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SOME CHARACTERIZATION OF CURVES IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$ SPACE

B. SENOUSSE, M. BEKKAR

ABSTRACT. In 1997 Emil Molnár introduced [15] the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. In this paper, we obtained characterizations of a curve with respect to the Frenet frame of $\widetilde{\mathbf{SL}_2\mathbb{R}}$. Rectifying curves are introduced in [3] as space curves whose position vector always lies in its rectifying plane. We characterize rectifying curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$.

Keywords: $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry, biharmonic curves, general helix, rectifying curve.

1. INTRODUCTION

First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds.

Harmonic maps $\psi : (M, g) \rightarrow (N, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$E_1 : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E_1(\psi) = \frac{1}{2} \int_M |d\psi|^2 v_g,$$

and is characterized by the vanishing of the first tension field

$$\tau_1(\psi) = \text{trace} \nabla d\psi.$$

We remind that the bienergy of ψ is given by

$$E_2 : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - \text{trace}_g R^N(d\psi, \tau(\psi))d\psi,$$

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where $\Delta^\psi = -\text{trace}(\nabla^\psi)^2 = -\text{trace}(\nabla^\psi \nabla^\psi - \nabla_{\nabla^\psi}^\psi)$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional E_2 . We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$(1) \quad \nabla_{\phi'}^3 \phi' - R(\phi', \nabla_{\phi'} \phi') \phi' = 0.$$

In 1997 Emil Molnár introduced [15] the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. In the theory of curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space, one of the important and interesting problems is the characterizations of a regular curve. In [12] T. Ikawa obtained the following equation

$$D_X^3 X - (k^2 - \tau^2) D_X X = 0,$$

for the circular helices which corresponds to the case that the curvatures k and τ of a timelike curve γ on the Lorentzian manifold M are constant.

In [9] N. Ekmekçi and H. H. Hacısalihoğlu generalized T. Ikawa's result to the case of general helices and gave the following characterization

$$D_X^3 X - \left(\frac{3k'}{k}\right) D_X^2 X - \left(\frac{k''}{k} - \frac{3k'}{k^2} + k^2 - \tau^2\right) D_X X = 0$$

for timelike curve with its tangent vector fields on any point. The authors in [7] determined the equation of the geodesic curve in the hyperboloid model. Recently, in [19] Y. Nakanishi prove the following lemma.

LEMMA 1.1. *A unit speed curve c in M_c is a helix if and only if there exist a constant λ such that*

$$D_X^3 X = \lambda D_X X.$$

2. PRELIMINARIES

2.1. Hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. In [15] Emil Molnár proposed a projective spherical model as unified geometrical model of homogeneous geometries. The hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry is described in details in [7, 11, 17, 18].

The real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the usual product operation, taken to act on row matrices as on point coordinates on the right as follows [5]

$$(2) \quad (z^0, z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (\omega^0, \omega^1)$$

with

$$\omega = \frac{\omega^1}{\omega^0} = \frac{b + \frac{z^1}{z^0} a}{d + \frac{z^1}{z^0} c} = \frac{b + za}{d + zc}$$

as right action on the complex projective line C^∞ . This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology. In order to model the above structure in the projective sphere $\mathcal{PS}^3$ and

in the projective space $\mathcal{P}^3$, we introduce the new projective coordinates (x^0, x^1, x^2, x^3) where

$$a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3,$$

with positive, then the non-zero multiplicative equivalence as a projective freedom in $\mathcal{PS}^3$ and in $\mathcal{P}^3$, respectively. Meanwhile we turn to the proportionality

$$\mathbf{SL}_2\mathbb{R} < \mathbf{PSL}_2\mathbb{R},$$

natural in this context. Then it follows that

$$(3) \quad 0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3$$

describes the interior of the above one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1; 0; 0; 0)$ and the ideal points of the axes

$E_1^\infty(0; 1; 0; 0)$, $E_2^\infty(0; 0; 1; 0)$, $E_3^\infty(0; 0; 0; 1)$. We consider the collineation group $\mathbf{G}_*$ that acts on the projective sphere $\mathcal{PS}^3$ and preserves a polarity, i.e. a scalar product of signature $(- - ++)$, this group leaves the one sheeted hyperboloid solid $\mathcal{H}$ invariant. We have to choose an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, then the universal covering group and space $\tilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$.

