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INITIAL-BOUNDARY VALUE PROBLEM
FOR A RADIATIVE TRANSFER EQUATION WITH
GENERALIZED MATCHING CONDITIONS

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ABSTRACT. We consider the Cauchy problem for a non-stationary radiative transfer equation in a three-dimensional multicomponent medium with generalized matching conditions. These matching conditions describe Fresnel and diffuse reflection and refraction at the interfaces. The existence and uniqueness of a solution of the initial-boundary value problem is proved. We construct a Monte-Carlo numerical method designed to find a solution that accounts for the space-time localization of radiation sources. Computational experiments were carried out and their results presented.

Keywords: radiative transfer equation, a Cauchy problem, Fresnel and diffuse matching conditions, Monte Carlo methods.

1. ВВЕДЕНИЕ

The study of the theoretical and numerical aspects of the initial-boundary value problems for kinetic equations has a rather long history (see, for example, [1–5]). Nevertheless, this trend has not lost its relevance to date. The rapid development of computational methods over the last decades allows for solving genuinely time-consuming computational problems of the radiation transfer theory. In turn, new sophisticated models, new statements of the direct, inverse, and the extremal problems require both the theoretical justification and the development of numerical methods for its solving [6–18].

Conditions at the media interfaces play a special role in the theory of radiation transfer. These conditions allow for the description of a number of physical effects

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not accounted for in the radiation transfer equation (reflection, refraction, surface sources, etc.). Neglecting the effects of this kind leads to the traditional matching conditions such as continuous solution «gluing» or «shot» [2, 6]. Solvability of the stationary boundary-value problems with the generalized conjugation conditions describing the Fresnel or diffuse reflection and refraction at the interfaces of the media have been investigated in [19–26]. Problems for the non-stationary radiation transfer equations with the generalized matching conjugation conditions in a three-dimensional bounded media and in a plane-parallel layered media have been considered in [27–35].

In this work, a problem for the non-stationary transport equation is studied in the case when matching operator is a linear combination of the Fresnel and Lambert operators. The correctness of the initial-boundary value problem is proved and a Monte Carlo method for the numerical solution is proposed.

Under more stringent constraint, the unique solvability of the initial-boundary value problem with the indicated conjugation conditions is proved in [35] with the methods of strongly continuous semigroups theory. Unlike the results of [35], in this paper we studied the problem with inhomogeneous matching conditions and much wider class of functions for the initial state of the process.

Previously, the initial-boundary problem with the Fresnel conditions at the interfaces was investigated in [34], where a branching modification of the Monte Carlo method was suggested and its complexity estimated. One of the drawbacks of the Monte Carlo methods with the reverse ray tracing is poor convergence in the case when radiation sources are localized in space and time. In this work, in addition to the inclusion of diffuse reflection and refraction in the model, we also attempted to build a method that accounts for a priori information about the distribution of the radiation sources.

Consider the following integro-differential equation [29–32, 34]

$$(1) \quad \left(\frac{1}{v(r)} \frac{\partial}{\partial t} + \omega \cdot \nabla_r + \mu(r) \right) I(r, \omega, t) = \\ = \sigma(r) \int_{\Omega} p(r, \omega \cdot \omega') I(r, \omega', t) d\omega' + J(r, \omega, t).$$

Equation (1) describes the non-stationary process of interaction of radiation with matter. The function $I(r, \omega, t)$ is interpreted as the particle radiation flux density at time $t \in [0, \infty)$, at the point $r \in \mathbb{R}^3$, moving at the speed v in the direction of the unit vector $\omega \in \Omega = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$. Functions μ, σ, p and J denote the attenuation and the scattering coefficients, indicatrix of the scattering and density of internal sources, respectively.

The process of radiation transfer occurs in the multicomponent media G which we describe as the union of a finite number of bounded and pairwise disjoint subregions $G_1, G_2, \dots, G_m$, with the closure $\overline{G}$ being a convex set in $\mathbb{R}^3$. The boundary of each subregion G_i , $\partial G_i = \overline{G}_i \setminus G_i$ belongs to the class C^1 . The surface $\partial \overline{G}$ will be called the outer boundary of the set G , and $\gamma = \partial G \setminus \partial \overline{G}$ denotes the inner boundary of the set G . For definiteness, the unit normal vector $n(z)$ to the surface $\partial \overline{G}$ at the point z is meant to be the outward normal vector. If z is a contact point of two adjacent subregions G_i and G_j , $1 \leq i < j \leq m$, then the normal $n(z)$ is chosen to be exterior to the surface having the greater index, that is, to ∂G_j .

For brevity, we introduce the following notation:

$$\begin{aligned} X &= G \times \Omega \times [0, \infty), \quad X_0 = \{(r, \omega, t) \in X : t = 0\}, \\ \Omega_{\pm}(z) &= \{\omega \in \Omega : \operatorname{sgn}(n(z) \cdot \omega) = \pm 1\}, \quad \Gamma_{ext}^{\pm} = \partial \overline{G} \times \Omega_{\pm}(z), \\ \Gamma_{int} &= \gamma \times \Omega, \quad \Gamma^{\pm} = \Gamma_{ext}^{\pm} \cup \Gamma_{int}, \quad Y_{int} = \Gamma_{int} \times [0, \infty), \quad Y_{ext}^{\pm} = \Gamma_{ext}^{\pm} \times [0, \infty), \\ X_{ext}^{-} &= X_0 \cup Y_{ext}^{-}, \quad X^{-} = X_{ext}^{-} \cup Y_{int}, \quad X^{+} = Y_{ext}^{+} \cup Y_{int}. \end{aligned}$$

In each subregion G_i the functions $v(r), \mu(r), \sigma(r), p(r, \omega \cdot \omega')$ do not depend on the variable r and take the following values: $v(r) = v_i > 0, \mu(r) = \mu_i > 0, \sigma(r) = \sigma_i, \sigma(r) \leq \mu(r), p(r, \omega \cdot \omega') = p_i(\omega \cdot \omega') \geq 0$, where the function $p_i(\omega \cdot \omega')$ is integrable over the segment $[-1, 1]$ and satisfies the normalization condition

$$\int_{\Omega} p_i(\omega \cdot \omega') d\omega' = 1.$$

The function J is non-negative and belongs to $L_{\infty}(X)$, where $L_{\infty}(X)$ is the space of real-valued, measurable and almost everywhere bounded functions on the set X with the norm

$$\|f\|_{L_{\infty}(X)} = \operatorname{ess\,sup}_{x \in X} |f(x)|.$$

Together with equation (1) we consider the following initial and boundary conditions:

$$(2) \quad I^{-} = h_0(r, \omega) \quad \text{for} \quad X_0,$$

$$(3) \quad I^{-} = h_{ext}(z, \omega, t) \quad \text{for} \quad Y_{ext}^{-},$$

$$(4) \quad I^{-} = \mathcal{B}I^{+} \quad \text{for} \quad Y_{int}.$$

In equations (2), (3), the function $h_0(z, \omega) \geq 0$ characterizes the state of the process at the initial time $t = 0$, the function $h_{ext}(z, \omega, t) \geq 0$ equals the density of the radiation flux entering G . The functions $I^{\pm}$ represent the limit values of the function I , $I^{\pm}(z, \omega, t) = \lim_{\epsilon \rightarrow 0} I(z \pm \epsilon \omega, \omega, t \pm \epsilon)$.

