keel tower provides a tool to generate semiorthogonal decompositions and exceptional collections for $D^b(\overline{M}_{0,n+1})$ from the corresponding objects for $D^b(\overline{M}_{0,m})$, $m \leq n$.

3.3. Exceptional collections for small n . On $\overline{M}_{0,4} \equiv \mathbf{P}^1$, the standard full strong exceptional collection is $(\mathcal{O}(-1), \mathcal{O})$.

If we represent $\overline{M}_{0,5}$ as a blow-up $p: \overline{M}_{0,5} \rightarrow \mathbf{P}^2$ at four points, and denote by l_i , $i = 1, \dots, 4$, the corresponding exceptional divisors, then for any choice (F_0, F_1, F_2) of a full strong exceptional collection on $\mathbf{P}^2$, for example, $F_i = \mathcal{O}(i - 2)$, Orlov’s theorem will provide a full strong exceptional collection on $\overline{M}_{0,5}$ of the form

$$(\mathcal{O}_{l_1}(-1), \dots, \mathcal{O}_{l_4}(-1), Lp^*F_0, Lp^*F_1, Lp^*F_2). \quad (3.8)$$

It was shown in [15] that a sequence of mutations turns such a sequence into a full exceptional collection of locally free sheaves (this trick actually works for all del Pezzo surfaces). Taking its $\boxtimes$ with the standard collection on $\mathbf{P}^1$ and then using Keel’s blow-up, one gets an analogue of (3.8) for $\overline{M}_{0,6}$.

In the remainder of this section we will show that under certain assumptions, one can find a sequence of mutations turning the collection thus obtained into a collection of locally free sheaves. In the next section we will demonstrate that this procedure can be applied to $\overline{M}_{0,6}$.

3.4. Preparation. Consider a blow-up diagram (1.2)

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{i} & \tilde{X} \\ \downarrow \pi & & \downarrow q \\ Y & \xrightarrow{j} & X \end{array}$$

For a general blow-up, $R^0 q_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X$ and $R^i q_* \mathcal{O}_{\tilde{X}} = 0$ (see [1, proof of Lemma 4.1]). Hence the projection formula and the Leray spectral sequence give $H^i(\tilde{X}, q^* \mathcal{E}) = H^i(X, \mathcal{E})$ for any locally free sheaf $\mathcal{E}$.

From now on we assume that $j(Y)$ is of codimension two in X . If $(F_1, \dots, F_r)$ (resp. $(E_1, \dots, E_n)$) is a full exceptional collection of locally free sheaves on Y (resp. on X), then Orlov's collection (1.6),

$$(Ri_*(\pi^* F_1 \otimes \mathcal{O}_\pi(-1)), \dots, Ri_*(\pi^* F_r \otimes \mathcal{O}_\pi(-1)), q^* E_1, \dots, q^* E_n), \quad (3.9)$$

is a full exceptional collection on $\tilde{X}$.

3.5. Proposition. *Let $(E_1, \dots, E_n)$ be a full exceptional collection of locally free sheaves on X . Assume moreover that $(j^* E_1, \dots, j^* E_r)$ forms a full exceptional collection on Y . If Y is of codimension 2 in X , then the collection*

$$(q^* E_1, q^* E_1(\tilde{Y}), \dots, q^* E_r, q^* E_r(\tilde{Y}), q^* E_{r+1}, \dots, q^* E_n) \quad (3.10)$$

is a full exceptional collection on $\tilde{X}$.

Here and below we use the abbreviation $q^* L(\tilde{Y})$ for $q^* L \otimes \mathcal{O}_{\tilde{X}}(i(\tilde{Y}))$.

PROOF. The strategy of our proof is simple. We start with the exceptional collection (3.9), with $F_a := q^* E_a$ for $1 \leq a \leq r$, and show that it can be transformed into (3.10) by an explicit sequence of mutations (cf. [9], [16]).

More precisely, for $1 \leq a \leq r$, put $A_a := Ri_*(\pi^* j^* E_a \otimes \mathcal{O}_\pi(-1))$, $B_a := q^* E_a$.

We will first check that

$$\mathrm{Hom}^\bullet(A_b, B_a) = 0 \quad \text{for } b > a, \quad (3.11)$$

so that the right mutation of such an exceptional pair simply reduces to the permutation $(A_b, B_a) \mapsto (B_a, A_b)$. This shows that we may consecutively move A_r in (3.9) to the right until it reaches the position directly to the left of B_r , then move A_{r-1} to the right, until it reaches the position directly to the left of B_{r-1} , and so on. The result will be the exceptional collection $(A_1, B_1; A_2, B_2; \dots; A_r, B_r; B_{r+1}, \dots, B_n)$.

