Math-Net.Ru

All Russian mathematical portal

K. Gongopadhyay, M. M. Mishra, D. Tiwari, On generalized Jørgensen inequality in $SL(2, \mathbb{C})$, *Sib. Elektron. Mat. Izv.*, 2019, Volume 16, 542–546

DOI: <https://doi.org/10.33048/semi.2019.16.035>

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use

<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 3.93.64.190

April 26, 2023, 14:50:54

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 16, стр. 542–546 (2019)

DOI 10.33048/semi.2019.16.035

УДК 515.124.2

MSC 30L99, 53C23, 54D10

ON GENERALIZED JØRGENSEN INEQUALITY IN $SL(2, \mathbb{C})$

K. GONGOPADHYAY, M.M. MISHRA, D. TIWARI

ABSTRACT. Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality in Proc. Indian Acad. Sci. Math. Sci., 123(2):245–251, 2013, for two generator subgroups of $SL(2, \mathbb{C})$ where one of the generators is loxodromic. We prove that their inequality is strict.

Keywords: Jørgensen inequality, discreteness.

1. INTRODUCTION

The group $SL(2, \mathbb{C})$ acts by Möbius transformations on the Riemann sphere $\mathbb{S}^2$ and the action extends to the three dimensional hyperbolic space $\mathbf{H}^3$ by orientation-preserving isometries. An element g in $SL(2, \mathbb{C})$ is elliptic if it has a fixed point on the hyperbolic space. It is parabolic, resp. loxodromic, if it is non-elliptic and has exactly one, resp. two fixed points on the boundary $\mathbb{S}^2$. It is a classical problem to understand discreteness of subgroups of $SL(2, \mathbb{C})$. The Jørgensen inequality is a celebrated result in this direction that provides necessary condition of discreteness for a two generator subgroup of $SL(2, \mathbb{C})$, see [4]. Extremality of the Jørgensen inequality has been investigated in [5]. In the literature there are several generalizations of the Jørgensen inequality, and extremalities of some of those inequalities have also been investigated, e.g. [2], [3], [6], [7]. In this note we investigate extremality of one such generalized Jørgensen inequality.

In [8], Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality for two generator subgroups of $SL(2, \mathbb{C})$ where one of the generators is loxodromic. We recall their result.

GONGOPADHYAY, K., MISHRA, M.M., TIWARI, D., ON GENERALIZED JØRGENSEN INEQUALITY IN $SL(2, \mathbb{C})$.

© 2019 GONGOPADHYAY K., MISHRA M.M., TIWARI D.

Gongopadhyay acknowledges partial support from SERB MATRICS grant MTR/2017/000355. Tiwari is supported by NBHM-SRF.

Received September, 17, 2018, published April, 19, 2019.

Theorem WJC. [8] *Let g, h are elements in $SL(2, \mathbb{C})$ such that g is loxodromic. Suppose that g, h are of the form: for $|\lambda| > 1$,*

$$(1.1) \quad g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let g be such that $M_g < 1$, where

$$M_g = |\lambda - 1| + |\lambda^{-1} - 1|.$$

If $\langle g, h \rangle$ is discrete and non-elementary, then

$$(1.2) \quad |abcd|^{\frac{1}{2}} \geq \frac{1 - M_g}{M_g^2}.$$

We prove that the above inequality is strict, see Theorem 2.1 below. Further, we note down a few generalized Jørgensen type inequalities which are also strict.

2. PROOF OF STRICTNESS OF THE INEQUALITY

Theorem 2.1. *Under the hypothesis of the above theorem, equality does not hold in (1.2).*

Доказательство. If possible suppose equality holds in (1.2), i.e. suppose

$$|abcd|^{\frac{1}{2}} = \frac{1 - M_g}{M_g^2}.$$

Let $h_1 = hgh^{-1}$. Then

$$\begin{aligned} h_1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad\lambda - bc\lambda^{-1} & -(\lambda - \lambda^{-1})ab \\ (\lambda - \lambda^{-1})cd & ad\lambda^{-1} - bc\lambda \end{pmatrix} \\ &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}. \end{aligned}$$

Note that

$$(2.1) \quad b_1c_1 = -(\lambda - \lambda^{-1})^2abcd = -(\lambda - \lambda^{-1})^2bc(1 + bc).$$

$$\begin{aligned} a_1d_1 &= (ad\lambda - bc\lambda^{-1})(ad\lambda^{-1} - bc\lambda) \\ &= (ad)^2 + (bc)^2 - abcd(\lambda^2 + \lambda^{-2}) \\ &= (ad - bc)^2 - abcd(\lambda^2 + \lambda^{-2} - 2) \\ &= 1 - (\lambda - \lambda^{-1})^2abcd. \end{aligned}$$