The matching operator $\mathcal{B}$ is a linear combination of the Fresnel and diffuse operators [35]:

$$(5) \quad \mathcal{B}I^{+} = \alpha_f \mathcal{B}_f I^{+} + \alpha_d \mathcal{B}_d I^{+},$$

where $\mathcal{B}_f$ is the Fresnel matching operator describing the mirror reflection and refraction according to Snell's law on the media interfaces [20, 22, 29, 31, 34, 35]:

$$(6) \quad (\mathcal{B}_f I^{+})(z, \omega, t) = R(z, \omega) I^{+}(z, \omega_{re}, t) + T(z, \omega) I^{+}(z, \omega_{tr}, t),$$

and $\mathcal{B}_d$ is the operator describing diffuse reflection and refraction [30, 32, 35] at the interfaces according to Lambert's law:

$$(7) \quad (\mathcal{B}_d I^{+})(z, \omega, t) = \frac{R_d(z, \omega)}{\pi} \int_{\Omega(z, -\omega)} |n(z) \cdot \omega'| I^{+}(z, \omega', t) d\omega' + \frac{T_d(z, \omega)}{\pi} \int_{\Omega(z, +\omega)} |n(z) \cdot \omega'| I^{+}(z, \omega', t) d\omega'.$$

The values $\alpha_f(z), \alpha_d(z) \geq 0, \alpha_f + \alpha_d \leq 1$ define the contributions of the Fresnel and diffuse reflection and refraction at the interface. If $\alpha_f + \alpha_d < 1$, then the surface γ is partially absorbing.

The following notation is used in equations (5), (6):

$$\begin{aligned}
\omega_{re} &= \omega - 2\nu n, \quad \omega_{tr} = \psi(z, \nu)n + \tilde{\kappa}(z, \nu)(\omega - \nu n), \quad \nu = \nu(z) = \omega \cdot n(z), \\
\tilde{\kappa}(z, \nu) &= \begin{cases} \kappa_i / \kappa_j, & \text{if } z \in \partial G_i \cap \partial G_j, \ 0 < \nu(z) \leq 1, \\ \kappa_j / \kappa_i, & \text{if } z \in \partial G_i \cap \partial G_j, \ -1 \leq \nu(z) < 0, \end{cases} \\
\psi(z, \nu) &= \begin{cases} \operatorname{sgn}(\nu) \sqrt{1 - \tilde{\kappa}^2(z, \nu)(1 - \nu^2)}, & \text{if } 1 - \tilde{\kappa}^2(z, \nu)(1 - \nu^2) \geq 0, \\ 0, & \text{else,} \end{cases} \\
R(z, \omega) &= \frac{1}{2}(R_{\parallel}^2(z, \nu) + R_{\perp}^2(z, \nu)), \quad T(z, \omega) = 1 - R(z, \omega), \\
R_{\parallel}(z, \nu) &= \frac{\tilde{\kappa}(z, \nu)\psi(z, \nu) - \nu}{\tilde{\kappa}(z, \nu)\psi(z, \nu) + \nu}, \quad R_{\perp}(z, \nu) = \frac{\psi(z, \nu) - \tilde{\kappa}(z, \nu)\nu}{\psi(z, \nu) + \tilde{\kappa}(z, \nu)\nu}, \\
R_d(z, \omega) &= \begin{cases} R_d^+(z), & \text{if } (z, \omega) \in \gamma \times \Omega_+(z), \\ R_d^-(z), & \text{if } (z, \omega) \in \gamma \times \Omega_-(z), \end{cases} \\
T_d(z, \omega) &= \begin{cases} 1 - R_d^+(z), & \text{if } (z, \omega) \in \gamma \times \Omega_+(z), \\ 1 - R_d^-(z), & \text{if } (z, \omega) \in \gamma \times \Omega_-(z), \end{cases} \\
\Omega(z, \omega) &= \{\omega' \in \Omega \mid (\omega \cdot n(z))(\omega' \cdot n(z)) > 0\}, \quad \Omega_{\pm}(z) = \Omega(z, \pm n(z)).
\end{aligned}$$

It is assumed that the functions $R_d^{\pm}(z)$, $T_d^{\pm}(z)$ and $\alpha_f(z)$, $\alpha_d(z)$ are constant for all $z \in \partial G_i$. At the outer boundary of $\partial \overline{G}$, the effects of reflection, refraction, and absorption are absent, that is, the surface $\partial \overline{G}$ is a virtual media interface.

2. FORMULATION OF THE CAUCHY PROBLEM

We will assume the validity of the generalized convexity condition [2, 6] for the set G : any line that has a common point with G intersects ∂G at a finite number of points.

Let Π_{ω} be the orthogonal projection of the set G onto a plane perpendicular to the direction ω and passing through a fixed point in $\mathbb{R}^3$, and let $\Pi_{\xi, \omega}$, where $\xi \in \Pi_{\omega}$, $\omega \in \Omega$, be the intersection of the line $\{\xi + \tau\omega, -\infty < \tau < +\infty\}$ and the set G . Then, by the generalized convexity condition, the one-dimensional open set $\Pi_{\xi, \omega}$ is the union of a finite number of intervals.

$$\begin{aligned}
\Pi_{\xi, \omega}^i &= \{\xi + \tau\omega, \tau_i(\xi, \omega) < \tau < \tau_{i+1}(\xi, \omega)\}, \quad i = 1, \dots, q(\xi, \omega), \\
-\infty &< \tau_1(\xi, \omega) < \tau_2(\xi, \omega) < \dots < \tau_{q+1}(\xi, \omega) < +\infty, \\
(8) \quad q(\xi, \omega) &\leq \bar{q} = \sup_{(\xi, \omega) \in \Pi_{\omega} \times \Omega} q(\xi, \omega) < \infty, \quad \Pi_{\xi, \omega} = \bigcup_{i=1}^{q(\xi, \omega)} \Pi_{\xi, \omega}^i.
\end{aligned}$$

Let us note that

$$\begin{aligned}
\Gamma_{ext}^- &= \{\xi + \tau_1(\xi, \omega)\omega, \xi \in \Pi_{\omega}\} \times \Omega, \quad \Gamma_{ext}^+ = \{\xi + \tau_{q+1}(\xi, \omega)\omega, \xi \in \Pi_{\omega}\} \times \Omega, \\
\Gamma^- &= \{\xi + \tau_i(\xi, \omega)\omega, \xi \in \Pi_{\omega}, i = \overline{1, q(\xi, \omega)}\} \times \Omega, \\
\Gamma^+ &= \{\xi + \tau_{i+1}(\xi, \omega)\omega, \xi \in \Pi_{\omega}, i = \overline{1, q(\xi, \omega)}\} \times \Omega.
\end{aligned}$$

For convenience, we introduce the function $h \in L_{\infty}(X^-)$ by:

$$h(z, \omega, t) = \begin{cases} h_0(z, \omega), & \text{if } (z, \omega, t) \in X_0, \\ h_{ext}(z, \omega, t), & \text{if } (z, \omega, t) \in Y_{ext}^-, \\ 0, & \text{if } (z, \omega, t) \in Y_{int}, \end{cases}$$

and define the matching operator assuming that $(\mathcal{B}\phi)(r, \omega, t) = 0$ for all $(r, \omega, t) \in X_{ext}^-$. Hence, the function h and the functions that belong to the domain of the operator $\mathcal{B}$, are defined on the same set X^- .

Set $d(r, -\omega, t) = \min\{d(r, -\omega), v_i t\}$, where the number $d(r, -\omega)$ is the distance between $r \in G_i \subset G$ and the boundary of the subregion G_i in the direction $-\omega$, that is $d(r, -\omega) = \sup_{\tau > 0} \{r - \tau'\omega \in G_i \text{ for all } \tau' \in [0, \tau)\}$.