Second, we will check that

$$R_{B_a}(A_a) \cong B_a(\tilde{Y}). \quad (3.12)$$

Thus a further r right mutations will transform the latter collection into

$$(B_1, B_1(\tilde{Y}), \dots, B_r, B_r(\tilde{Y}), B_{r+1}, \dots, B_n),$$

that is, into (3.10).

Proof of (3.11). Consider the isomorphism (1.17) $\cong$ (1.25) written for $F := j^* E_b$, $E := E_a[i]$, where i is an arbitrary shift and $b > a$. Its left hand side will represent one of the components of $\mathrm{Hom}^\bullet(A_b, B_a)$. Hence it suffices to prove that the right hand side vanishes. But it is simply $\mathrm{Hom}_{D(Y)}(j^*(E_b)[1], j^* E_a[i])$, $1 \leq a < b \leq r$, and all these groups vanish because we assumed that $(j^* E_1, \dots, j^* E_r)$ is an exceptional collection.

Proof of (3.12). Now consider the case $a = b$. First, recall that $R_{B_a}(A_a)$ is defined as the cone $C(\alpha_a)$ of the canonical morphism in $D(\tilde{X})$,

$$\alpha_a: A_a \rightarrow \mathrm{Hom}^\bullet(A_a, B_a)^t \otimes B_a, \quad (3.13)$$

where t means the linear dual in the category of graded linear spaces. In order to calculate $\mathrm{Hom}^\bullet(A_a, B_a)$, we note that A_a fits into an exact sequence, $0 \rightarrow q^*E_a \rightarrow q^*E_a(\tilde{Y}) \rightarrow A_a \rightarrow 0$ (cf. (1.13)), and thus A_a is quasi-isomorphic to its locally free resolution,

$$\mathcal{A}_a: 0 \rightarrow q^*E_a \rightarrow q^*E_a(\tilde{Y}) \rightarrow 0 \quad (3.14)$$

(with the first term in degree -1).

Since A_a is exceptional, $\mathrm{Hom}^\bullet(\mathcal{A}_a, B_a)$ is spanned by the canonical isomorphism $\mathcal{A}_a \rightarrow B_a[1]$, which is the identity in degree -1 , and the morphism (3.13) can be represented by the morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & q^*E_a & \xrightarrow{g} & q^*E_a(\tilde{Y}) & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & q^*E_a & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Its cone is the complex

$$0 \longrightarrow q^*E_a \xrightarrow{(\mathrm{id}, -g)} q^*E_a \oplus q^*E_a(\tilde{Y}) \longrightarrow 0,$$

where $q^*E_a \oplus q^*E_a(\tilde{Y})$ is in degree -1 . There exists a short exact sequence of sheaves

$$0 \longrightarrow q^*E_a \xrightarrow{(\mathrm{id}, -g)} q^*E_a \oplus q^*E_a(\tilde{Y}) \xrightarrow{\psi} q^*E_a(\tilde{Y}) \longrightarrow 0,$$

where $\psi(v, w) = g(v) + w$. Therefore, the cone is quasi-isomorphic to the complex with one non-zero term $q^*E(\tilde{Y})$ placed in degree -1 , that is, $q^*E_a(\tilde{Y})[1]$. Hence $R_{B_a}(A_a)[1] \cong q^*E_a(\tilde{Y})[1]$ and finally $R_{B_a}(A_a) \cong B_a(\tilde{Y})$.

This completes the proof of the proposition.

3.6. Corollary. *Let $(E_1, \dots, E_n)$ be a full exceptional collection of locally free sheaves on X . Assume moreover that for some $s \leq r$, $(j^*E_s, \dots, j^*E_r)$ forms a full exceptional collection on Y . If Y is of codimension 2 in X , then the collection*

$$(q^*E_1(\tilde{Y}), \dots, q^*E_{s-1}(\tilde{Y}), q^*E_s, q^*E_s(\tilde{Y}), \dots, q^*E_r, q^*E_r(\tilde{Y}), q^*E_{r+1}, \dots, q^*E_n)$$

is a full exceptional collection on $\tilde{X}$.