This implies,

$$\begin{aligned}
|a_1 d_1| &= |1 - (\lambda - \lambda^{-1})^2 abcd| \\
&\leq 1 + |\lambda - 1 + 1 - \lambda^{-1}|^2 |abcd| \\
&\leq 1 + (|\lambda - 1| + |\lambda^{-1} - 1|)^2 |abcd| \\
&\leq 1 + M_g^2 \frac{(1 - M_g)^2}{M_g^4} \\
&\leq 1 + \frac{(1 - M_g)^2}{M_g^2} \\
&\leq \frac{M_g^2 + (1 - M_g)^2}{M_g^2} \\
&\leq \frac{M_g^2 + (1 - M_g)^2 + 2M_g(1 - M_g)}{M_g^2}, \text{ since } M_g < 1, \\
&\leq \frac{(M_g + 1 - M_g)^2}{M_g^2}.
\end{aligned}$$

That is,

$$(2.2) \quad |a_1 d_1|^{\frac{1}{2}} \leq \frac{1}{M_g}.$$

Also, we have $b_1 = -(\lambda - \lambda^{-1})ab$. Thus

$$|b_1| \leq |\lambda - 1 + 1 - \lambda^{-1}| |ab| \leq M_g |ab|.$$

Similarly, $|c_1| \leq M_g |cd|$. Hence

$$|b_1 c_1| \leq M_g^2 |abcd| = M_g^2 \cdot \frac{(1 - M_g)^2}{M_g^4}, \text{ which implies}$$

$$(2.3) \quad |b_1 c_1|^{\frac{1}{2}} \leq \frac{1 - M_g}{M_g}.$$

In particular,

$$(2.4) \quad M_g(1 + |b_1 c_1|^{\frac{1}{2}}) \leq 1.$$

Now note that $\langle g, h_1 \rangle$ is discrete and non-elementary. Discreteness of $\langle g, h_1 \rangle$ is obvious, and if it was elementary, that would have implied that g and h had a common fixed point and hence, would have contradicted the assumption that $\langle g, h \rangle$ is non-elementary. So, applying Theorem WJC to $\langle g, h_1 \rangle$, we have

$$\begin{aligned}
\frac{1 - M_g}{M_g^2} &\leq |a_1 b_1 c_1 d_1|^{\frac{1}{2}} \leq |a_1 d_1|^{\frac{1}{2}} |b_1 c_1|^{\frac{1}{2}} \\
&\leq \frac{1 - M_g}{M_g^2}, \text{ by (2.2) and (2.3).}
\end{aligned}$$

This implies,

$$|a_1 b_1 c_1 d_1|^{\frac{1}{2}} = \frac{1 - M_g}{M_g^2}.$$

Next we observe that

$$\begin{aligned}
 \left(\frac{1-M_g}{M_g^2}\right)^2 &= |a_1 b_1 c_1 d_1| \\
 &\leq |1 + b_1 c_1| |b_1 c_1| \\
 &\leq (1 + |b_1 c_1|)(\lambda - \lambda^{-1})^2 |abcd|, \text{ from first equality in (2.1)} \\
 &\leq (\lambda - \lambda^{-1})^2 (1 + |b_1 c_1|) \left(\frac{1-M_g}{M_g^2}\right)^2 \\
 &\leq M_g^2 (1 + |b_1 c_1| + 2|b_1 c_1|^{\frac{1}{2}}) \left(\frac{1-M_g}{M_g^2}\right)^2 \\
 &\leq (M_g (1 + |b_1 c_1|^{\frac{1}{2}}))^2 \left(\frac{1-M_g}{M_g^2}\right)^2 \\
 &\leq \left(\frac{1-M_g}{M_g^2}\right)^2, \text{ by (2.4).}
 \end{aligned}$$

Hence we have

$$(2.5) \quad (\lambda - \lambda^{-1})^2 (1 + |b_1 c_1|) = 1.$$

Noting that

$$tr^2(g) - 4 = (\lambda - \lambda^{-1})^2, \text{ and } tr[g, h_1] - 2 = -(\lambda - \lambda^{-1})^2 b_1 c_1,$$

the above equality implies that $\langle g, h_1 \rangle$ satisfies equality in the classical Jørgensen inequality. By a theorem of Jørgensen and Kiikka, see [5, Theorem 2], this implies that g is either elliptic or parabolic, which is a contradiction. Hence equality can not hold in (1.2). $\square$

Combining Theorem WJC and Theorem 2.1, we can rephrase the generalized Jørgensen inequality as follows.

Theorem 2.2. *Let g, h are elements in $SL(2, \mathbb{C})$ such that g is loxodromic. Suppose that g, h are of the form (1.1). Let g be such that $M_g < 1$. If*

$$(2.6) \quad |abcd|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g^2},$$

then $\langle g, h \rangle$ is either elementary or non-discrete.