We will say that a function $f(r, \omega, t)$ belongs to $D(X)$ if, after a proper modification on a set of measure zero in X , the following conditions hold true: 1) for almost all $(r, \omega, t) \in G_i \times \Omega \times [0, \infty)$, $i = 1, \dots, m$, function $f(r + \tau\omega, \omega, t + \tau/v_i)$ is absolutely continuous in τ , $\tau \in (d(r, -\omega, t), d(r, \omega)]$, and the derivative of the function f at the point $(r, t) \in G_i \times [0, \infty)$ in the direction $(\omega_1, \omega_2, \omega_3, 1/v_i)$,

$$\left(\frac{1}{v_i} \frac{\partial}{\partial t} + \omega \cdot \nabla_r \right) f(r, \omega, t) = \frac{\partial}{\partial \tau} f(r + \tau\omega, \omega, t + \tau/v_i) \Big|_{\tau=0},$$

existing for almost all $(r, \omega, t) \in X$, belongs to the space $L_\infty(X)$;

2) $f \in L_\infty(X)$, $f^\pm \in L_\infty(X^\pm)$;

Let the operators $\mathcal{L} : D(X) \rightarrow L_\infty(X)$ and $\mathcal{S} : L_\infty(X) \rightarrow L_\infty(X)$ be defined by the formulas:

$$(9) \quad \mathcal{L}f = \left(\frac{1}{v} \frac{\partial}{\partial t} + \omega \cdot \nabla_r \right) f + \mu f,$$

$$(10) \quad \mathcal{S}f = \sigma(r) \int_{\Omega} p(r, \omega \cdot \omega') f(r, \omega', t) d\omega'.$$

Then a function $I \in D(X)$ is called a solution for the initial-boundary value problem (1) - (4) if it satisfies the following conditions

$$(11) \quad \mathcal{L}I = \mathcal{S}I + J \quad \text{almost everywhere on } X$$

and

$$(12) \quad I^- = \mathcal{B}I^+ + h \quad \text{almost everywhere on } X^-.$$

3. STUDYING THE SOLVABILITY OF THE INITIAL-BOUNDARY VALUE PROBLEM

We introduce the operators $\mathcal{P} : L_\infty(X^-) \rightarrow D(X)$ and $\mathcal{E} : L_\infty(X) \rightarrow D(X)$ by the following formulas:

$$(\mathcal{P}\phi)(r, \omega, t) = \phi^-(r - d(r, -\omega, t)\omega, \omega, t - d(r, -\omega, t)/v_i) \exp(-\mu_i d(r, -\omega, t)), \quad r \in G_i,$$

$$(\mathcal{E}\Phi)(r, \omega, t) = \int_0^{d(r, -\omega, t)} \exp(-\mu_i \tau) \Phi(r - \tau\omega, \omega, t - \tau/v_i) d\tau.$$

Since $\mathcal{L}\mathcal{P}\phi = 0$, $\mathcal{P}\phi|_{X^-} = \phi^-$ and $\mathcal{L}\mathcal{E}\Phi = \Phi$, $\mathcal{E}\Phi|_{X^-} = 0$, then the solvability of the initial-boundary value problem in D is equivalent to the solvability of the equation

$$(13) \quad I = \mathcal{P}(\mathcal{B}I^+ + h) + \mathcal{E}(\mathcal{S}I + J).$$

In this section we study the boundary value problem for the auxiliary equation

$$(14) \quad \mathcal{L}f = \Phi, \quad \Phi \in L_\infty(X), \quad f \in D(X),$$

with the boundary condition (12). Equation (13) implies that solving this problem is equivalent to solving the corresponding integral equation

$$(15) \quad f = \mathcal{P}(h + \mathcal{B}f^+) + \mathcal{E}\Phi.$$

In the sequel, we will need an additional restriction on the regularity of the boundary of the set G : there are δ, δ' , $0 < \delta < 1$, $\delta' > 0$, such that for any point $z \in \partial G$ the set $\{z + \tau\omega : \delta \leq |n(z) \cdot \omega| \leq 1, 0 < |\tau| < \delta'\}$ has no intersection with ∂G . Conditions of this kind are typical in the study of the properties of anisotropic Sobolev spaces.

Lemma 1. *Equation (14) has at most one solution in D .*

Proof. Let the functions $f_1, f_2 \in D$ be solutions of the inhomogeneous equation (14). Then the function $f = f_1 - f_2$ satisfies the homogeneous version of equation (14). Consequently, the function f is a solution of equation (14) for $\Phi = 0$, and f^+ satisfies the relation

$$(16) \quad f^+ = \mathcal{P}(\mathcal{B}f^+) \text{ on } X^+.$$

Let us show that $f^+ = 0$ almost everywhere on X^+ . Since $R + T \leq 1$, $R_d + T_d \leq 1$, the Fresnel and the diffusion matching conditions at the interfaces of the media G satisfy the inequalities $\|\mathcal{B}_f\|_{L_\infty(X^-) \rightarrow L_\infty(X^+)} \leq 1$, $\|\mathcal{B}_d\|_{L_\infty(X^-) \rightarrow L_\infty(X^+)} \leq 1$. Given this circumstance, we obtain from (16) that

$$(17) \quad \begin{aligned} \|\mathcal{B}f^+\|_{L_\infty(X^-)} &= \|\mathcal{B}f^+\|_{L_\infty(X^- \setminus X_{ext}^-)} = \|\mathcal{B}\mathcal{P}\mathcal{B}f^+\|_{L_\infty(X^- \setminus X_{ext}^-)} \leq \\ &\leq \|\alpha_f \mathcal{B}_f \mathcal{P} + \alpha_d \mathcal{B}_d \mathcal{P}\|_{L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)} \|\mathcal{B}f^+\|_{L_\infty(X^-)} \leq \\ &\leq \left\| \alpha_f + \alpha_d \|\mathcal{B}_d \mathcal{P}\|_{L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)} \right\|_{L_\infty(\gamma)} \|f^+\|_{L_\infty(X^+)}. \end{aligned}$$

Let us estimate the operator norm of $\mathcal{B}_d \mathcal{P} : L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)$. Since the function $d(r, -\omega, t)$ equals $d(r, -\omega)$ on the set $X^- \setminus X_{ext}^-$, and the operators $\mathcal{B}_d$ and $\mathcal{P}$ are non-negative, we get the following inequality

$$(18) \quad \begin{aligned} \|\mathcal{B}_d \mathcal{P}\|_{L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)} &\leq \\ &\leq \sup_{(z, \omega) \in \Gamma^+} \left| \frac{R_d(z, \omega)}{\pi} \int_{\Omega(z, -\omega)} e^{-\bar{\sigma}d(z, -\omega')} |n(z) \cdot \omega'| d\omega' + \right. \\ &\quad \left. + \frac{T_d(z, \omega)}{\pi} \int_{\Omega(z, +\omega)} e^{-\bar{\sigma}d(z, -\omega')} |n(z) \cdot \omega'| d\omega' \right|, \end{aligned}$$

where $\bar{\sigma} = \max_{i=1, m} \sigma_i$. Let $W_\pm$ denote the following integrals

$$W_\pm(z, \omega) = \frac{1}{\pi} \int_{\Omega(z, \pm\omega)} \exp(-\bar{\sigma}d(z, -\omega')) |n(z) \cdot \omega'| d\omega'.$$

We will estimate $W_+(z, \omega)$ for all $n(z) \cdot \omega > 0$. To this end introduce the Lebesgue integration on the sphere Ω , assuming $d\omega' = dv d\varphi$, where:

$$(19) \quad \begin{cases} \nu = n(z) \cdot \omega', & -1 \leq \nu \leq 1, \\ \varphi = \arctan(\omega'_2/\omega'_1), & 0 \leq \varphi < 2\pi, \end{cases} \begin{cases} \omega'_1 = \cos(\varphi)\sqrt{1-\nu^2}, \\ \omega'_2 = \sin(\varphi)\sqrt{1-\nu^2}, \\ \omega'_3 = \nu. \end{cases}$$

In view of the boundary regularity condition, we obtain

$$\begin{aligned}
 (20) \quad W_+(z, \omega) &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \exp(-\bar{\sigma}d(z, -\omega'(\nu, \varphi))) \nu d\nu d\varphi \leq \\
 &\leq \frac{1}{\pi} \int_0^{2\pi} \int_0^\delta \nu d\nu d\varphi + \frac{1}{\pi} \int_0^{2\pi} \int_\delta^1 \exp(-\bar{\sigma}\delta') \nu d\nu d\varphi \leq \\
 &\leq \delta^2 + (1 - \delta^2) \exp(-\bar{\sigma}\delta') = 1 - (1 - \delta^2)(1 - \exp(-\bar{\sigma}\delta')).
 \end{aligned}$$

