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SOME CALCULATIONS OF ORLICZ COHOMOLOGY AND POINCARÉ–SOBOLEV–ORLICZ INEQUALITIES

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ABSTRACT. We carry out calculations of Orlicz cohomology for some basic Riemannian manifolds (the real line, the hyperbolic plane, the ball). Relationship between Orlicz cohomology and Poincaré–Sobolev–Orlicz-type inequalities is discussed.

Keywords: differential form, Orlicz cohomology, torsion, Poincaré–Sobolev–Orlicz inequality

INTRODUCTION

The article continues the study of Orlicz cohomology of Riemannian manifolds initiated in [7, 8].

Orlicz cohomology is a natural generalization of L_{qp} -cohomology (for a detailed discussion of L_{qp} -cohomology, the reader is referred, for example, to [4]).

Like Orlicz function spaces, the Orlicz spaces L^Φ of differential forms are a natural nonlinear generalization of the spaces L^p . Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [6] and then by Agarwal, Ding, and Nolder in [1]. Orlicz forms on an arbitrary Riemannian manifold were apparently first examined by Kopylov and Panenko in [7].

In [4], Gol'dshtein and Troyanov demonstrated close relationship between L_{qp} -cohomology and Sobolev-type inequalities on Riemannian manifolds and, basing on this and some “almost duality” techniques, performed calculations of L_{qp} -cohomology for some basic manifolds. It turns out that, with some significant corrections and sometimes under additional constraints on the N -functions from which the Orlicz

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cohomology is constructed, these methods prove to be fruitful in computing Orlicz cohomology.

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we recall some basic information on abstract Banach complexes. Section 3 contains definitions concerning Orlicz spaces of differential forms on a Riemannian manifold, Orlicz cohomology, and its interpretation in terms of Poincaré–Sobolev–Orlicz inequalities (Theorems 3.3 and 3.4). Then we calculate the L_{Φ_1, Φ_2} -cohomology of $\mathbb{R}$ (Section 4) the hyperbolic plane (Section 5) and the L_Φ -cohomology of the ball (“ L^Φ -Poincaré inequality”, Section 6).

1. N -FUNCTIONS AND ORLICZ FUNCTION SPACES

Definition 1.1. A nonnegative function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is called an N -function if

- (i) Φ is even and convex;
- (ii) $\Phi(x) = 0 \iff x = 0$;
- (iii) $\lim_{x \rightarrow 0} \frac{\Phi(x)}{x} = 0$; $\lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty$.

An N -function Φ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [10, Theorem 1, p. 7]). The left derivative φ of Φ is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 : \varphi(t) > s\}, \quad s > 0,$$

is called the *left inverse* of φ .

The functions Φ, Ψ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t) dt, \quad \Psi(x) = \int_0^{|x|} \psi(t) dt$$

are called *complementary N -functions*.

The N -function Ψ complementary to an N -function Φ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

Throughout the article, given an N -function $\Phi : \mathbb{R} \rightarrow [0, \infty)$, we denote by Φ^{-1} its “positive” inverse $\Phi^{-1} : [0, \infty) \rightarrow [0, \infty)$.

N -functions are classified in accordance with their growth rates as follows:

Definition 1.2. An N -function Φ is said to satisfy the Δ_2 -condition (for all x), which is written as $\Phi \in \Delta_2$ if there exists a constant $K > 2$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$; Φ is said to satisfy the ∇_2 -condition (for all x), which is denoted symbolically as $\Phi \in \nabla_2$, if there is a constant $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for all $x \geq 0$.

It is not hard to see that an N -function Φ satisfies the ∇_2 -condition if and only if its dual N -function satisfies the Δ_2 -condition.

Henceforth, let Φ be an N -function and let (Ω, Σ, μ) be a measure space.

Definition 1.3. Given a measurable function $f : \Omega \rightarrow \mathbb{R}$, we put

$$\rho_\Phi(f) := \int_\Omega \Phi(f) d\mu.$$

Definition 1.4. The linear space

$$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \rho_\Phi(af) < \infty \text{ for some } a > 0\}$$

is called an *Orlicz space* on (Ω, Σ, μ) .

Let Ψ be the complementary N -function to Φ .

Below we as usual identify two functions equal outside a set of measure zero.

If $f \in L^\Phi$ then the functional $\|\cdot\|_\Phi$ (called *the Orlicz norm*) defined by

$$\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup \left\{ \left| \int_\Omega fg \, d\mu \right| : \rho_\Psi(g) \leq 1 \right\}$$

is a seminorm. It becomes a norm if μ satisfies the *finite subset property* (see [10, p. 59]): if $A \in \Sigma$ and $\mu(A) > 0$ then there exists $B \in \Sigma$, $B \subset A$, such that $0 < \mu(B) < \infty$.