2.1. Some more inequalities. The main idea in [8] was to embed $SL(2, \mathbb{C})$ into the isometry group $Sp(1, 1)$ of the one dimensional quaternionic hyperbolic space $\mathbf{H}_{\mathbb{H}}^1$, and then use quaternionic Jørgensen inequality of Cao and Parker, see [1, Theorem 1.1]. In view of the above theorem, following arguments as used in the proof of [1, Corollary 1.2], we note the following.

Corollary 2.3. *Let g, h are elements in $SL(2, \mathbb{C})$ such that g is loxodromic. Suppose that g, h are of the form (1.1). Let g be such that $M_g < 1$. If $\langle g, h \rangle$ is discrete and non-elementary, then each of the following strict inequalities holds.*

$$(2.7) \quad |bc|^{\frac{1}{2}} > \frac{1-M_g}{M_g}.$$

$$(2.8) \quad |1 + bc|^{\frac{1}{2}} > \frac{1-M_g}{M_g}.$$

$$(2.9) \quad |1 + bc| + |bc| > \frac{2(1-M_g)}{M_g^2}.$$

Доказательство. If possible, suppose $|bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g}$. Then

$$|ad|^{\frac{1}{2}}|bc|^{\frac{1}{2}} \leq (1 + |bc|^{\frac{1}{2}})|bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g^2}.$$

Using Theorem 2.2, $\langle g, h \rangle$ is either discrete or non-elementary.

If possible suppose $|1 + bc|^{\frac{1}{2}} \leq \frac{1-M_g}{M_g}$. Then it follows similarly as above noting that $|bc|^{\frac{1}{2}} \leq 1 + |ad|^{\frac{1}{2}}$ and $ad - bc = 1$.

Finally, if $|1 + bc| + |bc| \leq \frac{2(1-M_g)}{M_g^2}$, then

$$|ad|^{\frac{1}{2}}|bc|^{\frac{1}{2}} \leq \frac{1}{2}(|ad| + |bc|) \leq \frac{1-M_g}{M_g^2},$$

and the result follows from Theorem 2.2.

This completes the proof. $\square$

In view of the results noted in this communication, the following question is natural to ask.

Question 1. What are sharp bounds for the inequalities (1.2) and (2.7) – (2.9)?

REFERENCES

- [1] Wensheng Cao and John R. Parker, *Jørgensen's inequality and collars in n -dimensional quaternionic hyperbolic space*, Q. J. Math., **62**:3 (2011), 523–543. MR2825470
- [2] Krishnendu Gongopadhyay, *On generalized Jørgensen inequality in infinite dimension*, New York J. Math., **24** (2018), 865–869. MR3874953
- [3] Krishnendu Gongopadhyay and Abhishek Mukherjee, *Extremality of quaternionic Jørgensen inequality*, Hiroshima Math. J., **47**:2 (2017), 113–137. MR3679886
- [4] Troels Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math., **98**:3 (1976), 739–749. MR0427627
- [5] Troels Jørgensen and Maire Kiikka, *Some extreme discrete groups*, Ann. Acad. Sci. Fenn. Ser. A I Math., **1**:2 (1975), 245–248. Zbl 0345.20052
- [6] A. V. Masleĭ, *On the Gehring-Martin-Tan numbers and Tan numbers of the elementary subgroups of $\mathrm{PSL}(2, \mathbb{C})$* , Mat. Zametki, **102**:2 (2017), 255–269; translation in Math. Notes, **102**:1-2 (2017), 219–231. MR3682795
- [7] Dušan Repovš and Andrei Vesnin, *On Gehring-Martin-Tan groups with an elliptic generator*, Bull. Aust. Math. Soc., **94**:2 (2016), 326–336. MR3568927
- [8] Hua Wang, Yueping Jiang, and Wensheng Cao, *Notes on discrete subgroups of Möbius transformations*, Indian Acad. Sci. Math. Sci., **123**:2 (2013), 245–251. MR3055443

KRISHNENDU GONGOPADHYAY

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) MOHALI,
KNOWLEDGE CITY, SECTOR 81, SAS NAGAR,
PUNJAB, 140306, INDIA

E-mail address: krishnendu@iisermohali.ac.in, krishnendug@gmail.com

MUKUND MADHAV MISHRA

DEPARTMENT OF MATHEMATICS, HANSRAJ COLLEGE,
UNIVERSITY OF DELHI,
DELHI, 110007, INDIA

E-mail address: mukund.math@gmail.com

DEVENDRA TIWARI

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF DELHI,
DELHI, 110007, INDIA

E-mail address: devendra9.dev@gmail.com