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Optimization of S -boxes GOST R 34.12-2015 «Magma» quantum circuits without ancilla qubits

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Abstract. We study the ways to implement S -boxes by quantum circuits with a minimal number of logical qubits and logical quantum gates without using ancilla qubits. New quantum circuits that implement the S -boxes of the GOST R 34.12-2015 “Magma” on 4 logical qubits are constructed. It means that for substitutions $s \in S(V_n)$ with a large number of cycles there exist quantum circuits on n logical qubits that implement the substitution s with fewer logical quantum gates compared with substitutions $g \in S(V_n)$ with a small number of cycles.

Keywords: S -box, quantum circuit, resource estimates

Оптимизация квантовых схем для S -боксов ГОСТ Р 34.12-2015 «Магма» без вспомогательных кубитов

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Аннотация. Исследуются способы реализации S -боксов квантовыми схемами с минимальным количеством кубитов без использования вспомогательных кубитов. Получены новые квантовые схемы, реализующие S -боксы алгоритма GOST R 34.12-2015 «Магма» на 4 кубитах. Показано, что для подстановок $s \in S(V_n)$ с большим количеством циклов существуют квантовые схемы на n кубитах, реализующие подстановку s меньшим количеством квантовых вентилей, чем для подстановок $g \in S(V_n)$ с небольшим количеством циклов.

Ключевые слова: S -бокс, квантовые схемы, оценка квантовых ресурсов для реализации подстановок

1. Introduction

The theory of quantum computing has been developing since the end of the 20th century. A number of formal quantum computing models are constructed in which some computational problems, for example, [1–3], are solved more efficiently than in the classical computational model [4]. Actual information on the current level of quantum technologies development in the field of quantum computing is presented in [5, 6].

In [7] we have considered a method to implement S -boxes GOST R 34.12-2015 and AES by quantum circuits without using ancilla qubits, based on an algorithm for implementing an arbitrary unitary operator by means of a quantum circuit via decomposition of the corresponding unitary matrix into a product of two-level unitary matrices ([8, sec. 4.5]). Quantum circuits that implement S -boxes of GOST R 34.12-2015 and AES without ancilla qubits were constructed taking into account the optimization, based on the removal of sequences of quantum operations that are equivalent to the identical transformation.

For the implementation of GOST R 34.12-2015 "Magma" S -boxes 4 logical qubits are enough, and for the implementation of GOST R 34.12-2015 "Kuznechik" and AES S -boxes 8 logical qubit are sufficient, while in [9, 10] the implementation of AES S -box requires 40 logical qubits. In [10] a method for constructing a quantum circuit is described which implements the AES S -box on 9 logical qubits, but it is argued that in comparison with 40 logical qubits case, this implementation requires approximately three times more quantum gates.

In this paper to construct quantum circuits (Fig. 1–8) we use generalized CNOT($C|t$) gates ([8, 11]), in which qubit t is controlled by the set of qubits C . Generalized gates CNOT($C|t$) may be implemented without using ancilla qubits ([8, p. 184]), therefore, we will consider generalized gates CNOT($C|t$) as one self-independent logic gate, which is consistent with the techniques of assessing the quantum resources described in [12, 13].

2. Construction of quantum circuits implementing S -boxes without ancilla qubits

In this section, we present an algorithm for constructing quantum circuits that implement S -boxes without ancilla qubits, based on the decomposition of the substitution into independent cycles.

Table 1 presents the results of our implementation of Algorithm 1.

Algorithm 1**Input:** Substitution $s \in S(V_n)$.**Output:** Quantum circuit for $s \in S(V_n)$ without ancilla qubits.

- 1: Represent s as a product of independent cycles and remove fixed points. As a result, we obtain $k \in \mathbb{N}$ independent cycles, $s = s_1 s_2 \dots s_k$.
- 2: Consider each cycle s_i , $i \in \overline{1, k}$ as an independent substitution, find the decomposition of the unitary matrix U_{s_i} corresponding to the cycle s_i , $i \in \overline{1, k}$, into the product of two-level matrices (see [7, 8]):

$$U_{s_i} = V_1^i \cdot \dots \cdot V_t^i.$$

- 3: By the matrices $V_1^i, \dots, V_t^i$ we can determine all possible pairs of states

$$data_i = \{(x_j^i, y_j^i) : V_j^i |x_j^i\rangle = |y_j^i\rangle, i \in \overline{1, k}, j \in \overline{1, t}, x_j^i \neq y_j^i\}.$$

Lists $data_i$ may be simply written according to the cycles of s_i , $i \in \overline{1, k}$, by restoring the transition table of s_i . Description of formation of lists $data_i$ through two-level matrices is given in order to define $data_i$ strictly and unambiguously.