The equivalent *gauge* (or *Luxemburg*) *norm* of a function $f \in L^\Phi$ is defined by the formula

$$\|f\|_{(\Phi)} = \|f\|_{L^{(\Phi)}(\Omega)} = \inf \left\{ K > 0 : \rho_\Phi\left(\frac{f}{K}\right) \leq 1 \right\}.$$

This is a norm without any constraint on the measure μ (see [10, p. 54, Theorem 3]).

2. BANACH COMPLEXES

Like in the case of $L_{q,p}$ -cohomology, treated in [4], we apply some abstract facts about Banach complexes to the Orlicz cohomology of Riemannian manifolds.

In this section, we recall some definitions and assertions about abstract Banach complexes given in [4].

Definition 2.1. A *Banach complex* is a sequence $F^* = \{F^k, d_k\}_{k \in \mathbb{N}}$ where F^k is a Banach space and $d = d^k : F^k \rightarrow F^{k+1}$ is a bounded operator with $d^{k+1} \circ d^k = 0$.

Definition 2.2. Given a Banach complex $\{F^k, d\}$, introduce the vector spaces:

- $Z^k := \ker(d : F^k \rightarrow F^{k+1})$ (a closed subspace of F^k);
- $B^k := \text{Im}(d : F^{k-1} \rightarrow F^k) \subset Z^k$;
- $H^k(F^*) := Z^k / B^k$ is the *cohomology* of the complex $F^* = \{F^k, d\}$;
- $\overline{H}^k(F^*) := Z^k / \overline{B}^k$ is the *reduced cohomology* of the complex F^* ;
- $T^k(F^*) := \overline{B}^k / B^k = H^k / \overline{H}^k$ is the *torsion* of the complex F^* .

As was observed in [4], the following easy assertion holds:

- (a) $\overline{H}^k, Z^k$ and $\overline{B}^k$ are Banach spaces;
- (b) The natural (quotient) topology on $T^k := \overline{B}^k / B^k$ is coarse (any closed set is either empty or T^k);
- (c) there is a natural exact sequence

$$0 \rightarrow T^k \rightarrow H^k \rightarrow \overline{H}^k \rightarrow 0.$$

Lemma 2.3. [4, Lemma 4.4] *For any Banach complex $\{F^k, d\}$, the following are equivalent:*

- (i) $T^k = 0$;
- (ii) $\dim T_k < \infty$;
- (iii) H^k is a Banach space;
- (iv) $B^k \subset F^k$ is closed.

Lemma 2.4. [4, Proposition 4.5] *The following are equivalent:*

- (i) $H^k = 0$;
- (ii) The operator $d_{k-1} : F^{k-1}/Z^{k-1} \rightarrow Z^k$ admits a bounded inverse d_{k-1}^{-1} ;
- (iii) There exists a constant C_k such that if for any $\theta \in Z^k$ there is an element $\eta \in F^{k-1}$ with $d\eta = \theta$ and

$$\|\eta\|_{F^{k-1}} \leq C_k \|\theta\|_{F^k}.$$

Lemma 2.5. [4, Propositions 4.6 and 4.7] *The following conditions (i) and (ii) are equivalent:*

- (i) $T^k = 0$.
- (ii) The operator $d_{k-1} : F^{k-1}/Z^{k-1} \rightarrow B^k$ admits a bounded inverse d_{k-1}^{-1} .

Any of these conditions implies

- (iii) There exists a constant C'_k such that for any $\xi \in F^{k-1}$ there is an element $\zeta \in Z^{k-1}$ such that

$$(2.1) \quad \|\xi - \zeta\|_{F^{k-1}} \leq C'_k \|d\xi\|_{F^k}.$$

Moreover, if F^{k-1} is a reflexive Banach space then conditions (i)-(iii) are equivalent.

3. ORLICZ SPACES OF DIFFERENTIAL FORMS AND ORLICZ COHOMOLOGY

Let X be a Riemannian manifold of dimension n . Given $x \in X$, denote by $(\omega(x), \theta(x))$ the scalar product of exterior k -forms $\omega(x)$ and $\theta(x)$ on $T_x X$. This gives a function $x \mapsto (\omega(x), \theta(x))$ on X .