The specific isometries $\mathbf{S}(\phi)$ ($\phi \in \mathbb{R}$) constitute a one parameter group given by the matrices

$$(4) \quad \mathbf{S}(\phi) : (s_i^j(\phi)) = \begin{pmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}.$$

The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0; x^1; x^2; x^3) \in \tilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X . The coordinates of points lying on the fibre line through X can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$(5) \quad (x^0; x^1; x^2; x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0 \cos \phi - x^1 \sin \phi; x^0 \sin \phi + x^1 \cos \phi; x^2 \cos \phi + x^3 \sin \phi; -x^2 \sin \phi + x^3 \cos \phi)$$

for the Euclidean coordinates $x := \frac{x^1}{x^0}$, $y := \frac{x^2}{x^0}$, $z := \frac{x^3}{x^0}$, $x^0 \neq 0$ as well.

The π periodicity for the above coordinates in the above maps can be seen from the formula (5).

In (4) and (5) we can see the 2π periodicity of ϕ . Moreover, we see the (logical) extension to $\phi \in \mathbb{R}$, as real parameter, to have the universal covers $\tilde{\mathcal{H}}$ and $\widetilde{\mathbf{SL}_2\mathbb{R}}$, respectively, through the projective sphere $\mathcal{PS}^3$. The elements of the isometry group of $\mathbf{SL}_2\mathbb{R}$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL}_2\mathbb{R}}$) can be described by the matrix (a_i^j) .

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix},$$

where

$$(6) \quad \begin{aligned} -(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 &= -1, \\ -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 &= 1, \\ -a_0^0 a_2^0 - a_0^1 a_2^1 + a_0^2 a_2^2 + a_0^3 a_2^3 &= 0, \\ -a_0^0 a_2^1 + a_0^1 a_2^0 - a_0^2 a_2^3 + a_0^3 a_2^2 &= 0, \end{aligned}$$

and we allow positive proportionality, of course, as projective freedom. We define the translation group $\mathbf{G}_T$, as a subgroup of the isometry group of $\mathbf{SL}_2\mathbb{R}$, those isometries acting transitively on the points of $\mathcal{H}$ and by the above extension on the points of $\tilde{\mathcal{H}}$. $\mathbf{G}_T$ maps the origin $E_0(1; 0; 0; 0)$ onto $X(x^0; x^1; x^2; x^3)$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$(7) \quad \begin{aligned} \mathbf{T} &: (t_i^j) = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix} \\ \mathbf{T}^{-1} &: (T_j^k) = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}. \end{aligned}$$

Horizontal intersection of the hyperboloid solid $\mathcal{H}$ with the plane $E_0 E_2^\infty E_3^\infty$ provides the *base plane* of the model $\tilde{\mathcal{H}} = \widetilde{\mathbf{SL}_2\mathbb{R}}$. The fibre through X intersects the hyperbolic ($\mathbf{H}^2$) base plane $z^1 = x = 0$ in the foot point

$$(8) \quad Z(z^0 = x^0 x^0 + x^1 x^1; z^1 = 0; z^2 = x^0 x^2 - x^1 x^3; z^3 = x^0 x^3 + x^1 x^2).$$

We generally introduce a so-called hyperboloid parametrization as follows

$$(9) \quad \begin{aligned} x^0 &= \cosh r \cos \phi, \\ x^1 &= \cosh r \sin \phi, \\ x^2 &= \sinh r \cos(\vartheta - \phi), \\ x^3 &= \sinh r \sin(\vartheta - \phi), \end{aligned}$$

where (r, ϑ) are the polar coordinates of the $\mathbf{H}^2$ base plane, and ϕ is the fibre coordinate. We note that

$$-x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates in (10), which will play an important role in the later $\mathbb{E}^3$ - visualization of the prism tilings in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, are given by

$$(10) \quad \begin{aligned} x &= \frac{x^1}{x^0} = \tan \phi, \\ y &= \frac{x^2}{x^0} = \tanh r \frac{\cos(\vartheta - \phi)}{\cos \phi}, \\ z &= \frac{x^3}{x^0} = \tanh r \frac{\sin(\vartheta - \phi)}{\cos \phi}. \end{aligned}$$