It is easy to see that the value $C_1 = 1 - (1 - \delta^2)(1 - \exp(-\bar{\sigma}\delta'))$ is less than unity if $0 < \delta < 1$, $\delta' > 0$. Similarly, we can show that $W_+(z, \omega) \leq C_1$ if $n(z) \cdot \omega < 0$ and $W_-(z, \omega) \leq C_1$ for almost all (z, ω) . Thus, from (18) and (20) we find that

$$(21) \quad \|\mathcal{B}_d \mathcal{P}\|_{L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)} \leq C_1 \sup_{(z, \omega) \in \Gamma^+} |R_d(z, \omega) + T_d(z, \omega)| \leq C_1 < 1.$$

Relations (16) and (21) imply that

$$\begin{aligned}
 (22) \quad \|f^+\|_{L_\infty(X^+)} &\leq \|\mathcal{P} \mathcal{B} f^+\|_{L_\infty(X^+)} \leq \|\mathcal{B} f^+\|_{L_\infty(X^-)} \leq \\
 &\leq \|\alpha_f + \alpha_d \|\mathcal{B}_d \mathcal{P}\|_{L_\infty(X^- \setminus X_{ext}^-) \rightarrow L_\infty(X^+)} \|f^+\|_{L_\infty(X^+)} \leq \\
 &\leq \|\alpha_f + \alpha_d C_1\|_{L_\infty(\gamma)} \|f^+\|_{L_\infty(X^+)} = C_2 \|f^+\|_{L_\infty(X^+)}.
 \end{aligned}$$

By assumption the function $\alpha_d(z)$ is constant for $z \in \partial G_i$, $i = 1, \dots, m$, so if $\alpha_d(z) > 0$ for at least one boundary component $z \in \partial G_j$, then the constant $C_2 = \|\alpha_f + \alpha_d C_1\|_{L_\infty(\gamma)}$ in (22) is less than one. As $C_2 < 1$, inequality (22) is possible only when $f^+ = 0$ almost everywhere on X^+ and hence it follows from (15) that $f = 0$ almost everywhere on X .

Consider the case $\alpha_d(z) = 0$ everywhere on γ . In this case, the matching operator $\mathcal{B}$ involves only the Fresnel component $\mathcal{B}_f$.

Let the functions f_1 and f_2 belong to D and satisfy equation (14) and suppose that $\kappa(r) = \kappa_i$, $r \in G_i$ is the refractive index of the medium G . Then the function $f = (f_1 - f_2)/\kappa^2$ satisfies a homogeneous equation $\mathcal{L}f = 0$ with the following boundary conditions:

$$(23) \quad f^-(z, \omega, t) = R(z, \nu) f^+(z, \omega_{re}, t) + \tilde{\kappa}^2(z, \nu) T(z, \nu) f^+(z, \omega_{tr}, t), \quad (z, \omega, t) \in Y_{int},$$

$$(24) \quad f^-(z, \omega, t) = 0, \quad (z, \omega, t) \in X_{ext}^-.$$

Without loss of generality, define the function f to be equal to zero for $t < 0$ and write the solution of the equation $\mathcal{L}f = 0$, valid for almost all $(\xi, \tau, \omega, t) \in \Pi_\omega \times \Pi_{\xi, \omega} \times \Omega \times [0, \infty)$ as

$$f(\xi + \tau\omega, \omega, t + \tau/\tilde{v}_i) = f^-(\xi + \tau_i(\xi, \omega)\omega, \omega, t + \tau_i(\xi, \omega)/\tilde{v}_i) \exp\left(-\int_{\tau_i(\xi, \omega)}^\tau \mu(\xi + \tau'\omega) d\tau'\right),$$

where the value $\tilde{v}_i$ from $f(\xi + \tau\omega, \omega, t + \tau/\tilde{v}_i)$ equals v_j if the interval $\{\xi + \tau\omega, \tau \in (\tau_i(\xi, \omega), \tau_{i+1}(\xi, \omega))\}$ belongs to the subregion G_j .

From the last equation it follows that the function $f(\xi + \tau\omega, \omega, t + \tau/v)$ for $\tau \in (\tau_i(\xi, \omega), \tau_{i+1}(\xi, \omega)]$ does not change sign.

As $\mathcal{L}f = \Phi \in L_\infty(X)$ the function $\mathcal{L}f$ is bounded and measurable on the set $G \times \Omega$ almost everywhere if $t \in [0, \infty)$ by Fubini's theorem.

Multiply the equation $\mathcal{L}f = 0$ by the function $\text{sgn}(f(r, \omega, t))$ and integrate over the set $G \times \Omega$ to obtain

$$\begin{aligned}
 (25) \quad A &= \\
 &= \int_{\Omega} \int_G \text{sgn}(f(r, \omega, t)) \left(\frac{1}{v(r)} \frac{\partial f(r, \omega, t)}{\partial t} + \omega \cdot \nabla_r f(r, \omega, t) + \mu(r) f(r, \omega, t) \right) dr d\omega = \\
 &= \int_{\Omega} \int_{\Pi_{\omega}} \sum_{i=1}^{q(\xi, \omega)} \int_{\Pi_{\xi, \omega}^i} \frac{\partial |f(\xi + \tau\omega, \omega, t + \tau/\tilde{v}_i)|}{\partial \tau} d\tau d\xi d\omega + \int_{\Omega} \int_G \mu(r) |f(r, \omega, t)| dr d\omega = \\
 &= \int_{\Omega} \int_{\Pi_{\omega}} \{ |f|^+(\xi + \tau_{q+1}(\xi, \omega)\omega, \omega, t + \tau_{q+1}(\xi, \omega)/\tilde{v}_{q+1}) - \\
 &\quad - |f|^{-}(\xi + \tau_1(\xi, \omega)\omega, \omega, t + \tau_1(\xi, \omega)/\tilde{v}_1) + \\
 &\quad + \sum_{i=2}^{q(\xi, \omega)} (|f|^+ - |f|^{-})(\xi + \tau_i(\xi, \omega)\omega, \omega, t + \tau_i(\xi, \omega)/\tilde{v}_i) \} d\xi d\omega + \\
 &\quad + \int_{\Omega} \int_G \mu(r) |f(r, \omega, t)| dr d\omega = 0.
 \end{aligned}$$

Using boundary conditions (23), (24) and equation (25) and applying the change of variables in the surface integral ($d\xi = |n(z) \cdot \omega| ds_z$) we get:

$$\begin{aligned}
 (26) \quad A &= \int_{\partial \overline{G} \cap \Omega_-(z)} |f(z, \omega, t)|^+ |\nu| d\omega ds_z \int_{\gamma} \int_{\Omega} \{ |f(z, \omega, t)|^+ - \\
 &\quad - |R(z, \nu) f^+(z, \omega_{re}, t) + \tilde{\kappa}^2(z, \nu) T(z, \nu) f^+(z, \omega_{tr}, t)| \} |\nu| d\omega ds_z + \\
 &\quad + \int_{\Omega} \int_G \mu(r) |f(r, \omega, t)| dr d\omega = A_1 + A_2 + A_3, \quad \nu = n(z) \cdot \omega,
 \end{aligned}$$

where A_1, A_2, A_3 denote the corresponding terms from (24). Note immediately that

$$(27) \quad A_1 = \int_{\partial \overline{G} \cap \Omega_-(z)} |f(z, \omega, t)|^+ |\nu| d\omega ds_z \geq 0, \quad \nu = n(z) \cdot \omega.$$

As $R, T \geq 0$ we have

$$\begin{aligned}
 (28) \quad &|f(z, \omega, t)|^+ - |R(z, \nu) f^+(z, \omega_{re}, t) + \tilde{\kappa}^2(z, \nu) T(z, \nu) f^+(z, \omega_{tr}, t)| \geq \\
 &\geq |f(z, \omega, t)|^+ - R(\nu) |f(z, \omega_{re}, t)|^+ - \tilde{\kappa}^2(z, \nu) T(\nu) |f(z, \omega_{tr}, t)|^+,
 \end{aligned}$$

so that

$$\begin{aligned}
 (29) \quad A_2 &\geq \int_{\gamma} \int_{\Omega} |f(z, \omega, t)|^+ |\nu| d\omega ds_z - \int_{\gamma} \int_{\Omega} R(z, \nu) |f(z, \omega_{re}, t)|^+ |\nu| d\omega ds_z - \\
 &\quad - \int_{\gamma} \int_{\Omega} \tilde{\kappa}^2(z, \nu) T(z, \nu) |f(z, \omega_{tr}, t)|^+ |\nu| d\omega ds_z = A_{2,1} - A_{2,2} - A_{2,3}.
 \end{aligned}$$