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be two complementary N -functions. Given a measurable k -form ω , we put

$$\rho_\Phi(\omega) := \int_X \Phi(|\omega(x)|) d\mu_X.$$

Here $d\mu_X$ stands for the volume element of the Riemannian manifold X . We will identify k -forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold X , introduce the space $L^\Phi(X, \Lambda^k)$ as the class of all measurable k -forms ω satisfying the condition

$$\rho_\Phi(\alpha\omega) < \infty \text{ for some } \alpha > 0.$$

As in the case of Orlicz function spaces, the space $L^\Phi(X, \Lambda^k)$ is endowed with two equivalent norms: the *gauge norm*

$$\|\omega\|_{(\Phi)} = \inf \left\{ K > 0 : \rho_\Phi \left(\frac{\omega}{K} \right) \leq 1 \right\}.$$

and the *Orlicz norm* (Ψ is the complementary N -function to Φ):

$$\|\omega\|_\Phi = \sup \left\{ \left| \int_X (\omega(x), \theta(x)) d\mu_X \right| : \rho_\Psi(\theta) \leq 1 \right\}$$

As in the case of function spaces, it can be proved that $L^\Phi(X, \Lambda^k)$ endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a k -form ω is nothing but the gauge norm of its modulus function $|\omega|$. The same holds for the Orlicz norm ([7, Lemma 2.1]).

Unless otherwise specified, we endow the L^Φ spaces with the gauge norms; the quotient (semi)norm on each of the cohomology spaces to be defined below depends on the choice of the norms on L^{Φ_I} and $L^{\Phi_{II}}$ but the resulting topology does not.

Definition 3.1. A form $\theta \in L^{j+1}_{1,\text{loc}}(X)$ is called the (weak) differential $d\omega$ of $\omega \in L^j_{1,\text{loc}}(X)$ if

$$\int_U \omega \wedge du = (-1)^{j+1} \int_U \theta \wedge u$$

for every orientable domain $U \subset \text{Int } X$ and every form $u \in D^{n-j-1}(X)$ having support in U .

Let Φ_I and Φ_{II} be N -functions. For $0 \leq k \leq n$, put

$$\Omega^k_{\Phi_I, \Phi_{II}}(X) = \{\omega \in L^{\Phi_I}(X, \Lambda^k) : d\omega \in L^{\Phi_{II}}(X, \Lambda^{k+1})\}.$$

This is a Banach space with the norm

$$\|\omega\|_{(\Phi_I), (\Phi_{II})} = \|\omega\|_{(\Phi_I)} + \|d\omega\|_{(\Phi_{II})}.$$

Consider also the spaces

$$\begin{aligned} Z^k_{\Phi_{II}}(X) &= \{\omega \in L^{\Phi_{II}}(X, \Lambda^k) : d\omega = 0\}; \\ B^k_{\Phi_I, \Phi_{II}}(X) &= \{\omega \in L^{\Phi_{II}}(X, \Lambda^k) : \omega = d\beta \text{ for some } \beta \in L^{\Phi_I}(X, \Lambda^{k-1})\}. \end{aligned}$$

Denote by $\overline{B}^k_{\Phi_I, \Phi_{II}}(X)$ the closure of $B^k_{\Phi_I, \Phi_{II}}(X)$ in $L^{\Phi_{II}}(X, \Lambda^k)$.

Definition 3.2. The quotient spaces

$$H^k_{\Phi_I, \Phi_{II}}(X) := Z^k_{\Phi_{II}}(X) / B^k_{\Phi_I, \Phi_{II}}(X)$$

and

$$\overline{H}^k_{\Phi_I, \Phi_{II}}(X) := Z^k_{\Phi_{II}}(X) / \overline{B}^k_{\Phi_I, \Phi_{II}}(X)$$

are called the k th $L_{\Phi_I, \Phi_{II}}$ -cohomology and the k th reduced $L_{\Phi_I, \Phi_{II}}$ -cohomology of the Riemannian manifold X , the latter cohomology being a Banach space. Define the $L_{\Phi_I, \Phi_{II}}$ -torsion as

$$T^k_{\Phi_I, \Phi_{II}}(X) := \overline{B}^k_{\Phi_I, \Phi_{II}}(X) / B^k_{\Phi_I, \Phi_{II}}(X).$$

The torsion $T^k_{\Phi_I, \Phi_{II}}(X)$ can be either $\{0\}$ or infinite-dimensional. In fact, if $\dim T^k_{\Phi_I, \Phi_{II}}(X) < \infty$ then $B^k_{\Phi_I, \Phi_{II}}(X)$ is closed, hence $T^k_{\Phi_I, \Phi_{II}}(X) = \{0\}$. In particular, if $\dim T^k_{\Phi_I, \Phi_{II}}(X) \neq 0$ then $\dim H^k_{\Phi_I, \Phi_{II}}(X) = \infty$.

If $\Phi_I = \Phi_{II} = \Phi$ then we use the notations $\Omega^k_\Phi(X)$, $H^k_\Phi(X)$, and $\overline{H}^k_\Phi(X)$ instead of $\Omega^k_{\Phi, \Phi}(X)$, $H^k_{\Phi, \Phi}(X)$, and $\overline{H}^k_{\Phi, \Phi}(X)$ respectively. Thus, the L_Φ -cohomology $H^k_\Phi(X)$ (respectively, the reduced L_Φ -cohomology $\overline{H}^k_\Phi(X)$) is the k th cohomology (respectively, the k th reduced cohomology) of the cochain complex $\{\Omega^*_\Phi(X), d\}$.