An invariant infinitesimal arc length square in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, is given by [15]

$$ds^2 = \lambda(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

where $\lambda = [-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2]^{-2}$ and

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = (x^0, x^1, x^2, x^3) \begin{pmatrix} -dx^1 & -dx^2 & -dx^3 \\ dx^0 & -dx^3 & dx^2 \\ dx^3 & -dx^0 & dx^1 \\ -dx^2 & -dx^1 & -dx^0 \end{pmatrix}.$$

By using (9) we obtain the following result

$$ds^2 = dr^2 + (\cosh r \sinh r)^2 d\vartheta^2 + (d\phi + \sinh^2 r d\vartheta)^2.$$

If

$$\{\theta^1 = dr, \theta^2 = \cosh r \sinh r d\vartheta, \theta^3 = d\phi + \sinh^2 r d\vartheta\}$$

is an orthonormal coframe then the dual orthonormal frame fields are given by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sinh 2r} & -\tanh r \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\vartheta \\ \partial_\phi \end{pmatrix},$$

where $\partial_r = \frac{\partial}{\partial r}$, $\partial_\vartheta = \frac{\partial}{\partial \vartheta}$, $\partial_\phi = \frac{\partial}{\partial \phi}$.

One can easily check that $g(e_i, e_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

The symmetric metric tensor field g is given by

$$G = (g_{ij})_{1 \leq i, j \leq 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\sinh r)^2((\cosh r)^2 + (\sinh r)^2) & (\sinh r)^2 \\ 0 & (\sinh r)^2 & 1 \end{pmatrix}.$$

The inverse matrix $(g^{ij})_{1 \leq i, j \leq 3}$ of $(g_{ij})_{1 \leq i, j \leq 3}$ is given by

$$G^{-1} = \frac{1}{(\sinh r \cosh r)^2} \begin{pmatrix} (\sinh r \cosh r)^2 & 0 & 0 \\ 0 & 1 & -(\sinh r)^2 \\ 0 & -(\sinh r)^2 & (\sinh r)^2((\cosh r)^2 + (\sinh r)^2) \end{pmatrix}.$$

The Levi-Civita connection ∇ is defined by $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$, where the Cristoffel symbols Γ_{ij}^k are given by

$$\begin{aligned} \Gamma_{ij}^1 &= -\frac{1}{2} \sinh 2r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + 4(\sinh r)^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \Gamma_{ij}^2 &= \frac{2}{\sinh 2r} \begin{pmatrix} 0 & 1 + 3(\sinh r)^2 & 1 \\ 1 + 3(\sinh r)^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \Gamma_{ij}^3 &= -\tanh r \begin{pmatrix} 0 & 2(\sinh r)^2 & 1 \\ 2(\sinh r)^2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The Levi-Civita connection ∇ is given by

$$\begin{aligned} \begin{pmatrix} \nabla_{e_1} e_1 \\ \nabla_{e_1} e_2 \\ \nabla_{e_1} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_2} e_1 \\ \nabla_{e_2} e_2 \\ \nabla_{e_2} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 2 \coth 2r & 1 \\ -2 \coth 2r & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_3} e_1 \\ \nabla_{e_3} e_2 \\ \nabla_{e_3} e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned}$$

The characterising properties of this algebra are the following commutation relations:

$$[e_1, e_2] = -2 \coth(2r) e_2 - 2e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

The components of the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

are [10]

$$\begin{aligned} (11) \quad R(e_1, e_2)e_1 &= 7e_2 & R(e_1, e_2)e_2 &= -7e_1 & R(e_1, e_3)e_1 &= -e_3 \\ R(e_1, e_3)e_3 &= e_1 & R(e_3, e_2)e_2 &= e_3 & R(e_3, e_2)e_3 &= -e_2. \end{aligned}$$

The Riemannian curvature tensor field R is described by

$$R_{1212} = -7, \quad R_{1313} = R_{2323} = 1.$$

The product

$$(12) \quad g_\lambda(X \wedge Y, Z) = [X, Y, Z]$$

is called the mixed product.

