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**On quantiles of minimal codeword weights
of random linear codes over $\mathbb{F}_p$**

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Abstract. We propose explicit equations for two-sided estimates of quantiles of minimal non-zero codeword weight distributions for random equiprobable linear code with given dimension over the prime field $\mathbb{F}_p$. It is shown that the differences between quantiles of these distributions are bounded by values depending only on p and levels of quantiles.

Key words: random linear codes, distributions of minimal codeword weights, quantile estimates

**О квантилях минимальных весов кодовых слов в случайных
линейных кодах над $\mathbb{F}_p$**

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Аннотация. Получены явные уравнения для двусторонних оценок квантилей распределений минимальных весов ненулевых кодовых слов в случайных равновероятных линейных кодах заданной размерности над простым полем $\mathbb{F}_p$. Показано, что разности между квантилями этих распределений ограничены величинами, зависящими только от p и уровней квантилей.

Ключевые слова: случайные линейные коды, распределения минимальных весов кодовых слов, оценки квантилей

Let p be a prime number, $\mathbf{F}_p$ be a prime field, $\mathbf{F}_p^N = \{X = (x_1, \dots, x_N) : x_1, \dots, x_N \in \mathbf{F}_p\}$ be N -dimensional linear space over $\mathbf{F}_p$. Any k -dimensional subspace $L \subset \mathbf{F}_p^N$ is k -dimensional linear code. Weight $w(X) = \sum_{k=1}^N I\{x_k \neq 0\}$ of the vector $X = (x_1, \dots, x_N) \in \mathbf{F}_p^N$ is the number of its non-zero coordinates. Let $\mu_*(L) = \min\{w(X) : X \in L \setminus \{0\}\}$ be the minimal weight of non-zero code-words in a linear code L .

The statement below is a direct consequence of results obtained in [7].

Theorem. *If $L \subset \mathbf{F}_p^N$ is a random linear k -dimensional code in $\mathbf{F}_p^N$ having the uniform distribution on the set of all such codes, then*

$$\frac{1}{1 + \frac{p-1}{(p^k-1)\Sigma_{N,p}^{1,s}}} \leq \mathbf{P}\{\mu_*(L) \leq s\} \leq p^k \Sigma_{N,p}^{1,s}, \quad \Sigma_{N,p}^{a,b} \stackrel{\text{def}}{=} \sum_{u=a}^b C_N^u \frac{(p-1)^u}{p^N}. \quad (1)$$

Fig 1. Two-sided bounds for $\mathbf{P}\{\mu_*(L) \leq s\}$, $N = 128$, $\dim L = 96, 64, 32$

Fig 2. Two-sided bounds for $\mathbf{P}\{\mu_*(L) \leq s\}$, $N = 1024$, $\dim L = 768, 512, 256$

For the case $\mathbf{F}_2^N$ estimates (1) are illustrated graphically on Fig. 1 and Fig. 2 where the lower and upper bounds for probabilities $\mathbf{P}\{\mu_*(L) \leq s\}$ are presented correspondingly for $N = 128$, $N = 1024$ and several values of $\dim L$. Figures 1 and 2 show that distributions of $\mu_*(L)$ are highly concentrated. In what follows we propose an approach to estimate the quantiles of these distributions without calculating binomial sums.

It follows from [6] that if $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2} du$ and

$$H(x, r) = x \ln \frac{x}{r} + (1-x) \ln \frac{1-x}{1-r},$$

then

$$\Phi\left(-\sqrt{2NH\left(\frac{s}{N}, \frac{p-1}{p}\right)}\right) < \Sigma_{N,p}^{0,s} = \Sigma_{N,p}^{1,s} + \frac{1}{p^N} < \Phi\left(-\sqrt{2NH\left(\frac{s+1}{N}, \frac{p-1}{p}\right)}\right). \quad (2)$$

Let us define values $z_C > 0$ and $x_C < 1 - \frac{1}{p}$ as solutions of equations

$$\Phi(-\sqrt{2z_C}) = \frac{C}{p^k}, \quad NH\left(x_C, 1 - \frac{1}{p}\right) = z_C. \quad (3)$$