In [4], Gol'dshtein and Troyanov realized the k th $L_{q,p}$ -cohomology as the k th cohomology of some Banach complex. Here we apply this approach to $L_{\Phi_I, \Phi_{II}}$ -cohomology.

Fix an $(n+1)$ -tuple of N -functions $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ and put

$$\Omega^k_{\mathcal{F}}(X) = \Omega^k_{\Phi_k, \Phi_{k+1}}(X);$$

Since the weak exterior differential is a bounded operator $d : \Omega_{\mathcal{F}}^k(X) \rightarrow \Omega_{\mathcal{F}}^{k+1}(X)$, we obtain a Banach complex

$$0 \rightarrow \Omega_{\mathcal{F}}^0(X) \rightarrow \Omega_{\mathcal{F}}^1(X) \rightarrow \cdots \rightarrow \Omega_{\mathcal{F}}^k(X) \rightarrow \cdots \rightarrow \Omega_{\mathcal{F}}^n(X) \rightarrow 0.$$

The $L_{\mathcal{F}}$ -cohomology $H_{\mathcal{F}}^k(X)$ (respectively, the reduced $L_{\mathcal{F}}$ -cohomology $\overline{H}_{\mathcal{F}}^k(X)$) of X is the k th cohomology (respectively, the k th reduced cohomology) of the Banach complex $(\Omega_{\mathcal{F}}^*, d)$.

The above-defined cohomology spaces $H_{\mathcal{F}}^k(X)$ and $\overline{H}_{\mathcal{F}}^k(X)$ in fact depend only on Φ_{k-1} and Φ_k :

$$\begin{aligned} H_{\mathcal{F}}^k(X) &= H_{\Phi_{k-1}, \Phi_k}^k(X) = Z_{\Phi_k}^k(X) / B_{\Phi_{k-1}, \Phi_k}^k; \\ \overline{H}_{\mathcal{F}}^k(X) &= \overline{H}_{\Phi_{k-1}, \Phi_k}^k(X) = Z_{\Phi_k}^k(X) / \overline{B}_{\Phi_{k-1}, \Phi_k}^k. \end{aligned}$$

The results on abstract Banach complexes by Gol'dshstein and Troyanov enable us to interpret Orlicz cohomology in terms of a Poincaré–Sobolev–Orlicz type inequality for differential forms on a Riemannian manifold X :

Theorem 3.3. $H_{\Phi_I, \Phi_{II}}^k(X) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed differential form $\omega \in L^{\Phi_{II}}(X, \Lambda^k)$ there exists a differential form $\theta \in L^{\Phi_I}(X, \Lambda^{k-1})$ such that $d\theta = \omega$ and

$$\|\theta\|_{L(\Phi_I)} \leq C \|\omega\|_{L(\Phi_{II})}.$$

This result is an immediate consequence of Lemma 2.4.

Theorem 3.4. (A) If $T_{\Phi_{II}}^k(X) = 0$ then there exists a constant C' such that for any differential form $\theta \in \Omega_{\Phi_I, \Phi_{II}}^{k-1}(X)$ there exists a closed form $\zeta \in Z_{\Phi_I}^{k-1}(X)$ such that

$$(3.1) \quad \|\theta - \zeta\|_{L(\Phi_I)} \leq C' \|d\theta\|_{L(\Phi_{II})}.$$

(B) Conversely, if $\Phi_{II} \in \Delta_2 \cap \nabla_2$ and there exists a constant C' such that for any form $\theta \in \Omega_{\Phi_I, \Phi_{II}}^{k-1}(X)$ there exists $\zeta \in Z_{\Phi_I}^{k-1}(X)$ such that (3.1) holds then $T_{\Phi_I, \Phi_{II}}(X) = 0$.

Proof. Considering the Banach complex $\Omega_{\mathcal{F}}^*$ with $\mathcal{F} = \{\Phi_{II}, \dots, \Phi_{II}, \Phi_I, \dots, \Phi_I\}$, where Φ_{II} changes to Φ_I at the k th position, we get

$$H_{\mathcal{F}}^k(X) = H_{\Phi_I, \Phi_{II}}^k(X); \quad \overline{H}_{\mathcal{F}}^k(X) = \overline{H}_{\Phi_I, \Phi_{II}}^k(X).$$

Since $\Phi_I \in \Delta_2 \cap \nabla_2$, the Banach space $\Omega_{\Phi_{II}, \Phi_{II}}^{k-1}(X)$ is reflexive. Theorem 3.4 now stems from Lemma 2.5. $\square$

4. THE L_{Φ_1, Φ_2} -COHOMOLOGY OF $\mathbb{R}$

Let Φ_1 and Φ_2 be N -functions.