Insofar as the vectors $\omega, \omega_{re}, \omega_{tr}$ belong to a single plane

$$\nu_{re} = n(z) \cdot \omega_{re} = -n(z) \cdot \omega = -\nu, \quad \nu_{tr} = n(z) \cdot \omega_{tr} = \psi(z, \nu),$$

we can get the following expressions for $A_{2,2}$, $A_{2,3}$ after parametrization of the sphere by formulas (19):

$$A_{2,2} = \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 R(z, \nu) |f(z, \omega(-\nu, \varphi), t)|^+ |\nu| d\nu d\varphi ds_z,$$

$$A_{2,3} = \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 \tilde{\kappa}^2(z, \nu) T(z, \nu) |f(z, \omega(\psi(\nu), \varphi), t)|^+ |\nu| d\nu d\varphi ds_z.$$

Substitute $\nu' = -\nu$ in the integral representing $A_{2,2}$ and $\nu' = \psi(z, \nu)$ in the integral representing $A_{2,3}$. Since the functions $\nu' = \psi(z, \nu)$ and ν have the same sign, $\tilde{\kappa}(z, \nu') = \tilde{\kappa}(z, \nu)$, $\tilde{\kappa}(z, \nu') = \tilde{\kappa}(z, \nu)$. Hence,

$$\psi'(z, \nu) = \frac{\text{sgn}(z, \nu) \cdot \nu \cdot \tilde{\kappa}^2(z, \nu)}{\sqrt{1 - \tilde{\kappa}^2(z, \nu)(1 - \nu^2)}} = \frac{\nu \cdot \tilde{\kappa}^2(z, \nu)}{\psi(z, \nu)} = \frac{\nu \cdot \tilde{\kappa}^2(z, \nu)}{\nu'} = \frac{\nu \cdot \tilde{\kappa}^2(z, \nu')}{\nu'}$$

and

$$|\nu| d\nu = \frac{|\nu| \cdot \nu' d\nu'}{\nu \cdot \tilde{\kappa}^2(z, \nu')} = \frac{\text{sgn}(\nu) \cdot \nu' d\nu'}{\tilde{\kappa}^2(z, \nu')} = \frac{\text{sgn}(\nu') \cdot \nu' d\nu'}{\tilde{\kappa}^2(z, \nu')} = \frac{|\nu'| d\nu'}{\tilde{\kappa}^2(z, \nu')}.$$

Returning to the old notation for the variable ν we get

$$A_{2,2} = \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 R(z, -\nu) |f(z, \omega(\nu, \varphi), t)|^+ |\nu| d\nu d\varphi ds_z,$$

$$A_{2,3} = \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 T(z, \psi^{-1}(z, \nu)) |f(z, \omega(\nu, \varphi), t)|^+ |\nu| d\nu d\varphi ds_z.$$

In view of the equations

$$\kappa(z, \nu) = 1/\kappa(z, -\nu), \quad \psi^{-1}(z, \nu) = -\psi(z, -\nu), \quad \text{if } 1 - \kappa^2(z, \nu)(1 - \nu^2) \geq 0,$$

$$R_{\parallel}(z, -\nu) = \frac{\tilde{\kappa}(z, -\nu)\psi(z, -\nu) + \nu}{\tilde{\kappa}(z, -\nu)\psi(z, -\nu) - \nu} = \frac{-\psi^{-1}(z, \nu)/\tilde{\kappa}(z, \nu) + \nu}{-\psi^{-1}(z, \nu)/\tilde{\kappa}(z, \nu) - \nu} =$$

$$= -\frac{\nu\tilde{\kappa}(z, \nu) - \psi^{-1}(z, \nu)}{\nu\tilde{\kappa}(z, \nu) + \psi^{-1}(z, \nu)} = -R_{\parallel}(\psi^{-1}(z, \nu)),$$

$$R_{\perp}(z, -\nu) = \frac{\psi(z, -\nu) - \tilde{\kappa}(-z, \nu)\nu}{\psi(z, -\nu) - \tilde{\kappa}(z, -\nu)\nu} = \frac{-\psi^{-1}(z, \nu) + \nu/\tilde{\kappa}(z, \nu)}{-\psi^{-1}(z, \nu) - \nu/\tilde{\kappa}(z, \nu)} =$$

$$= -\frac{\nu - \psi^{-1}(z, \nu)\tilde{\kappa}(z, \nu)}{\nu + \psi^{-1}(z, \nu)\tilde{\kappa}(z, \nu)} = -R_{\perp}(\psi^{-1}(z, \nu)),$$

$$R(z, -\nu) = 0.5\{R_{\parallel}^2(z, -\nu) + R_{\perp}^2(z, -\nu)\} =$$

$$= 0.5\{R_{\parallel}^2(z, \psi^{-1}(z, \nu)) + R_{\perp}^2(z, \psi^{-1}(z, \nu))\} = R(\psi^{-1}(z, \nu)),$$

$$T(\psi^{-1}(z, \nu)) = 1 - R(z, \psi^{-1}(z, \nu)) = 1 - R(z, -\nu) = T(z, -\nu),$$

we find from (29) that

$$\begin{aligned}
 (30) \quad A_2 &\geq A_{2,1} - A_{2,2} - A_{2,3} = \int_{\gamma} \int_{\Omega} |f(z, \omega, t)|^+ |\nu| d\omega ds_z - \\
 &\quad - \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 R(z, -\nu) |f(z, \omega(\nu, \varphi), t)|^+ |\nu| d\nu d\varphi ds_z - \\
 &\quad - \int_{\gamma} \int_0^{2\pi} \int_{-1}^1 T(z, -\nu) |f(z, \omega(\nu, \varphi), t)|^+ |\nu| d\nu d\varphi ds_z = 0.
 \end{aligned}$$

From (26),(27) it follows that $A = A_1 + A_2 + A_3 \geq A_3$ for almost all $t \in [0, \infty)$. According to (25) the function $A = 0$, hence $A_3 = 0$ for almost all $t \in [0, \infty)$. The function $\mu(r) > 0$, therefore $f = 0$ almost everywhere in $G \times \Omega \times [0, \infty)$.

This completes the proof of the lemma. ■

Lemma 2. *A solution of the equation*

$$(31) \quad f = \mathcal{P}(h + \mathcal{B}f^+) + \mathcal{E}\Phi$$

exists, is unique and can be estimated by

$$(32) \quad \|f\|_{L_{\infty}(X)} \leq \max \left\{ \|h\|_{L_{\infty}(X_{ext}^-)}, \left\| \frac{\Phi}{\mu} \right\|_{L_{\infty}(X)} \right\}.$$

Proof. The uniqueness of the solution of equation (31) follows from lemma 1, let us show its existence.

Introduce the notation $\Phi_{\pm}, h_{\pm}$ for the following functions:

$$\Phi_+(r, \omega, t) = \begin{cases} \Phi(r, \omega, t), & \text{if } \Phi(r, \omega, t) \geq 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$\Phi_-(r, \omega, t) = \begin{cases} \Phi(r, \omega, t), & \text{if } \Phi(r, \omega, t) \leq 0; \\ 0, & \text{otherwise,} \end{cases}$$

$$h_+(z, \omega, t) = \begin{cases} h(z, \omega, t), & \text{if } h(z, \omega, t) \geq 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$h_-(z, \omega, t) = \begin{cases} h(z, \omega, t), & \text{if } h(z, \omega, t) \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $f_{\pm,0} = \mathcal{P}h_{\pm} + \mathcal{E}\Phi_{\pm}$ and consider the iterative process

$$(33) \quad f_{\pm,n} = \mathcal{P}\mathcal{B}f_{\pm,n-1}^+ + f_{\pm,0} \quad n = 1, 2, \dots$$

As the operators $\mathcal{P}, \mathcal{E}, \mathcal{B}$ are nonnegative, it follows that $f_{+,0}, f_{+,1}, \dots, f_{+,n}, \dots$ is a monotonically increasing sequence of functions and $f_{-,0}, f_{-,1}, \dots, f_{-,n}, \dots$ is a monotonically decreasing sequence. Let us show that the sequence $\{f_{+,n}\}$ is bounded from above and the sequence $\{f_{-,n}\}$ is bounded from below.