Then according to (2)

$$p^k \left(\Sigma_{N,p}^{1, \lceil Nx_C \rceil - 1} + \frac{1}{p^N} \right) < C = p^k \Phi\left(-\sqrt{2NH\left(x_C, \frac{p-1}{p}\right)}\right) < p^k \left(\Sigma_{N,p}^{1, \lceil Nx_C \rceil} + \frac{1}{p^N} \right), \quad (4)$$

where $\lceil x \rceil$ denotes the minimal integer in $[x, \infty)$ and $\lfloor x \rfloor$ denotes the maximal integer in $(-\infty, x]$. Combining these inequalities with (1) we find that

$$1 - \frac{p-1}{C+p-1} = \frac{1}{1 + \frac{p-1}{C}} \approx \frac{1}{1 + \frac{p-1}{(p^k-1)\Sigma_{N,p}^{1, \lceil Nx_C \rceil}}} \leq \mathbf{P}\{\mu_*(L) \leq \lceil Nx_C \rceil\}, \quad (5)$$

$$\mathbf{P}\{\mu_*(L) \leq \lfloor Nx_C \rfloor - 1\} \leq p^k \Sigma_{N,p}^{1, \lfloor Nx_C \rfloor - 1} < C, \quad (6)$$

the approximation in the left hand side of (5) is good if p^k is large.

So, for small α the approximate value of α -quantile of $\mu_*(L)$ distribution is Nx_C with $C = \alpha$ and the approximate value of $(1 - \alpha)$ -quantile is Nx_C with $C = (p - 1)/\alpha$. For specific values of p, N, k, C equations (3) may be solved numerically.

Approximate formula for the solution of equation $\Phi(-z) = q$ with the accuracy $|\varepsilon(q)| < 0.003$ may be found in [1]:

$$z(q) = t - \frac{2.30753 + 0.27061 t}{1 + 0.99229 t + 0.04481 t^2} + \varepsilon(q), \quad t = \sqrt{\ln q^{-2}}, \quad 0 < q \leq \frac{1}{2}. \quad (7)$$

Applying (7) to the first equation $\Phi(-\sqrt{2z_C}) = \frac{C}{p^k}$ in (3) we find that

$$z_C \approx \frac{1}{2} \left(t_C - \frac{2.30753 + 0.27061 t_C}{1 + 0.99229 t_C + 0.04481 t_C^2} \right)^2, \quad t_C = \sqrt{\ln \frac{p^{2k}}{C^2}} = \sqrt{2(k \ln p - \ln C)}.$$

If $|\ln C_1| \leq |\ln C_2| \leq \theta k \ln p$ for $\theta < 1$, then

$$|t_{C_1} - t_{C_2}| = \frac{|\ln(C_1/C_2)|}{t_{C_1} + t_{C_2}}$$

and the order of difference $|z_{C_1} - z_{C_2}|$ is

$$|t_{C_1}^2 - t_{C_2}^2| = \left| \ln \frac{C_1}{C_2} \right|.$$

Therefore the order of difference $x_{C_1} - x_{C_2}$ of roots of equations

$$H\left(x_{C_i}, 1 - \frac{1}{p}\right) = \frac{1}{N} z_{C_i}, \quad i = 1, 2,$$

from (3) is $N^{-1} |\ln(C_1/C_2)|$. Further, the difference between $(1-\alpha)$ - and α -quantiles $Nx_{(p-1)/\alpha}$ and Nx_α of $\mu_*(L)$ distribution has the order $|\ln \frac{\alpha^2}{p-1}|$. This crude estimates supports the validity of observation on the high concentration of $\mu_*(L)$ distribution near the value $Nx_{1/2}$. If $N, k, N-k \rightarrow \infty$ then $x_{1/2}$ may be approximated by the root of equation

$$H\left(x_{1/2}, \frac{p-1}{p}\right) = \frac{k \ln p}{N};$$

this corresponds to the results of [3].

The case of maximal codeword weight of random linear k -dimensional code may be considered analogously.

The estimates of the minimal weights of nonzero codewords may be used to choose the parameters of McEliece cryptosystem [5]. If McEliece cryptosystem is used and someone has intercepted single encrypted message, he faces the problem of decoding of seemingly random linear code, such a problem is equivalent ([3], p. 368) to a problem of finding codeword of minimal weight in extended code with dimension increased by 1. It was shown in [2] that general problem of determining minimal weight of codeword for given code is NP -hard.

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