Proposition 4.1. $T_{\Phi_1, \Phi_2}^1(\mathbb{R}) \neq 0$.

Proof. Suppose on the contrary that $T_{\Phi_1, \Phi_2}^1(\mathbb{R}) = 0$. In accordance with Theorem 3.4, then there is a Sobolev inequality for functions on $\mathbb{R}$

$$(4.1) \quad \inf_{z \in \mathbb{R}} \|f - z\|_{(\Phi_1)} \leq C \|f'\|_{(\Phi_2)}$$

for some real positive constant C .

Consider the function

$$\theta(x) = \omega_{1/2}(x) = \begin{cases} Ce^{-\frac{1}{1-4x^2}} & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| > 1/2. \end{cases}$$

Here the constant C is chosen so that

$$\int_{-\infty}^{\infty} \theta(x) dx = \frac{c}{2} \int_{-1}^1 e^{-\frac{1}{1-t^2}} dt = 1.$$

Now, consider the family of smooth functions with compact support $\{f_a : \mathbb{R} \rightarrow \mathbb{R} : a > 1\}$, where

$$f_a(x) = \int_{-\infty}^x \left(\theta\left(x + \frac{3}{2}\right) + \theta\left(-x + a + \frac{1}{2}\right) \right) dx$$

(we owe this construction to [2, pp. 8-9]). Then $f_a(x) = 1$ if $x \in [1, a]$, $f_a(x) = 0$ if $x \notin [0, a+1]$, and $\|f'_a\|_{L^\infty} =: L < \infty$. Clearly, $\|f_a - z\|_{(\Phi_1)}$ is finite only for $z = 0$. Estimate the Orlicz norms involved in (4.1). We have

$$\rho_{\Phi_1}\left(\frac{f_a}{K}\right) = \int_{-\infty}^{\infty} \Phi_1\left(\frac{f_a(x)}{K}\right) dx \geq \int_1^a \Phi_1\left(\frac{1}{K}\right) dx = (a-1)\Phi_1\left(\frac{1}{K}\right).$$

If $\rho_{\Phi_1}\left(\frac{f_a}{K}\right) \leq 1$ then $(a-1)\Phi_1\left(\frac{1}{K}\right) \leq 1$, which is equivalent to

$$K \geq \frac{1}{\Phi_1^{-1}\left(\frac{1}{a-1}\right)}.$$

Hence,

$$\|f_a\|_{(\Phi)} = \inf \left\{ K : \rho_{\Phi_1}\left(\frac{f_a}{K}\right) \leq 1 \right\} \geq \frac{1}{\Phi_1^{-1}\left(\frac{1}{a-1}\right)}.$$

On the other hand,

$$\begin{aligned} \rho_{\Phi_2}\left(\frac{f'_a}{K}\right) &= \int_{-\infty}^{\infty} \Phi_2\left(\frac{f'_a(x)}{K}\right) dx \\ &= \int_0^1 \Phi_2\left(\frac{f'_a(x)}{K}\right) dx + \int_a^{a+1} \Phi_2\left(\frac{f'_a(x)}{K}\right) dx \leq 2\Phi_2\left(\frac{L}{K}\right). \end{aligned}$$

We have

$$2\Phi_2\left(\frac{L}{K}\right) \leq 1 \iff K \geq \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}.$$

Put $\mathcal{M}_{f',a} = \left\{ K : \rho_{\Phi_2}\left(\frac{f'_a}{K}\right) \leq 1 \right\}$. We have shown that if $K \geq \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}$ then $K \in \mathcal{M}_{f',a}$. Therefore,

$$\|f'_a\|_{(\Phi_2)} = \inf \mathcal{M}_{f',a} \leq \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}.$$

Thus,

$$C \geq \frac{\Phi_2^{-1}\left(\frac{1}{2}\right)}{L\Phi_1^{-1}\left(\frac{1}{a-1}\right)} \rightarrow \infty \quad \text{as } a \rightarrow \infty.$$

The obtained contradiction proves the proposition. $\square$

Corollary 4.2. *If Φ_1 and Φ_2 are N -functions then the space $H_{\Phi_1, \Phi_2}^1(\mathbb{R})$ is not separated; in particular, $H_{\Phi_1, \Phi_2}^1(\mathbb{R}) \neq 0$.*

Proposition 4.3. *If Φ_1 and Φ_2 are N -functions and $\Phi_2 \in \Delta_2$ then $\overline{H}_{\Phi_1, \Phi_2}^1(\mathbb{R}) = 0$.*