For the function $f_{+,0}$ estimate (32) ensues from the following chain of inequalities, valid for almost all $(r, \omega, t) \in X$,

$$\begin{aligned}
 (34) \quad f_{+,0} &\leq \max_{1 \leq i \leq m} \|h_+(r - d(r, -\omega, t)\omega, \omega, t - d(r, -\omega, t)/v_i) \exp(-\mu_i d(r, -\omega, t)) + \\
 &\quad + \int_0^{d(r, -\omega, t)} \exp(-\mu_i \tau) \Phi_+(r - \tau\omega, \omega, t - \tau/v_i) d\tau \|_{L_\infty(G_i \times \Omega \times [0, \infty))} \leq \\
 &\leq \max_{1 \leq i \leq m} \|\exp(-\mu_i d(r, -\omega, t)) h\|_{X_{ext}^-} + \\
 &+ (1 - \exp(-\mu_i d(r, -\omega, t))) \left\| \frac{\Phi_+}{\mu} \right\|_{L_\infty(X)} \leq \max \left\{ \|h\|_{X_{ext}^-}, \left\| \frac{\Phi_+}{\mu} \right\|_{L_\infty(X)} \right\}.
 \end{aligned}$$

Assuming that the function $f_{+,n-1}$ satisfies inequality (32), we will now verify that it is also true for the function $f_{+,n}$. Indeed, since the supports of the functions h and $\mathcal{B}f_{+,n}^+$ do not intersect and $\|\mathcal{B}\| \leq 1$, it follows that for almost all $(r, \omega, t) \in X$ we get from (34) that

$$\begin{aligned}
 (35) \quad f_{+,n}(r, \omega, t) &\leq \max_{1 \leq i \leq m} \left\| \max \left\{ \|h\|_{L_\infty(X_{ext}^-)}, \|\mathcal{B}f_{+,n-1}^+\|_{L_\infty(X^-)} \right\} \times \right. \\
 &\times \exp(-\mu_i d(r, -\omega, t)) + \left. \left\| \frac{\Phi_+}{\mu} \right\|_{L_\infty(X)} (1 - \exp(-\mu_i d(r, -\omega, t))) \right\|_{L_\infty(G_i \times \Omega \times [0, \infty))} \leq \\
 &\leq \max_{1 \leq i \leq m} \left\| \max \left\{ \|h\|_{L_\infty(X_{ext}^-)}, \max \left\{ \|h\|_{L_\infty(X_{ext}^-)}, \left\| \frac{\Phi_+}{\mu} \right\|_{L_\infty(X)} \right\} \right\} \times \right. \\
 &\times \exp(-\mu_i d(r, -\omega, t)) + \left. \left\| \frac{\Phi_+}{\mu} \right\|_{L_\infty(X)} (1 - \exp(-\mu_i d(r, -\omega, t))) \right\|_{G_i \times \Omega \times [0, \infty)} \leq \\
 &\leq \max \left\{ \|h\|_{L_\infty(X_{ext}^-)}, \left\| \frac{\Phi}{\mu} \right\|_{L_\infty(X)} \right\}.
 \end{aligned}$$

Thus, the monotonic sequence $f_{+,0}, f_{+,1}, \dots, f_{+,n}, \dots$ is bounded from above, and hence it has a limit $f_+ = \lim_{n \rightarrow \infty} f_{+,n}$ in the space $L_\infty(X)$. Similarly, it can be shown that

$$f_{-,n}(r, \omega, t) \geq -\max \left\{ \|h\|_{L_\infty(X_{ext}^-)}, \left\| \frac{\Phi}{\mu} \right\|_{L_\infty(X)} \right\}$$

for almost all $(r, \omega, t) \in X$.

As relation (33) is also valid for the function $f \in D$, repeating the proof it is easy to verify the existence of the limit functions $f_\pm^+ \in L_\infty(X^+)$.

Passing to the limit in equality (33), we conclude that the functions $f_\pm$ satisfy equation (31) with $\Phi = \Phi_\pm$. Hence, we have shown the existence of the solution f of equation (31) satisfying estimate (32). ■

From the lemmas 1 and 2, it follows that an inverse to the operator $\mathcal{L}$ exists and is bounded. Introduce a norm on the linear set D by the formula

$$(36) \quad \|f\|_D = \max \left\{ \|f^-\|_{L_\infty(X_{ext}^-)}, \left\| \frac{\mathcal{L}f}{\mu} \right\|_{L_\infty(X)} \right\}.$$

As $\mu \geq \text{conts} > 0$ inequality (32) implies that

$$(37) \quad \|f\|_{L_\infty(X)} \leq \|f\|_D.$$

Thus convergence of a sequence of functions in the norm of D implies its convergence in the spaces $L_\infty(X^+)$ and $L_\infty(X)$. It follows that the set $D \subset L_\infty(X)$ with the norm (36) forms a functional Banach space.

Theorem 1. *If*

$$(38) \quad \bar{\lambda} = \max_{1 \leq i \leq m} \lambda_i < 1 \quad \lambda_i = \sigma_i / \mu_i,$$

then a solution I of equation (13) in D exists and is unique.

Proof. The function p is non-negative and satisfies the normalization condition and, moreover, inequality (37) holds, therefore for $\|SI/\mu\|$ we get

$$(39) \quad \left\| \frac{SI}{\mu} \right\|_{L_\infty(X)} = \left\| \frac{\sigma(r)}{\mu(r)} \int_{\Omega} p(r, \omega \cdot \omega') I(r, \omega', t) d\omega' \right\|_{L_\infty(X)} \leq \bar{\lambda} \|I\|_{L_\infty(X)} \leq \bar{\lambda} \|I\|_D.$$

By construction $\mathcal{L}(\mathcal{PB} + \mathcal{ES})I = SI$, and $(\mathcal{PBI} + \mathcal{ESI})^- = 0$ for X_{ext}^- , hence, from lemma 2 we find that

$$\|(\mathcal{PB} + \mathcal{ES})I\|_D = \max \left\{ 0, \left\| \frac{SI}{\mu} \right\|_{L_\infty(X)} \right\} \leq \bar{\lambda} \|I\|_D.$$

From (39) and the last inequality it follows that

$$\|\mathcal{PB} + \mathcal{ES}\|_{D \rightarrow D} \leq \bar{\lambda} < 1.$$

Since the norm of the operator $\mathcal{PB} + \mathcal{ES}$ acting in the Banach space $\mathcal{D}$ is less than one, it implies that equation (13) with condition (38) is uniquely solvable, and the solution can be found by successive approximations of the form:

$$I_n = (\mathcal{PB} + \mathcal{ES})I_{n-1} + I_0, \quad n = 1, 2, \dots$$

This completes the proof of Theorem 1. ■

Note, that condition (38) ensures the existence and uniqueness of a bounded solution for $t \in [0, \infty)$. For functions that admit unbounded growth as $t \rightarrow \infty$, or in the case of a finite time interval ($t \in [0, t^*], t^* < \infty$), validity of the condition $\bar{\lambda} < 1$ is not required for the well-posedness of the problem. Indeed, if $\sigma_i \geq \mu_i$, then after substitution $I = I_\lambda e^{\lambda t}$, where λ is an arbitrary number satisfying the conditions $\frac{\sigma_i}{\mu_i + \lambda/v_i} < 1$, $i = 1, 2, \dots, m$, the initial data of the initial-boundary value problem for the function I_λ will satisfy condition (38) automatically. Theorem 1 implies the existence and uniqueness of the solution I_λ , and hence, the solution I of the original problem.

4. MONTE CARLO METHOD OF NUMERICAL SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM

There is a large variety of numerical methods for solving the radiation transfer equations [1–5, 12–19]. However, there is virtually no alternative to Monte Carlo methods in case of general multidimensional space, especially when solving the real-life applied problems [15–17, 36, 37].