- 4: For each pair $(x_j^i, y_j^i) \in data_i$, $i \in \overline{1, k}, j \in \overline{1, t}$, define the list of bit numbers $numb_{(x_j^i, y_j^i)} \subset \{1, 2, \dots, n\}$ containing j iff x_j^i is different from y_j^i .
- 5: Denote $data = \bigcup_{i=1}^k data_i$. It is required to sort the elements of $data$ in such a way that

$$\sum_{w=1}^{|data|-1} |numb_{(x_w, y_w)} \cap numb_{(x_{w+1}, y_{w+1})}| \rightarrow \max,$$

moreover, the transitions (x_j^i, y_j^i) obtained through the cycle s_i must preserve the relative order (otherwise, instead of the cycle s_i we will get some other cycle s'). This stage may be implemented using a random search with restrictions. As a result of this sorting of $data$, we get $data'$.

- 6: To each element $(x', y') \in data'$ still corresponds some two-level matrix, i.e. each element of $data'$ may be easily implemented using some simple quantum circuit (see [7, 8]). The implementation of two-level matrices in the form of a quantum circuit may be ambiguous. Let $d_{(x', y')}$ be the Hamming distance between the binary representations x' and y' , then there exists exactly $d_{(x', y')}$ various quantum circuits consisting of quantum gates CNOT and generalized CNOT($C|t$) that implement the transition (x', y') (see [8]). Among them there are only $d_{(x', y')}$ quantum circuits that differ significantly, which are determined by the number t of the controlled qubit in quantum gates CNOT($C|t$) that occurs during the implementation of two-level matrices.

-
- 7: We assume that the quantum circuit that implements the transitions of $data'$ may be optimized by independent parts. Then the search for an optimized quantum circuit for $s \in S(V_n)$ may be organized using the following procedure:
- a: Initialize an array of $|data'|$ elements $memory = \{1, \dots, 1\}$. In $memory[iter]$ we will write the number of implementation of the two-level matrix that implements the transition $(x'_{iter}, y'_{iter}) \in data'$, $iter \in \overline{1, |data'|}$. Set $iter = 1$.
  - b: Set the search depth, for example, $depth = 3$. Until $iter < |data'|$ do:
    - i: Search for a quantum circuit with minimum length that implements transitions $(x'_{iter}, y'_{iter}), \dots, (x'_{iter+depth-1}, y'_{iter+depth-1})$ by iterating over all possible variants of quantum circuits that implements $(x'_{iter}, y'_{iter}), \dots, (x'_{iter+depth-1}, y'_{iter+depth-1})$. Write the obtained numbers of implementations of the two-level matrices to $memory[iter], \dots, memory[iter + depth - 1]$.
    - ii: $iter = iter + depth$; The depth of the search is determined by the available computing power.
  - c: Repeat this procedure starting at $iter = 2$ and the same value of $depth$.
- 8: As a result, we obtain a quantum circuit that implements $s \in S(V_n)$ without using ancilla qubits, with the minimal number of quantum gates.
-

Table 1: Data from algorithm 1 for implementing S -boxes without ancilla qubits

S -box	data'-sequences of elementary qubit states transformations
π_0	$(14,15)(13,14)(11,14)(10,13)(9,14)(8,14)(7,9)(6,11)(12,0)(4,10)(3,6)(2,6)(1,4)$
π_1	$(10,14)(11,12)(9,14)(7,12)(1,8)(6,10)(5,10)(4,9)(6,0)$
π_2	$(1,3)(9,1)(13,9)(5,15)(7,13)(10,7)(6,10)(14,6)(2,5)(4,2)(8,14)(11,4)(0,11)$
π_3	$(14,15)(13,14)(12,15)(11,13)(9,12)(8,15)(7,15)(6,15)(5,13)(4,13)(3,8)(1,8)(12,0)$
π_4	$(5,1)(0,7)(2,5)(8,0)(14,2)(4,8)(13,4)(10,3)(11,14)(12,11)(15,12)$
π_5	$(12,13)(11,13)(10,11)(9,10)(8,11)(7,10)(5,15)(6,12)(4,9)(2,15)(3,6)(5,0)(1,13)$
π_6	$(13,15)(9,14)(12,13)(6,14)(5,9)(11,15)(10,11)(4,6)(3,5)(8,15)(1,14)(7,12)(8,0)$
π_7	$(7,3)(1,7)(0,1)(4,0)(8,4)(6,8)(11,6)(14,11)(2,14)(15,2)(9,15)(12,9)(13,12)$

In Fig. 1–8 the quantum circuits are presented that implement the S -boxes of GOST R 34.12-2015 "Magma". Optimization of quantum circuits that implement S -boxes GOST R 34.12-2015 "Magma" was carried out by "brute force", i.e. for "Magma" in Algorithm 1, we have set

$depth = |data'| - 1$. Comparison of the number of quantum gates in quantum circuits, at [7] and pic. 1–8, that implement the S -box without ancilla qubits is given in table 2.