Proof. Let $\omega = a(x)dx \in L^{\Phi_2}(\mathbb{R})$. For each n , put

$$C_m = \int_{-m}^m a(x) dx.$$

If $C_m = 0$ then put $\lambda_m(x) \equiv 0$ for all $x \in \mathbb{R}$. If $C_m \neq 0$ then put

$$\lambda_m(x) = \text{sign } C_m \varepsilon_m \chi \left[-\frac{|C_m|}{2\varepsilon_m}, \frac{|C_m|}{2\varepsilon_m} \right],$$

where $\varepsilon_m = t_m/m$ and t_m is the only root of the equation

$$\frac{\Phi_2(t_m)}{t_m} = \frac{1}{m|C_m|}.$$

(The function $t \mapsto \Phi_2(t)/t$ is strictly increasing; see, for example, [9]). We obviously have

$$\int_{\mathbb{R}} \lambda_m(x) dx = C_m = \int_{-m}^m a(x) dx.$$

Compute the norm $\|\lambda_m\|_{(\Phi_2)}$. We have

$$\rho_{\Phi_2} \left(\frac{\lambda_m}{K} \right) = \int_{-|C_m|/2\varepsilon_m}^{|C_m|/2\varepsilon_m} \Phi_2 \left(\frac{\varepsilon_m}{K} \right) dx = \frac{|C_m|}{\varepsilon_m} \Phi_2 \left(\frac{\varepsilon_m}{K} \right).$$

Thus,

$$\begin{aligned} \rho_{\Phi_2} \left(\frac{\lambda_m}{K} \right) \leq 1 &\iff \frac{|C_m|}{\varepsilon_m} \Phi_2 \left(\frac{\varepsilon_m}{K} \right) \leq 1 \iff \Phi_2 \left(\frac{\varepsilon_m}{K} \right) \leq \frac{\varepsilon_m}{|C_m|} \\ &\iff \frac{\varepsilon_m}{K} \leq \Phi_2^{-1} \left(\frac{\varepsilon_m}{|C_m|} \right) \iff K \geq \frac{\varepsilon_m}{\Phi_2^{-1} \left(\frac{\varepsilon_m}{|C_m|} \right)}. \end{aligned}$$

Here Φ_2^{-1} stands for the positive inverse function to $\Phi_2 : [0, \infty) \rightarrow [0, \infty)$. Hence, $\|\lambda_m\|_{(\Phi_2)} = \frac{\varepsilon_m}{\Phi_2^{-1}(\varepsilon_m/|C_m|)}$. By the choice of ε_m ,

$$\frac{\Phi_2(m\varepsilon_m)}{m\varepsilon_m} = \frac{1}{m|C_m|},$$

and so $\|\lambda_m\|_{(\Phi_2)} = \frac{1}{m}$.

Let $b_m(x) := \int_{-\infty}^x (\chi_{[-m, m]}(t)a(t) - \lambda_m(t)) dt$. Since b_m has compact support, $b_m \in L^{\Phi_1}(\mathbb{R})$ for each m . Furthermore, $\|db_m - \omega\|_{(\Phi_2)} \leq \|a\|_{L^{\Phi_2}(\mathbb{R} \setminus [-m, m])} + \|\lambda_m\|_{L^{\Phi_2}(\mathbb{R})} \rightarrow 0$ as $m \rightarrow \infty$ since for $\Phi_2 \in \Delta_2$ all functions in L^{Φ_2} have absolutely continuous norm ([9, Theorem 10.3]). Thus, $\overline{H}_{\Phi_1, \Phi_2}^1(\mathbb{R}) = 0$. $\square$

All the results of this section are also valid for the half-line $\mathbb{R}_+$ (with similar proofs).

5. THE L_{Φ_1, Φ_2} -COHOMOLOGY OF THE HYPERBOLIC PLANE

We will need the following Orlicz versions of Propositions 8.3 and 8.4 in [4], which are proved in absolutely the same manner:

Proposition 5.1. *Let M be a complete manifold of dimension n and let (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z_{\Phi_2}^k(X)$ and there exists a smooth closed $(n-k)$ -form γ such that $\gamma \in Z_{\Psi_1}^{n-k}(X)$, $\gamma \wedge \alpha \in L^1(X, \Lambda^n)$, and*

$$\int_M \gamma \wedge \alpha \neq 0,$$

then $\alpha \notin B_{\Phi_1, \Phi_2}^k(X)$. In particular, $H_{\Phi_1, \Phi_2}^k(X) \neq 0$.