2.1.1. Translation curves. Let us consider for a given vector $(q : u : v : w)$ a curve $C(t) = (x^0(t) : x^1(t) : x^2(t) : x^3(t))$, $t \geq 0$, in $\mathcal{H}$ starting at the origin: $C(0) = E_0(1; 0; 0; 0)$ and such that

$$\dot{C}(0) = (\dot{x}^0(0) : \dot{x}^1(0) : \dot{x}^2(0) : \dot{x}^3(0)) = (q : u : v : w),$$

where $\dot{C}(0)$ is the tangent vector at any point of the curve.

For $t \geq 0$ there exists a matrix

$$\mathbf{T}(t) = \begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \\ -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\ x^2(t) & x^3(t) & x^0(t) & x^1(t) \\ x^3(t) & -x^2(t) & -x^1(t) & x^0(t) \end{pmatrix}$$

which defines the translation from $C(0)$ to $\dot{C}(0)$.

DEFINITION 2.1 ([5]). The curve $C(t)$, $t \geq 0$, is said to be a translation curve if

$$\dot{C}(0) \cdot \mathbf{T}(t) = \dot{C}(t), \quad t \geq 0.$$

The solution, depending on $(q : u : v : w)$ had been determined in [16].

2.1.2. *Geodesic curves.* The second order differential equation system of the $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geodesic curve is the following: [5]

$$\begin{cases} \ddot{r} = \dot{\vartheta}\dot{\phi}\sinh(2r) + \frac{1}{2}(\sinh(4r) - \sinh(2r))\dot{\vartheta}\dot{\vartheta} \\ \ddot{\vartheta} = -\frac{2\dot{r}}{\sinh(2r)}(\dot{\vartheta}(3\cosh(2r) - 1) + 2\dot{\phi}) \\ \ddot{\phi} = 2\dot{r}(\dot{\phi} + 2\dot{\vartheta}\sinh^2 r)\tanh(r). \end{cases}$$

The parametrization of a geodesic curve in the hyperboloid model with the geographical sphere coordinates (λ, α) , as longitude and altitude, $(-\pi < \lambda \leq \pi, -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $s \geq 0$, has been determined in [7].

3. BIHARMONIC CURVES IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$ SPACE

In this section we study biharmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. Z. Erjavec and D. Horvat in [8] introduced the notion of biharmonic maps as a natural generalization of the well-known harmonic maps. Chen and Ishikawa in [4] classified biharmonic curves in semi-Euclidean 3-space. The biharmonic curves in the Heisenberg group $\mathbb{H}_3$ are investigated in [1].

Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic timelike biharmonic curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ parametrized by arclength and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ϕ in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ such that $T = \phi'$ is the unit vector field tangent to ϕ , N is the unit vector field in the direction $\nabla_T T$ normal to ϕ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$(13) \quad \begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$k^2 = g(\nabla_T T, \nabla_T T),$$

is the curvature of ϕ and τ is its torsion.

The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane.

From (13) we have

$$(14) \quad \nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B,$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$.

Using (11) one obtains [10]

$$(15) \quad R(T, N, T, N) = 1 - 8B_3^2, \quad R(T, N, T, B) = 8B_3N_3,$$

where

$$\begin{cases} T = T_1e_1 + T_2e_2 + T_3e_3 \\ N = N_1e_1 + N_2e_2 + N_3e_3 \\ B = T \wedge N = B_1e_1 + B_2e_2 + B_3e_3. \end{cases}$$

PROPOSITION 3.1 ([10]). *Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parametrized by arc length. Then ϕ is a proper non-geodesic biharmonic curve if and only if*

$$(16) \quad \begin{cases} k = \text{constant} \neq 0 \\ k^2 + \tau^2 = 1 - 8B_3^2 \\ \tau' = 8N_3B_3. \end{cases}$$

COROLLARY 3.2. *If $\tau = 0$ and $k = \text{constant} \neq 0$ for a timelike curve ϕ . ϕ is a non-geodesic biharmonic curve if and only if*

$$\begin{cases} k^2 = 1 - 8B_3^2 \\ B_3N_3 = 0. \end{cases}$$

4. GENERAL HELIX IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$

In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [20] for details).

DEFINITION 4.1. Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ and $\{T, N, B\}$ be the Frenet frame on $\widetilde{\mathbf{SL}_2\mathbb{R}}$ along ϕ .