PROOF. First, let us recall a general fact. Let $\mathcal{D}$ be a triangulated category with a Serre functor $S: \mathcal{D} \rightarrow \mathcal{D}$ and $\mathcal{A}$ an admissible subcategory (cf. [12], [16]). In this situation we have two semi-orthogonal decompositions $\langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$ and $\langle \mathcal{A}^\perp, \mathcal{A} \rangle$. By [12, Proposition 3.6], we obtain that

$$R_{\perp \mathcal{A}}(\mathcal{A}) = S^{-1}(\mathcal{A}), \quad (3.15)$$

$$L_{\mathcal{A}^\perp}(\mathcal{A}) = S(\mathcal{A}). \quad (3.16)$$

Our proof of the corollary will consist of applications of formulae (3.15), (3.16) and Proposition 3.5.

Let $\mathcal{D} = D(X)$, $\mathcal{A} = \langle E_1, \dots, E_{s-1} \rangle$, ${}^\perp\mathcal{A} = \langle E_s, \dots, E_n \rangle$ and the Serre functor is $S_X = \cdot \otimes \omega_X[\dim X]$. Applying formula (3.15) we get a full exceptional collection

$$(E_s, \dots, E_n, S_X^{-1}(E_1), \dots, S_X^{-1}(E_{s-1})).$$

To this collection we can apply Proposition 3.5, and we get a full exceptional collection on $\tilde{X}$

$$(q^*E_s, q^*E_s(\tilde{Y}), \dots, q^*E_r, q^*E_r(\tilde{Y}), q^*E_{r+1}, \dots, \dots, q^*E_n, Lq^* \circ S_X^{-1}(E_1), \dots, Lq^* \circ S_X^{-1}(E_{s-1})).$$

Now let $\mathcal{D} = D(\tilde{X})$, $\mathcal{A} = \langle Lq^*S_X^{-1}(E_1), \dots, Lq^*S_X^{-1}(E_{s-1}) \rangle$ and the Serre functor is $S_{\tilde{X}} = \cdot \otimes \omega_{\tilde{X}}[\dim \tilde{X}]$. Applying formula (3.16) and using the fact that

$$S_{\tilde{X}} \circ Lq^* \circ S_X^{-1} \simeq Lq^* \otimes \mathcal{O}_{\tilde{X}}(\tilde{Y}),$$

we get the desired assertion.

§ 4. Example: the moduli space $\overline{M}_{0,6}$

4.1. Preparation: the moduli space $\overline{M}_{0,5}$. Let $S = \{1, 2, 3, 4, 5\}$, $S' = \{1, 2, 3, 4\}$ and $P = \{1, 2, 3\}$. Consider the Keel tower

$$\begin{array}{c} B_{2,S'} = \overline{M}_{0,S} \\ \downarrow q_{1,S'} \\ B_{1,S'} = \overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}} \end{array}$$

The map $q_{1,S'}$ contracts three boundary divisors D_{σ_i} corresponding to the unordered partitions

$$\sigma_1 = (5, 1, 4|2, 3), \quad \sigma_2 = (5, 2, 4|1, 3), \quad \sigma_3 = (5, 3, 4|2, 1).$$

Let us identify $\overline{M}_{0,S'}$ with $\overline{M}_{0,P \sqcup \{5\}}$ using the bijection of labels that is the identity on $\{1, 2, 3\}$ and maps 4 to 5. Then the images of D_{σ_i} become three points on the diagonal, corresponding to the partitions with one part $\{i, 4\}$ (resp. $\{i, 5\}$). Let us imagine $\overline{M}_{0,P \sqcup \{5\}}$ as the horizontal axis $\mathbf{P}^1$ and $\overline{M}_{0,S'}$ as the vertical axis. Let H_i (resp. V_i) be the horizontal (resp. vertical) fibre passing through the image of D_{σ_i} .

Now denote by $\tilde{H}_i$ and $\tilde{V}_i$ the proper transforms of H_i (resp. V_i) in $\overline{M}_{0,S}$, and let $\tilde{Z}$ be the proper transform of the diagonal Z . The divisors D_{σ_i} , $\tilde{H}_i$, $\tilde{V}_i$ and $\tilde{Z}$ are all isomorphic to $\mathbf{P}^1$ and have self-intersection (-1) . There are precisely 10 such curves on $\overline{M}_{0,S}$.