Proposition 5.2. *Let M be a complete manifold of dimension n and let (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z_{\Phi_2}^k(X)$ and there exists a smooth closed $(n-k)$ -form $\gamma \in Z_{\Psi_1}^{n-k}(X) \cap Z_{\Psi_2}^{n-k}(X)$ such that*

$$\int_M \gamma \wedge \alpha \neq 0,$$

then $\alpha \notin \overline{B}_{\Phi_1, \Phi_2}^k(X)$. In particular, $\overline{H}_{\Phi_1, \Phi_2}^k(X) \neq 0$.

The hyperbolic plane $\mathbb{H}^2$ is the Riemannian manifold that can be modelled as the space $\mathbb{R}^2$ endowed with the Riemannian metric

$$ds^2 = e^{2z} dy^2 + dz^2.$$

For an N -function Φ , introduce the condition

$$\int_0^1 \frac{\Phi(v)}{v^2} dv < \infty. \quad (A)$$

(The upper integration limit 1 can be replaced by any positive number.)

Theorem 5.3. *If Φ_1 and Φ_2 are N -functions such that their complementary N -functions Ψ_1 and Ψ_2 and the function Φ_2 satisfy condition (A) then*

$$\dim(\overline{H}_{\Phi_1, \Phi_2}^1(\mathbb{H}^2)) = \infty.$$

We will need the following lemma, which is in fact Lemma 10.2 in [4]:

Lemma 5.4. *There exist two smooth functions f and g on $\mathbb{H}^2$ such that*

- (1) f and g are nonnegative;
- (2) $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;
- (3) df and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;
- (4) the support of $df \wedge dg$ is contained in $\{(y, z) : |y| \leq 1, 0 \leq z \leq 1\}$;
- (5) $df \wedge dg \geq 0$;
- (6) $\int_{\mathbb{H}^2} df \wedge dg = 1$;
- (7) $\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial y} \in L^\infty(\mathbb{H}^2)$;
- (8) $\frac{\partial f}{\partial z}$ and $\frac{\partial g}{\partial z}$ have compact support.

We will also need the following generalization of item (3) above:

Lemma 5.5. *If Φ is an N -function satisfying condition (A) then $df, dg \in L^\Phi(\mathbb{H}^2, \Lambda^1)$.*

Proof. Recall the construction of the functions f and g of [4, Lemma 10.2].

Choose smooth functions h_1 , h_2 , and $k : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- (1) h_1 , h_2 , and k are nonnegative;
- (2) $h_i(y) = 0$ if $|y| \geq 1$;
- (3) $h'_1(y)h_2(y) \geq 0$ and $h_1(y)h'_2(y) \leq 0$ for all y ;
- (4) the support of the function $h'_1(y)h_2(y) - h_1(y)h'_2(y)$ is not empty;
- (5) $k'(z) \geq 0$ for all z ;
- (6) $k(z) = 1$ if $z \geq 1$ and $k(z) = 0$ if $z \leq 0$.

Then f and g are defined as $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$ respectively.

We will now prove that $df \in L^\Phi$ by modifying the argument of the proof of [4, Lemma 10.2].

Indeed,

$$df = h_1(y)k'(z)dz + k(z)h'_1(y)dy.$$

The first summand $h_1(y)k'(z)dz$ has compact support, and the second summand $k(z)h'_1(y)dy$ is zero outside the infinite rectangle $Q = \{|y| \leq 1; z \geq 0\}$.

Choose $D < \infty$ such that $|k(z)h'_1(y)| \leq D$ on Q . We have

$$|k(z)h'_1(y)dy| \leq D|dy| = De^{-z}.$$

Since the area element of $\mathbb{H}^2$ is $dA = e^z dy dz$, for any $a > 0$ we infer

$$\int_{\mathbb{H}^2} \Phi(a|k(z)h'_1(y)dy|)dA \leq \int_Q \Phi(aDe^{-z})e^z dy dz = 2aD \int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}} dz.$$

Putting $aDe^{-z} = v$ in the last integral, we get

$$2aD \int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}} dz = 2aD \int_0^{aD} \frac{\Phi(v)}{v^2} dv < \infty.$$

Thus, $\rho_\Phi(ak(z)h'_1(y)dy) < \infty$ for any a . Consequently, $k(z)h'_1v \in L^\Phi(\mathbb{H}^2, \Lambda^1)$. Thus, $df = h_1(y)k'(z)dz + k(z)h'_1(y)dy$ also lies in $L^\Phi(\mathbb{H}^2, \Lambda^k)$.

The lemma is proved. $\square$

Proof of Theorem 5.3. Take the functions f and g on $\mathbb{H}^2$ defined in Lemma 5.4 and consider the 1-forms $\alpha = df$ and $\gamma = dg$ on $\mathbb{H}^2$. Obviously, $d\alpha = d\gamma = 0$. By Lemmas 5.4 and 5.5, $\alpha \in L^\Phi$ for any N -function Φ such that $\int_0^1 \Phi(v)/v^2 dv < \infty$ and γ is smooth and $\gamma \in L^{\Psi_1} \cap L^{\Psi_2}$ if $\int_0^1 \Psi_1(v)/v^2 dv < \infty$ and $\int_0^1 \Psi_2(v)/v^2 dv < \infty$.