Figure 1: Quantum circuit for $\pi_0 = (12, 4, 6, 2, 10, 5, 11, 9, 14, 8, 13, 7, 0, 3, 15, 1)$

Figure 2: Quantum circuit for $\pi_1 = (6, 8, 2, 3, 9, 10, 5, 12, 1, 14, 4, 7, 11, 13, 0, 15)$

Figure 3: Quantum circuit for $\pi_2 = (11, 3, 5, 8, 2, 15, 10, 13, 14, 1, 7, 4, 12, 9, 6, 0)$

Figure 4: Quantum circuit for $\pi_3 = (12, 8, 2, 1, 13, 4, 15, 6, 7, 0, 10, 5, 3, 14, 9, 11)$

Figure 5: Quantum circuit for $\pi_4 = (7, 15, 5, 10, 8, 1, 6, 13, 0, 9, 3, 14, 11, 4, 2, 12)$

Figure 6: Quantum circuit for $\pi_5 = (5, 13, 15, 6, 9, 2, 12, 10, 11, 7, 8, 1, 4, 3, 14, 0)$

Figure 7: Quantum circuit for $\pi_6 = (8, 14, 2, 5, 6, 9, 1, 12, 15, 4, 11, 0, 13, 10, 3, 7)$

Figure 8: Quantum circuit for $\pi_7 = (1, 7, 14, 13, 0, 5, 8, 3, 4, 15, 10, 6, 9, 12, 11, 2)$

Source code for verifying the correctness of quantum circuits in the quantum simulator *Quipper* [14], example for π_4 .

```
import Quipper
import QuipperLib.Simulation
import System.Random
import Quipper.Printing
import Quipper.QData
-----
sub1 :: ([Qubit]) -> Circ ([Qubit])
sub1 (input) = do
  let [q0, q1, q2, q3] = input
  qnot_at q1 'controlled' (q0==.0,q2==.0,q3)
  qnot_at q2 'controlled' [q1]
  qnot_at q3 'controlled' [q1]
  qnot_at q1 'controlled' (q0==.0,q2==.0,q3==.0)
  qnot_at q1 'controlled' (q0==.0,q2,q3==.0)
  qnot_at q3 'controlled' [q1]
  qnot_at q2 'controlled' [q1]
  qnot_at q0 'controlled' (q1==.0,q2==.0,q3==.0)
  qnot_at q1 'controlled' [q0]
  qnot_at q0 'controlled' (q1==.0,q2,q3==.0)
  qnot_at q0 'controlled' (q1,q2==.0,q3==.0)
  qnot_at q1 'controlled' [q0]
  qnot_at q3 'controlled' [q0]
  qnot_at q0 'controlled' (q1,q2==.0,q3==.0)
  qnot_at q0 'controlled' (q1==.0,q2,q3)
  qnot_at q3 'controlled' [q0]
  qnot_at q1 'controlled' [q3]
  qnot_at q3 'controlled' (q0,q1,q2)
  qnot_at q2 'controlled' [q3]
  qnot_at q3 'controlled' (q0,q1,q2==.0)
  qnot_at q1 'controlled' [q3]
  qnot_at q3 'controlled' (q0,q1,q2==.0)
  qnot_at q2 'controlled' [q3]
```

```

return ([q0, q1, q2, q3])

test1_circuit :: IO ()
test1_circuit = do
  putStrLn "Quantum Circuit in Adobe Reader..."
  print_generic Preview (sub1) (replicate 4 qubit)

test1_exec :: IO ()
test1_exec = do
  putStrLn "Substitution functionality test:"
  print_generic GateCount sub1 (replicate 4 qubit)
  g <- newStdGen
  --Here are only 2 states, but you could check all 16:
  print $ run_generic g(0.0::Double) sub1 ([True,True,True,False])
  print $ run_generic g(0.0::Double) sub1 ([True,True,True,True])

--Run--
main = do
  test1_circuit
  test1_exec

```

Table 2: Comparison of the number of quantum gates in quantum circuits, at [7] and Fig. 1-8, that implement the *S*-box without ancilla qubits

<i>S</i> -box	Number of cycles in <i>S</i> -box	Total gates in quantum circuit [7]	Total gates in new quantum circuit
π_0	2	33	29
π_1	3	29	23
π_2	2	37	27
π_3	1	29	29
π_4	3	31	23
π_5	2	35	29
π_6	2	31	25
π_7	1	31	29

3. Conclusion

We have obtained new quantum circuits implementing GOST R 34.12-2015 "Magma" *S*-boxes on 4 logical qubits with fewer logical quantum gates than in [7]. The obtained results allow to suppose that the more cycles in

substitution, the less the length of the quantum circuit implementing this substitution may be.