Representation in the form of a truncated Neumann series

$$(40) \quad I_N = \sum_{n=0}^N (\mathcal{P}\mathcal{B} + \mathcal{E}\mathcal{S})^n (\mathcal{P}h + \mathcal{E}J)$$

and its recurrence analogue

$$(41) \quad I_0 = \mathcal{P}h + \mathcal{E}J,$$

$$(42) \quad I_n = (\mathcal{P}\mathcal{B} + \mathcal{E}\mathcal{S})I_{n-1} + I_0, \quad n = 1, 2 \dots N$$

form the basis for the construction of numerical algorithms for calculating the unknown function I . This form of unknown function has a vivid physical representation. Each term of the sequence I_n is an approximation to the function I that accounts for the radiation that interacted with the medium no more than n times. The application of the $\mathcal{P}\mathcal{B} + \mathcal{E}\mathcal{S}$ operator to the term I_{n-1} of the sequence determines the contribution of radiation that experienced from 1 to n acts of interaction with the environment, and the term I_0 accounts for the contribution of the radiation coming from sources without interaction with the medium.

According to the Monte Carlo method, an approximation to I_N is calculated as the average value of a sample of size M drawn from a random variable $\mathcal{I}_N$ whose expectation equals I_N , so that

$$(43) \quad I_N \approx \bar{\mathcal{I}}_N.$$

To find the value of $\mathcal{I}_n$ at the point $x = (r, \omega, t) \in G_i \times \Omega \times [0, \infty)$ we calculate the function $\mathcal{I}_{n-1}$ at four other points:

$$(44) \quad \begin{aligned} x_{re} &= (r_h, \omega_{re}, t_h), & r_h &= r - d(r, -\omega, t)\omega, \\ & & \omega_{re} &= \omega - 2\nu n(r_h), \quad \nu = n(r_h) \cdot \omega, \\ & & t_h &= t - d(r, -\omega, t)/v_i, \\ x_{tr} &= (r_h, \omega_{tr}, t_h), & \omega_{tr} &= \psi(\nu)n(r_h) + \tilde{\kappa}(\nu)(\omega - \nu n(r_h)), \\ x_{di} &= (r_h, \omega_{di}, t_h), \\ x_{sc} &= (r_{sc}, \omega_{sc}, t_{sc}), & r_{sc} &= r - \tau\omega, \quad t_{sc} = t - \tau/v_i. \end{aligned}$$

Further, the functions h and J are computed at the points

$$x_h = (r_h, \omega, t_h), \quad x_s = (r_{sc}, \omega, t_{sc}).$$

Put

$$(45) \quad \mathcal{I}_n(x) = \exp(-\mu_i d(r, -\omega, t))$$

$$(h(x_h) + \alpha_f(x_h)(R(x_h)\mathcal{I}_{n-1}^+(x_{re}) + T(x_h)\mathcal{I}_{n-1}^+(x_{tr})) + \frac{\alpha_d(x_h)K_d(\omega, x_{di})}{f_{\omega_{di}}}\mathcal{I}_{n-1}^+(x_{di})) + \frac{1 - \exp(-\mu_i d(r, -\omega, t))}{\mu_i} \left(J(x_s) + \frac{\sigma_i p_i(\omega \cdot \omega_{sc})}{f_{\omega_{sc}}}\mathcal{I}_{n-1}(x_{sc}) \right),$$

$$(46) \quad \mathcal{I}_0(x) = \exp(-\mu_i d(r, -\omega, t))h(x_h) + \frac{1 - \exp(-\mu_i d(r, -\omega, t))}{\mu_i} J(x_s),$$

where

$$K_d(\omega, x_{di}) = \begin{cases} \frac{|n(r_h) \cdot \omega_{di}|}{\pi} T_d(x_{di}) & (\omega, \omega_{di}) > 0, \\ \frac{|n(r_h) \cdot \omega_{di}|}{\pi} R_d(x_{di}) & (\omega, \omega_{di}) < 0. \end{cases}$$

The random variable τ is distributed according to the probability density

$$f_\tau = \frac{\exp(-\mu_i \tau)}{(1 - \exp(-\mu_i d(r, -\omega, t)))}.$$

To sample the random vectors ω_{sc}, ω_{di} we will try to use the distribution minimizing the variance of the resulting estimate.

The random variables $\omega_{di}, \tau, \omega_{sc}$ are independent implying that the variance of the random variable $\mathcal{I}_N(x)$ is equal to

$$(47) \quad D(\mathcal{I}_n(x)) = \exp(-\mu_i d(r, -\omega, t)) \frac{\alpha_d(x_h)K_d(\omega, x_{di})}{f_{\omega_{di}}} D(\mathcal{I}_{n-1}^+(x_{di})) + \frac{1 - \exp(-\mu_i d(r, -\omega, t))}{\mu_i p_i(\omega \cdot \omega_{sc})} D(J(x_s) + \sigma_i \mathcal{I}_{n-1}(x_{sc})).$$

The following functions are often used as the probability densities of ω_{di}, ω_{sc} : $f_{\omega_{di}} = K_d(\omega, x_{di}), f_{\omega_{sc}} = p_i(\omega \cdot \omega_{sc})$ [34]. This choice is optimal in the case when the function $\mathcal{I}_{n-1}$ is constant in the domain under examination:

$$(48) \quad \mathcal{I}_{n-1}(x) = \text{const.}$$

Indeed,

$$(49) \quad \bar{\mathcal{I}}_{n-1}(x_{di}) = \int_{\Omega} K_d(\omega, x_{di}) \mathcal{I}_{n-1}(r_h, \omega_{di}, t_h) d\omega_{di},$$

$$(50) \quad \bar{\mathcal{I}}_{n-1}(x_{sc}) = \int_0^{d(r, -\omega, t)} \int_{\Omega} \exp(-\mu_i \tau) p(r, \omega \cdot \omega_{sc}) \mathcal{I}_{n-1}(r - \tau\omega, \omega_{sc}, t - \tau/v_i) d\omega_{sc} d\tau,$$

so that the probability densities of x_{di}, x_{sc} are equal up to a normalizing factor to the corresponding integrands which, in turn, is a minimality criterion for the variances of the random variables of the form (49), (50). The same remark is true for the random variable $J(x_s)$.

This choice of functions $f_{\omega_{di}}, f_{\omega_{sc}}$ also simplifies the calculation formula as follows

$$(51) \quad \mathcal{I}_n(x) = \exp(-\mu_i d(r, -\omega, t)) \\ (h(x_h) + \alpha_d(x_h)(R(x_h)\mathcal{I}_{n-1}^+(x_{re}) + T(x_h)\mathcal{I}_{n-1}^+(x_{tr}) + \alpha_d(x_h)\mathcal{I}_{n-1}^+(x_{di})) + \\ + \frac{1 - \exp(-\mu_i d(r, -\omega, t))}{\mu_i} (J(x_s) + \sigma_i \mathcal{I}_{n-1}(x_{sc}))).$$

Note also that assumption (48) expresses, in some sense, the absence of a priori information about the function $\mathcal{I}_{n-1}$ at the moment of calculation of the value of $\mathcal{I}_n(x)$.

In the case when the density function of external sources $h(x)$ is a priori known, we could use this information to reduce the variance of the generated random variables.

Let us consider the branching chains of a random variables, with transition functions given by relations (44). We introduce the notation for the elements of these chains according to the following recursive rule: the element of the chain at the N -th level is denoted by $x_{in_1, in_2, \dots, in_n}$, if it was obtained from the element $x_{in_1, in_2, \dots, in_{n-1}}$ after interaction of the type in_n , where $in_i \in \{tr, re, sc, di, h, s\}$.