Since $\int_{\mathbb{H}^2} \alpha \wedge \gamma \neq 0$, Proposition 5.1 shows that $\alpha \notin \overline{B}_{\Phi_1, \Phi_2}^1(\mathbb{H}^2)$.

Now, using the isometry group of $\mathbb{H}^2$, we obtain an infinite family of linearly independent classes in $\overline{H}_{\Phi_1, \Phi_2}^1(\mathbb{H}^2)$. $\square$

6. THE L_Φ -COHOMOLOGY OF THE BALL

In this section, we prove the “ L^Φ -Poincaré lemma”, i.e., the vanishing of the L^Φ -cohomology of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$.

Since $\mathbb{B}^n$ has finite volume, $H_{\Phi_1, \Phi_2}^0(\mathbb{B}^n) = \overline{H}_{\Phi_1, \Phi_2}^0(\mathbb{B}^n) = \mathbb{R}$ for any N -functions Φ_1 and Φ_2 .

For the case of L^p spaces, Gol'dshtein, Kuz'minov, and Shvedov proved the vanishing of the L^p -cohomology of the ball in [3, Lemma 3.2]; for $p \neq q$, Gol'dshtein and Troyanov found necessary and sufficient conditions on p and q for the vanishing

of the $L^{q,p}$ -cohomology of $\mathbb{B}^n$. Their proof is based on the following fact, established by Iwaniec and Lutoborski in [5]:

Proposition 6.1. *For any bounded convex domain $U \subset \mathbb{R}^n$ and any $k = 1, 2, \dots, n$, there exists an operator*

$$T = T_U : L^1_{loc}(U, \Lambda^k) \rightarrow L^1_{loc}(U, \Lambda^{k-1})$$

with the following properties:

- (a) $T(d\theta) + dT\theta = \theta$ (in the sense of currents);
- (b) $|T\theta(x)| \leq C \int_U \frac{|\theta(y)|}{|y-x|^{n-1}} dy$.

□

We prove

Corollary 6.2. *If Φ is an N -function then the operator T maps $L^\Phi(U, \Lambda^k)$ continuously into $L^\Phi(U, \Lambda^{k-1})$.*

Proof. The following Orlicz space version of Young's inequality for convolution holds (see the proof of Corollary 7 in [10, pp. 230–231]¹: If $f \in L^\Phi$ and $g \in L^1$ then $f * g \in L^\Phi$ and

$$\|f * g\|_\Phi \leq \|f\|_\Phi \|g\|_1.$$

Applying this inequality to $f = |\theta|$ and $g(x) = |x|^{1-n}$, we obtain the corollary from Proposition 6.1. In the Orlicz norms, the norm of the operator T is bounded by $\|g\|_1$. □

Corollary 6.3. *The operator $T : \Omega_\Phi(U, \Lambda^k) \rightarrow \Omega_\Phi(U, \Lambda^{k-1})$ is bounded and $Td\omega + dT\omega = \omega$ for any $\omega \in \Omega_\Phi^k(U)$.*

Corollary 6.3 gives the following

Theorem 6.4. *If Φ is an N -function then $H_\Phi^k(\mathbb{B}^n) = 0$ for all $k = 1, \dots, n$.*

Proof. Let $\omega \in Z_\Phi^k(\mathbb{B}^n)$. By Corollary 6.3, $T\omega \in L^\Phi(\mathbb{B}^n, \Lambda^{k+1})$. Since $\omega = dT\omega + Td\omega = d(T\omega)$, we conclude that $[\omega] = [dT\omega] = 0 \in H_\Phi^k(\mathbb{B}^n)$ and so $H_\Phi^k(\mathbb{B}^n) = 0$. □

Remark 6.5. The Sobolev space analog of Theorem 6.4, Theorem 11.5 in [4] (see also [4, Proposition 11.4]) gives the following criterion: for $1 < p, q \leq \infty$, the $L_{q,p}$ -cohomology spaces $H_{q,p}^k(\mathbb{B}^n)$, $k = 1, 2, \dots, n$ are trivial if and only if $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. It would be interesting to obtain a criterion like [4, Theorem 11.5] for the Orlicz cohomology $H_{\Phi_1, \Phi_2}^k(\mathbb{B}^n)$ for different functions Φ_1 and Φ_2 .

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¹Though it is required in Corollary 7 in [10, pp. 230–231] that $\Phi \in \Delta_2$, the proof of Young's inequality works for general N -functions Φ .

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