Let (F_0, F_1) and (G_0, G_1) be full exceptional collections of locally free sheaves on $\overline{M}_{0,S'}$ and $\overline{M}_{0,P \sqcup \{5\}}$ respectively. It is well known that F_i , G_i are invertible sheaves. We know that

$$(F_0 \boxtimes G_0, F_1 \boxtimes G_0, F_0 \boxtimes G_1, F_1 \boxtimes G_1)$$

is a full exceptional collection on $\overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}}$. Denote it by

$$(L_0, L_1, L_2, L_3). \quad (4.1)$$

Consider the decomposition $q_{1,S'} = f_1 \circ f_2 \circ f_3$, where f_i contracts only D_{σ_i} . Each f_i is a blow-up of a surface at a point. Applying Corollary 3.6 to the blow-up f_3 and using the restriction of L_0 as a full exceptional collection in $D(\text{pt})$, we obtain a full exceptional collection on the resulting surface,

$$(f_3^* L_0, f_3^* L_0(D_{\sigma_3}), f_3^* L_1, f_3^* L_2, f_3^* L_3).$$

Continuing in the same way and always using restriction of the first element as a full exceptional collection in $D(\text{pt})$, we obtain a full exceptional collection on $\overline{M}_{0,S}$,

$$(q_{1,S'}^* L_0, q_{1,S'}^* L_0(D_{\sigma_1}), q_{1,S'}^* L_0(D_{\sigma_2}), q_{1,S'}^* L_0(D_{\sigma_3}), q_{1,S'}^* L_1, q_{1,S'}^* L_2, q_{1,S'}^* L_3). \quad (4.2)$$

Of course, this is just an example of a full exceptional collection on $\overline{M}_{0,S}$. If we had used restrictions of other elements of (4.1) we would have obtained a different answer.

4.2. Preparation: the moduli space $\overline{M}_{0,6}$. Here we will introduce convenient notation which will be used later to write down a full exceptional collection on $\overline{M}_{0,6}$.

Let $S = \{1, 2, 3, 4, 5\}$, $\{\bullet\} = \{6\}$ and $P = \{1, 2, 3\}$. Consider the Keel tower

$$\begin{array}{c} B_{3,S} = \overline{M}_{0,S \sqcup \{6\}} \\ \downarrow q_{2,S} \\ B_{2,S} \\ \downarrow q_{1,S} \\ B_{1,S} = \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{6\}} \end{array} \quad (4.3)$$

The map $q_{1,S} \circ q_{2,S}$ contracts 10 boundary divisors E_i , $1 \leq i \leq 10$. At the level of height 3, it contracts 7 boundary divisors corresponding to the following unordered partitions:

$$\begin{aligned} E_4 &\leftrightarrow (6, 1, 4|5, 2, 3), & E_6 &\leftrightarrow (6, 2, 4|5, 1, 3), & E_8 &\leftrightarrow (6, 3, 4|5, 2, 1), \\ E_5 &\leftrightarrow (6, 1, 5|4, 2, 3), & E_7 &\leftrightarrow (6, 2, 5|4, 1, 3), & E_9 &\leftrightarrow (6, 3, 5|4, 2, 1), \end{aligned}$$

and $E_{10} \leftrightarrow (6, 4, 5|1, 2, 3)$. At the level of height 2, it contracts images under $q_{2,S}$ of 3 boundary divisors corresponding to the following unordered partitions:

$$E_1 \leftrightarrow (6, 1, 4, 5|2, 3), \quad E_2 \leftrightarrow (6, 2, 4, 5|1, 3), \quad E_3 \leftrightarrow (6, 3, 4, 5|2, 1).$$

The divisors E_1, E_2, E_3 are pairwise disjoint and $E_4, \dots, E_{10}$ are pairwise disjoint as well. We list below all non-empty intersections:

$$\begin{aligned} E_1 \cdot E_4 &= P_1, & E_1 \cdot E_5 &= Q_1, \\ E_2 \cdot E_6 &= P_2, & E_2 \cdot E_7 &= Q_2, \\ E_3 \cdot E_8 &= P_3, & E_3 \cdot E_9 &= Q_3, \\ E_1 \cdot E_{10} &= R_1, & E_2 \cdot E_{10} &= R_2, & E_3 \cdot E_{10} &= R_3, \end{aligned}$$

where P_i, Q_i, R_i are isomorphic to $\mathbf{P}^1$ and all intersections are transversal.