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Appendix 1

The algorithm for constructing quantum circuits of an arbitrary unitary operator is described in [8], Section 4.5.

Definition. Let $N = 2^n$, $n \in \mathbb{N}$, and $e_1, e_2, \dots, e_N$ be the basis of the vector space $L_{\mathbb{C}^N}$ over the field of complex numbers $\mathbb{C}$. The unitary matrices $U \in \mathbb{C}_{2^n, 2^n}$, nontrivially acting on no more than two basis vectors $e_1, e_2, \dots, e_N$, are called *two-level unitary matrices* (see [8], section 4.5.1).

Let's construct a quantum circuit that implements

$$\pi_1 = (6, 8, 2, 3, 9, 10, 5, 12, 1, 14, 4, 7, 11, 13, 0, 15).$$

The substitution $\pi_1 \in S(V_4)$. Denote $y = \pi_1(x)$, $x, y \in V_4$. The states $|x\rangle, |y\rangle$ are vector-columns from $L_{\mathbb{C}^{2^4}}$, the action of the operator $U|x\rangle = |y\rangle$ is a multiplication of the column vector $|x\rangle$ by the matrix $U \in \mathbb{C}_{2^4, 2^4}$.

1. The unitary matrix for π_1 :

$$U_{\pi_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. The matrix U_{π_1} may be represented as a product of two-level unitary matrices:

$$U_{\pi_1} = V_1 \cdot V_2 \cdot V_3 \cdot V_4 \cdot V_5 \cdot V_6 \cdot V_7 \cdot V_8 \cdot V_9.$$

The table 3 contains two-level matrices $V_1, \dots, V_9$, participating in the decomposition U_{π_1} , states s and t , on which two-level matrices act non-trivially, and quantum circuits implementing two-level matrices $V_1, \dots, V_9$. Matrices are written as a list of strings, each row is a vector $v_i \in V_{16}$, $\|v_i\| = 1$, $i \in \overline{1, 16}$ and written in hexadecimal notation.

Table 3: Representation of the matrix U_{π_1} as a product of two-level matrices

$$U_{\pi_1} = V_1 \cdot V_2 \cdot V_3 \cdot V_4 \cdot V_5 \cdot V_6 \cdot V_7 \cdot V_8 \cdot V_9$$

$V_1 = \{200, 4000, 2000, 1000, 800, 400, 8000, 100, 80, 40, 20, 10, 8, 4, 2, 1\}$	$s = 0110\rangle$ $t = 0000\rangle$	
$V_2 = \{8000, 80, 2000, 1000, 800, 400, 200, 100, 4000, 40, 20, 10, 8, 4, 2, 1\}$	$s = 0001\rangle$ $t = 1000\rangle$	
$V_3 = \{8000, 4000, 2000, 1000, 40, 400, 200, 100, 80, 800, 20, 10, 8, 4, 2, 1\}$	$s = 0100\rangle$ $t = 1001\rangle$	
$V_4 = \{8000, 4000, 2000, 1000, 800, 20, 200, 100, 80, 40, 400, 10, 8, 4, 2, 1\}$	$s = 0101\rangle$ $t = 1010\rangle$	
$V_5 = \{8000, 4000, 2000, 1000, 800, 400, 20, 100, 80, 40, 200, 10, 8, 4, 2, 1\}$	$s = 0110\rangle$ $t = 1010\rangle$	
$V_6 = \{8000, 4000, 2000, 1000, 800, 400, 200, 8, 80, 40, 20, 10, 100, 4, 2, 1\}$	$s = 0111\rangle$ $t = 1100\rangle$	
$V_7 = \{8000, 4000, 2000, 1000, 800, 400, 200, 100, 80, 2, 20, 10, 8, 4, 40, 1\}$	$s = 1001\rangle$ $t = 1110\rangle$	
$V_8 = \{8000, 4000, 2000, 1000, 800, 400, 200, 100, 80, 40, 2, 10, 8, 4, 20, 1\}$	$s = 1010\rangle$ $t = 1110\rangle$	
$V_9 = \{8000, 4000, 2000, 1000, 800, 400, 200, 100, 80, 40, 20, 8, 10, 4, 2, 1\}$	$s = 1011\rangle$ $t = 1100\rangle$	

Since $|y\rangle = U|x\rangle$, $|y\rangle = V_1 \cdot (\dots \cdot (V_8 \cdot (V_9 \cdot |x\rangle)) \dots)$, the quantum circuit of U_{π_1} is as follows:

Figure 9: Quantum circuit $\pi_1 = (6, 8, 2, 3, 9, 10, 5, 12, 1, 14, 4, 7, 11, 13, 0, 15)$

After optimization of the quantum circuit in the Fig. 9, we obtain the quantum circuit in the Fig. 2.