1) If both k and τ are constant along ϕ , then is called circular helix with respect to Frenet frame.

2) A curve ϕ such that

$$\frac{k}{\tau} = c, \quad c \in \mathbb{R},$$

is called a general helix with respect to Frenet frame.

If $k = \text{constant} \neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

THEOREM 4.2 ([3]). *A twisted curve $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve if and only if the ratio $\frac{\tau}{k}$ is a nonconstant linear function in arclength function s .*

THEOREM 4.3. *Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. Then ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$ if and only if*

1)

$$(17) \quad \nabla_T^3 T - \sigma_1(s) \nabla_T T = 3k' \nabla_T N,$$

where $\sigma_1(s) = \frac{k''}{k} - (k^2 + \tau^2)$.

2)

$$(18) \quad \nabla_T^3 B - \sigma_2(s) \nabla_T B + 3\tau' \nabla_T N = 0,$$

where $\sigma_2(s) = \tau^2 - k^2 - \frac{\tau''}{\tau}$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T, N, B\}$.

Then from (13), we have

$$(19) \quad N = \left(\frac{1}{k}\right) \nabla_T T, \quad B = \left(\frac{1}{\tau}\right) \nabla_T N + \left(\frac{k}{\tau}\right) T.$$

Since ϕ is general helix, we have

$$(20) \quad k' \tau = k \tau'.$$

If we substitute the equations (19) and (20) in (14), we obtain (17).

Conversely let us assume that the equation (17) holds. Differentiating covariantly (19) we obtain

$$\nabla_T N = -\left(\frac{k'}{k^2}\right) \nabla_T T + \left(\frac{1}{k}\right) \nabla_T^2 T$$

and so

$$\begin{aligned}
 \nabla_T^2 N &= \left(-\frac{k'}{k^2}\right)' \nabla_T T - 2 \left(\frac{k'}{k^2}\right) \nabla_T^2 T + \left(\frac{1}{k}\right) \nabla_T^3 T \\
 (21) \qquad &= -k'T - (k^2 + \tau^2)N + \left(\frac{k'\tau}{k}\right) B.
 \end{aligned}$$

Also we obtain

$$(22) \qquad \nabla_T^2 N = -k'T - (k^2 + \tau^2)N + \tau'B.$$

Since (21) and (22) are equal, then

$$(23) \qquad \tau' = \frac{k'\tau}{k}.$$

From (23), we obtain

$$\frac{k}{\tau} = \text{constant}.$$

This means that ϕ is a general helix.

2) Suppose that ϕ is general helix with respect to the Frenet frame. Then, from (13), we have

$$(24) \qquad \nabla_T^3 B = (k'\tau + 2\tau'k)T + (\tau^3 + k^2\tau - \tau'')N - 3\tau\tau'B$$

$$(25) \qquad N = -\left(\frac{1}{\tau}\right) \nabla_T B, \quad B = \frac{1}{\tau} \nabla_T N + \frac{k}{\tau} T.$$

Now we replace (25) in the above expression of $\nabla_T^3 B$, and we obtain (18).

Conversely (18) holds.

Differentiating covariantly of

$$N = -\left(\frac{1}{\tau}\right) \nabla_T B,$$

we obtain

$$(26) \qquad \nabla_T^2 N = -\left(\frac{1}{\tau}\right)'' \nabla_T B - 2 \left(\frac{1}{\tau}\right)' \nabla_T^2 B - \left(\frac{1}{\tau}\right) \nabla_T^3 B.$$

If we use (13) and (18) we get,

$$(27) \qquad \nabla_T^2 N = -\left(\frac{\tau'k}{\tau}\right) T - (k^2 + \tau^2)N + \tau'B.$$

Also we obtain

$$(28) \qquad \nabla_T^2 N = -k'T - (k^2 + \tau^2)N + \tau'B.$$

By comparing (27) and (28), we obtain

$$\left(\frac{k}{\tau}\right)' = 0.$$

From this, we have

$$\frac{k}{\tau} = \text{constant}.$$

Then ϕ is a general helix.