Let $(L_0, \dots, L_6)$ be an exceptional collection of invertible sheaves on $\overline{M}_{0,S}$ and (G_0, G_1) an exceptional collection of invertible sheaves on $\overline{M}_{0,P \sqcup \{6\}}$. From them we construct a collection on $\overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{6\}}$:

$$(L_0 \boxtimes G_0, \dots, L_6 \boxtimes G_0, L_0 \boxtimes G_1, \dots, L_6 \boxtimes G_1). \quad (4.4)$$

We will need some assumptions on the collection $(L_0, \dots, L_6)$. We use the notation for $\overline{M}_{0,S'}$ introduced in § 4.1.

ASSUMPTION 1. (L_1, L_2) restricted to D_{σ_1} forms a full exceptional collection on D_{σ_1} ; (L_2, L_3) restricted to D_{σ_2} forms a full exceptional collection D_{σ_2} ; (L_3, L_4) restricted to D_{σ_3} forms a full exceptional collection D_{σ_3} .

ASSUMPTION 2. (L_0, L_1) restricts to a full exceptional collection on $\tilde{H}_1$ and $\tilde{V}_1$. The same holds for (L_1, L_2) on $\tilde{H}_2, \tilde{V}_2$, (L_2, L_3) on $\tilde{H}_3, \tilde{V}_3$ and (L_5, L_6) on $\tilde{Z}$.

These assumptions are satisfied, for example, for

$$(\mathcal{O}, \mathcal{O}(D_{\sigma_1}), \mathcal{O}(D_{\sigma_2}), \mathcal{O}(D_{\sigma_3}), q_{1,S'}^* \mathcal{O}(0, 1), q_{1,S'}^* \mathcal{O}(1, 0), q_{1,S'}^* \mathcal{O}(1, 1)),$$

using the notation of § 4.1 and the identification of $\overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}}$ with $\mathbf{P}^1 \times \mathbf{P}^1$.

4.3. Collection. As in § 4.1, in view of the Keel tower (4.3), we can consecutively apply Corollary 3.6 to the full exceptional collection (4.4) to obtain a full exceptional collection on $\overline{M}_{0,6}$. We start by making a list of its elements, and afterwards give some indications of how it was obtained.

Let $q = q_{1,S} \circ q_{2,S}$ and $E_{\geq i} = \sum_{k=i}^{10} E_k$. The exceptional collection is

$$\begin{aligned} & q^*(L_0 \boxtimes G_0)(E_{\geq 1}), \quad q^*(L_1 \boxtimes G_0)(E_{\geq 2}), \quad q^*(L_1 \boxtimes G_0)(E_{\geq 1}), \\ & q^*(L_2 \boxtimes G_0)(E_{\geq 2}), \quad q^*(L_2 \boxtimes G_0)(E_1 + E_{\geq 3}), \quad q^*(L_2 \boxtimes G_0)(E_{\geq 1}), \\ & q^*(L_3 \boxtimes G_0)(E_{\geq 3}), \quad q^*(L_3 \boxtimes G_0)(E_2 + E_{\geq 4}), \quad q^*(L_3 \boxtimes G_0)(E_{\geq 2}), \\ & \quad q^*(L_4 \boxtimes G_0)(E_{\geq 4}), \quad q^*(L_4 \boxtimes G_0)(E_{\geq 3}), \\ & \quad q^*(L_5 \boxtimes G_0)(E_{\geq 4}), \\ & \quad q^*(L_6 \boxtimes G_0)(E_{\geq 4}), \\ & q^*(L_0 \boxtimes G_1)(E_{\geq 5}), \quad q^*(L_0 \boxtimes G_1)(E_4 + E_{\geq 6}), \quad q^*(L_0 \boxtimes G_1)(E_{\geq 4}), \\ & q^*(L_1 \boxtimes G_1)(E_{\geq 6}), \quad q^*(L_1 \boxtimes G_1)(E_{\geq 5}), \quad q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 7}), \\ & \quad q^*(L_1 \boxtimes G_1)(E_4 + E_6 + E_{\geq 8}), \quad q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 6}), \\ & q^*(L_2 \boxtimes G_1)(E_{\geq 8}), \quad q^*(L_2 \boxtimes G_1)(E_{\geq 7}), \quad q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 9}), \\ & \quad q^*(L_2 \boxtimes G_1)(E_6 + E_8 + E_{10}), \quad q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 8}), \\ & q^*(L_3 \boxtimes G_1)(E_{10}), \quad q^*(L_3 \boxtimes G_1)(E_{\geq 9}), \quad q^*(L_3 \boxtimes G_1)(E_8 + E_{10}), \\ & \quad q^*(L_4 \boxtimes G_1)(E_{10}), \\ & \quad q^*(L_5 \boxtimes G_1), \quad q^*(L_5 \boxtimes G_1)(E_{10}), \\ & \quad q^*(L_6 \boxtimes G_1), \quad q^*(L_6 \boxtimes G_1)(E_{10}). \end{aligned}$$

Below we describe the algorithm that was used to obtain this collection.