Substitute the definition $\mathcal{I}_{n-1}(x)$ from (41), (42) into (49), (50):

$$(52) \quad \bar{\mathcal{I}}_{n-1}(x_{di}) = \mathcal{P}B_d I_{n-1} = \mathcal{P}B_d(\mathcal{P}h + \mathcal{E}J + (\mathcal{P}\mathcal{B} + \mathcal{E}\mathcal{S})I_{n-2}),$$

$$(53) \quad \sigma_i \bar{\mathcal{I}}_{n-1}(x_{sc}) = \mathcal{E}S I_{n-1} = \mathcal{E}S(\mathcal{P}h + \mathcal{E}J + (\mathcal{P}\mathcal{B} + \mathcal{E}\mathcal{S})I_{n-2}).$$

Let us write the terms responsible for the contribution of the radiation coming from the source after a single interaction with the medium in greater detail:

$$(54) \quad \bar{\mathcal{I}}_{n-2}^{d,h} = \mathcal{P}B_d(\mathcal{P}h) = \exp(-\mu_i d(r_h, -\omega_{di}, t_h)) \int_{\Omega} K_d(\omega, x_{di}) h(x_{di,h}) d\omega_{di},$$

$$(55) \quad \bar{\mathcal{I}}_{n-2}^{sc,h} = \mathcal{E}S(\mathcal{P}h) = \sigma(r) \int_0^{d(r, -\omega, t)} \int_{\Omega} \exp(-\mu_i(d(r - \tau\omega, -\omega_{sc}, t - \tau/v_i) + \tau)) \\ p_i(\omega \cdot \omega_{sc}) h(x_{sc,h}) d\tau d\omega_{sc}.$$

The minimal variance of these random variables is attained at

$$(56) \quad f_{\omega_{di}} = \frac{K_d(\omega, x_{di}) h(x_{di,h})}{\int_{\Omega} K_d(\omega, x'_{di}) h(x'_{di,h}) d\omega'_{di}},$$

$$(57) \quad f_{\omega_{sc}} = \frac{\exp(-\mu_i(d(r - \tau\omega, -\omega_{sc}, t - \tau/v_i) + \tau)) p_i(\omega \cdot \omega_{sc}) h(x_{sc,h})}{\int_{\Omega} \exp(-\mu_i(d(r - \tau\omega, -\omega'_{sc}, t - \tau/v_i) + \tau)) p_i(\omega \cdot \omega'_{sc}) h(x'_{sc,h}) d\omega'_{sc}}.$$

In the case when the choice of the distribution laws $f_{\omega_{di}}, f_{\omega_{sc}}$ affects only the variance of the relevant terms $\bar{\mathcal{I}}_{n-2}^{di,h}, \bar{\mathcal{I}}_{n-2}^{sc,h}$, this distribution will also optimize sampling of the original random vector $\mathcal{I}_N(x)$. Assumption (48) actually expresses the absence a priori information about the distribution of the values of the function $\mathcal{I}_{n-2}$.

5. COMPUTATIONAL EXPERIMENTS

To verify the constructed model of the radiation transfer process, as well as to determine the effectiveness of the proposed computational method for solving the visualization problems of three-dimensional non-stationary scenes, some computational experiments were carried out. In these experiments the perspective projections for given three-dimensional scenes were calculated at different points of time. Comparing the results obtained with the natural ideas about the propagation of visible light, we were able to give an indirect assessment of the proposed methods.

Let us make a few remarks about the color spectrum of the reconstructed three-dimensional scenes. Strictly speaking, the considered model does not address the questions of interdependence between the radiation propagation and its frequency characteristics, although the resulting images are colored. To obtain a colored image, we carried out calculations separately for each of the RGB channels, using different sets of optical characteristics of the medium. On the one hand, our combined image gives us a clear and natural presentation of the results of multiple interactions of radiation with the medium. On the other hand, due to color contrast, the spatial distribution of the error caused by a particular implementation of the Monte Carlo method is much more visible on the constructed images.

Let us describe the experiments. The purpose of the first experiment is to visualize the evolutionary process of radiation propagation in the medium. The medium in question is located above the diffusely reflecting plane ($\alpha_f = 0, R_d(z) = 1$). In each diffuse interaction the radiation decays with the coefficients $\alpha_d = (0.7, 0.2, 0.2)$, for the red, green and blue channels, respectively.

The following objects are located on the plane:

several diffusely reflecting balls with the boundary characteristics equal to those of the plane itself;

several balls reflecting only the radiation in the green channel ($\alpha_f = (0, 1, 0), \alpha_d = 0, \kappa(z) = 3$, the coefficients μ and σ inside the specular and diffuse balls are equal to the corresponding coefficients of the surrounding medium);

balls with a transparent surface ($\alpha_f = 1, \alpha_d = 0, \kappa(z) = 1$) which are filled with strongly scattering substance with an interaction coefficient $\mu(r) = 2.3$ and $\sigma(r) = (0.1, 0.7, 0.8)$ for each channel.

There is a statuette [38] at the center of the scene, its surface has the following characteristics: $\alpha_f = (0.05, 0.45, 0.05)$, $\alpha_d = (0.35, 0.1, 0.1)$, $R_d(z) = 1$, $\kappa(z) = 1.3$, $\mu(r) = 2.3$, $\sigma(r) = (0.1, 0.7, 0.8)$.

The radiation source is located at the boundary of the region under examination and its density decreases exponentially both in time and in spatial coordinates staring at the point (z_s, t_s) , i.e $h(z, \omega, t) = \exp(-|z - z_s|^2 - |t - t_s|^2)$.

Figure 1 presents the results of the experiment for the points of time $t = 110, 115, 120, \dots$. We can see how the radiation successively illuminates more and more distant parts of the scene, while the time coordinate increases. This effect, as well as shadows, translucency of the scattering balls, glares on the mirror surfaces, absence of glares at diffuse boundaries and the mutual reflection of objects in each other indicate the ability of the algorithms developed here to generate accurate, photo-realistic reconstructions.

In the second experiment, we tried to compare the efficiency of the algorithms with the sampling of the random vector ω_{di}, ω_{sc} optimized by using formulas (56), (57) and the standard method with probability densities $f_{\omega_{di}} = K_d(\omega, x_{di})$. We fixed

FIGURE 1. The evolutionary process of solving the non-stationary problem for the radiative transfer equation

the value of the time coordinate at $t = 155$ and saved the images at certain regular intervals during the running time of the algorithm. Fig. 2 shows the calculation results for the two modifications of the Monte Carlo algorithm. The first pair of images is taken after 2^1 sec., the second — after 2^3 sec., the third — after 2^5 sec. etc. The upper images results from the standard method, while the lower — from the optimized one.

In order to simplify sampling instead of the law given by formulas (56), (57) in the optimized method we used a much simpler distribution law, namely:

$$f_{\omega_{di}} = \frac{h(x_{di,h})}{\int_{\Omega} h(x'_{di,h}) d\omega'_{di}}, \quad f_{\omega_{sc}} = \frac{h(x_{sc,h})}{\int_{\Omega} h(x'_{sc,h}) d\omega'_{sc}}.$$

Thus, for the selected distribution function of the boundary sources generation was reduced to the sampling from the random variables normally distributed on the source plane.

Analyzing the results, it is noteworthy that despite rather rough assumptions, some positive effect of calculation optimization clearly takes place. The quality of the first images obtained by the optimized method far exceeds that of the images obtained by the unoptimized method. However, we can also observe the main drawbacks of our optimization. First of all, these are blue and green spurious emissions on a diffusely reflecting surface, the absence of reflection of mirror balls directly below them, poor rendering of the figures behind a strongly scattering medium, and other image details resulting from the repeated interaction of the radiation with the medium. These drawbacks are direct consequences of the idea our method is based on. At each step of the calculations, we use the radiation source distribution function as an approximation to the solution from the previous step. Obviously, within this approach, the radiation that has repeatedly interacted with the medium will be accounted for less accurately, or even lost completely.

FIGURE 2. Comparison of standard and optimized methods

This analysis allows determining conditions of effectiveness for each method. The optimized sampling allows obtaining a sufficiently high-quality image of the corresponding areas in a short time when modeling diffusely reflected and scattered radiation. However, with intermediate terms of rendering times, if there is a strong repeatedly interacted radiation on the scene, the use of the optimized algorithm can lead to spurious emissions. In the limit, as the running time increases, both methods give similar results.

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