COROLLARY 4.4. Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$. ϕ is a circular helix with respect to the Frenet frame $\{T, N, B\}$ if and only if

1)

$$\nabla_T^3 B = \tau(\tau^2 + k^2)N,$$

2)

$$\nabla_T^3 T = -k(\tau^2 + k^2)N.$$

THEOREM 4.5. Let ϕ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$. ϕ is a general helix with respect to the Frenet frame $\{T, N, B\}$, then

$$(29) \quad \nabla_T^3 T + \sigma_1(s)\nabla_T B = 3k'\nabla_T N,$$

where $\sigma_1(s) = \frac{1}{\tau}(k'' - k^3 - k\tau^2)$.

Proof. Suppose that ϕ is general helix with respect to the Frenet frame $\{T, N, B\}$. If we substitute $\tau'k = k'\tau$ in (14), we obtain (29).

THEOREM 4.6. Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. Then ϕ is a general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ if and only if

1)

$$(30) \quad [\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0,$$

2)

$$(31) \quad [\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = 0,$$

3)

$$(32) \quad [\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0.$$

Proof. 1) From (12) and

$$\begin{pmatrix} \nabla_T T \\ \nabla_T^2 T \\ \nabla_T^3 T \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k^2 & k' & \tau k \\ -(3kk') & k'' - k^3 - k\tau^2 & 2\tau k' + k\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$(33) \quad [\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = g(\nabla_T T \wedge \nabla_T^2 T, \nabla_T^3 T) = k^5 \left(\frac{\tau}{k}\right)'.$$

Since ϕ is general helix and (33), it follows

$$[\nabla_T T, \nabla_T^2 T, \nabla_T^3 T] = 0.$$

The proof is completed.

2) From (12) and

$$\begin{pmatrix} \nabla_T B \\ \nabla_T^2 B \\ \nabla_T^3 B \end{pmatrix} = \begin{pmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ k'\tau + 2k\tau' & -\tau'' + \tau^3 + \tau k^2 & -3\tau\tau' \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

we have

$$[\nabla_T B, \nabla_T^2 B, \nabla_T^3 B] = \tau^5 \left(\frac{\tau}{k}\right)' = 0.$$

The proof is completed.

3) From (12) and

$$\begin{pmatrix} \nabla_T N \\ \nabla_T^2 N \\ \nabla_T^3 N \end{pmatrix} = \begin{pmatrix} -k & 0 & \tau \\ -k' & -(k^2 + \tau^2) & \tau' \\ -k'' + k^3 + k\tau^2 & -3(kk' + \tau\tau') & \tau'' - \tau^3 - \tau k^2 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

we have

$$(34) \quad [\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = (\tau^2 + k^2)(k\tau'' - \tau k'') - 3(kk' + \tau\tau')(\tau'k - k'\tau).$$

Since ϕ is general helix, we have

$$k\tau'' - \tau k'' = 0, \quad k'\tau - \tau'k = 0.$$

The proof is completed.

THEOREM 4.7. *Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. If ϕ is a general helix in $\mathbf{SL}_2\mathbb{R}$, then*

$$(35) \quad \det(\nabla_T^2 T, \nabla_T^3 T, \nabla_T^4 T) = 0.$$

Proof. From (13)

$$\begin{pmatrix} \nabla_T^2 T \\ \nabla_T^3 T \\ \nabla_T^4 T \end{pmatrix} = \begin{pmatrix} -k^2 & k' & \tau k \\ -(3kk') & k'' - k^3 - k\tau^2 & 2\tau k' + k\tau' \\ a(s) & b(s) & c(s) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$\begin{aligned} a(s) &= k^4 + k^2\tau^2 - 4kk'' - 3k'^2 \\ b(s) &= k''' - 3\tau(k'\tau + k\tau') - 6k'k^2 \\ c(s) &= 3k'\tau' + 3k''\tau - k\tau(k^2 + \tau^2) + k\tau'', \end{aligned}$$

we have

$$(36) \quad \begin{aligned} \det(\nabla_T^2 T, \nabla_T^3 T, \nabla_T^4 T) &= (k'''k^2 - 2k^4k' + 6k'^3 - 4kk'k'' - 3k^3\tau\tau' + k^2k'\tau^2)(k\tau' - k'\tau) + \\ &\quad (k''k^2 - k^3\tau^2 - k^5)(k''\tau - \tau''k) + 3k^2k'(k'\tau'' - k''\tau'). \end{aligned}$$

Since ϕ is a general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, we have

$$(37) \quad k\tau' = k'\tau, \quad k\tau'' = k''\tau.$$

Putting (37) in (36), we get (35).