4.3.1. *Algorithm, step I.* In view of Assumption 1, the restriction of the pair

$$(L_1 \boxtimes G_0, L_2 \boxtimes G_0)$$

to $q(E_1)$ is a full exceptional collection. The same holds for the pairs $(L_2 \boxtimes G_0, L_3 \boxtimes G_0)$ on $q(E_2)$ and $(L_3 \boxtimes G_0, L_4 \boxtimes G_0)$ on $q(E_3)$.

Represent $q_{1,S} = f_1 \circ f_2 \circ f_3$ as a composition of three blow-ups (cf. § 4.1) in such a way that the (pre-image of) $q(E_i)$ is blown up at the i -th step for $1 \leq i \leq 3$. At the first step we use the pair $(L_1 \boxtimes G_0, L_2 \boxtimes G_0)$ to apply Corollary 3.6. At the second step we use the pair

$$(f_1^*(L_2 \boxtimes G_0)(f_2 \circ f_3 \circ q_{2,S}(E_1)), f_1^*(L_3 \boxtimes G_0)),$$

which restricts to a full exceptional collection on $f_2 \circ f_3 \circ q_{2,S}(E_2)$ because $q(E_1)$ and $q(E_2)$ are disjoint and $(L_2 \boxtimes G_0, L_3 \boxtimes G_0)$ restricts to an exceptional collection on $q(E_2)$ as was pointed out above.

One proceeds similarly at the third step and obtains the following exceptional collection on $B_{2,S}$:

$$\begin{aligned} & q_{1,S}^*(L_0 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\ & q_{1,S}^*(L_1 \boxtimes G_0)(E'_2 + E'_3), \quad q_{1,S}^*(L_1 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\ & q_{1,S}^*(L_2 \boxtimes G_0)(E'_2 + E'_3), \quad q_{1,S}^*(L_2 \boxtimes G_0)(E'_1 + E'_3), \quad q_{1,S}^*(L_2 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\ & q_{1,S}^*(L_3 \boxtimes G_0)(E'_3), \quad q_{1,S}^*(L_3 \boxtimes G_0)(E'_2), \quad q_{1,S}^*(L_3 \boxtimes G_0)(E'_2 + E'_3), \\ & q_{1,S}^*(L_4 \boxtimes G_0), \quad q_{1,S}^*(L_4 \boxtimes G_0)(E'_3), \\ & q_{1,S}^*(L_5 \boxtimes G_0), \quad q_{1,S}^*(L_6 \boxtimes G_0), \\ & q_{1,S}^*(L_0 \boxtimes G_1), \quad q_{1,S}^*(L_1 \boxtimes G_1), \quad \dots, \end{aligned}$$

where $E'_i = q_{2,S}(E_i)$.

4.3.2. *Algorithm, step II.* In view of Assumption 2, the restriction of the pair

$$(L_0 \boxtimes G_1, L_1 \boxtimes G_1)$$

to $q_{2,S}(E_4)$ and $q_{2,S}(E_5)$ gives full exceptional collections on them. The same holds for the pair $(L_1 \boxtimes G_1, L_2 \boxtimes G_1)$ on $q_{2,S}(E_6)$ and $q_{2,S}(E_7)$, for $(L_2 \boxtimes G_1, L_3 \boxtimes G_1)$ on $q_{2,S}(E_8)$ and $q_{2,S}(E_9)$, and for $(L_5 \boxtimes G_1, L_6 \boxtimes G_1)$ on $q_{2,S}(E_{10})$.

Represent $q_{2,S} = g_4 \circ \dots \circ g_{10}$ as the composition of blow-downs g_i of E_i for $4 \leq i \leq 10$. To apply Corollary 3.6 to a single blow-up, one needs to choose an exceptional pair. We always choose pairs related to those described at the beginning of this section (in fact, they are pullbacks of those followed by a twist with $\mathcal{O}(D)$, where D is a divisor disjoint from the exceptional divisor considered at this step).

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