5. RECTIFYING CURVE IN $\widetilde{\mathbf{SL}_2\mathbb{R}}$

Rectifying curves in the three-dimensional sphere, Euclidean space and Minkowski space are studied in [2, 14, 3, 6, 13]. If the position vector ϕ lies on its rectifying plane then $\phi(s)$ is called rectifying curve. Therefore, the position vector ϕ of a rectifying curve satisfies by definition [3] the equation

$$(38) \quad \phi(s) = \rho(s)T(s) + \sigma(s)B(s),$$

for some differentiable functions $\rho(s)$ and $\sigma(s)$.

THEOREM 5.1. *Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parameterized by arc length. If ϕ is a rectifying curve in $\mathbf{SL}_2\mathbb{R}$, then*

$$(39) \quad [\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = -ak^3(k'(1 + (as + b)^2) + 3ak(as + b)).$$

Proof. ϕ is a rectifying curve if and only if the ratio $\frac{\tau}{k}$ satisfies

$$(40) \quad \frac{\tau}{k} = as + b,$$

for some constants a, b with a $a \neq 0$.

By differentiating (40) we find

$$(41) \quad \begin{cases} \tau'k - k'\tau = ak^2 \\ \frac{\tau''}{\tau} - \frac{k''}{k} = 2a\frac{k'}{\tau}. \end{cases}$$

Moreover, using (34), (40) and (41), we get (39).

COROLLARY 5.2. Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve. Then

$$[\nabla_T N, \nabla_T^2 N, \nabla_T^3 N] = 0$$

if and only if

$$(42) \quad k = \frac{\lambda}{(1 + (as + b)^2)^{\frac{3}{2}}}, \quad a\lambda \neq 0.$$

Proof. From (39) we get

$$(43) \quad k'(1 + (as + b)^2) + 3ak(as + b) = 0.$$

After solving (43) we find (42).

COROLLARY 5.3. From (42) and (40) we get

$$\tau = \frac{\lambda(as + b)}{(1 + (as + b)^2)^{\frac{3}{2}}}, \quad a\lambda \neq 0.$$

THEOREM 5.4. Let $\phi : I \rightarrow \widetilde{\mathbf{SL}_2\mathbb{R}}$ is a rectifying curve. Then

$$g(\phi, \nabla_T^3 T) = k^2, \quad g(\phi, \nabla_T^3 B) = -\tau k, \quad g(\phi, \nabla_T^3 N) = 2k'.$$

Proof. Let us suppose that $\phi = \phi(s)$ is a unit speed rectifying curve. Then the position vector ϕ of the curve satisfies the equation (38).

Differentiating (38) with respect to s and applying (13) gives

$$\begin{cases} \rho' = 1 \\ \rho k - \sigma \tau = 0 \\ \sigma' = 0. \end{cases}$$

Therefore, it follows that

$$(44) \quad \begin{cases} \rho(s) = s + c_1 \\ \sigma(s) = c_2 \\ \rho(s)k - \sigma(s)\tau = 0. \end{cases}$$

Then

$$(45) \quad \phi(s) = (s + c_1)T(s) + c_2B(s).$$

From the first and third equations of (44) we get

$$(46) \quad \frac{\tau}{k} = \frac{1}{c_2}s + \frac{c_1}{c_2}, \quad c_1, c_2 \in \mathbb{R}, \quad c_2 \neq 0.$$

Then from (14) and (46) we have $g(\phi, \nabla_T^3 T) = k^2$.

From (45), (46) and (24) we have $g(\phi, \nabla_T^3 B) = -\tau k$.

From (13) we obtain

$$(47) \quad \nabla_T^3 N = (-k'' + k^3 + k\tau^2)T - 3(kk' + \tau\tau')N + (\tau'' - \tau^3 - \tau k^2)B.$$

From the equations (45) and (47) we easily find $g(\phi, \nabla_T^3 N) = 2k'$.

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