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UNIQUE DETERMINATION OF CONFORMAL TYPE
FOR DOMAINS. II

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ABSTRACT. The article is the second part of a review series entitled “Unique determination of conformal type for domains,” initiated by the author’s eponymous paper, published in *Sib. Elektron. Mat. Izv.*, **16**, 692–708 (2019). The main result of the present article is that any convex bounded polyhedral domain in the three-dimensional Euclidean space is uniquely determined by the relative conformal moduli of its boundary condensers.

Keywords: p -modulus of a family of paths, boundary condenser, quasiconformal mapping, conformal mapping, isometric mapping, unique determination.

1. INTRODUCTION

In [1], we observed that the main result of [1] is Theorem 1.1, discussed in detail in [2]–[4]. In accordance with this theorem, *if $n \geq 4$ then any bounded convex polyhedral domain $U \subset \mathbb{R}^n$ (i.e., a nonempty bounded intersection of finitely many open n -dimensional half-spaces) is uniquely determined by the relative conformal moduli of its boundary condensers in the class $\mathcal{P} = \mathcal{P}(n)$ of all bounded convex polyhedral domains $V \subset \mathbb{R}^n$* . In this article, we keep to the notations and use the notions and assertions of [1]. For this reason, we refer the reader for them to [1].

In the present article, we fully expose an analogous result for the case of bounded convex domains in the three-dimensional Euclidean space. Moreover, we consider

KOPYLOV, A.P., UNIQUE DETERMINATION OF CONFORMAL TYPE FOR DOMAINS.

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the question of unique determination of isometric type for domains in $\mathbb{R}^3$ on the basis of the notion of the p -moduli of a family of paths.

2. UNIQUE DETERMINATION OF THREE-DIMENSIONAL CONVEX POLYHEDRAL DOMAINS BY THE RELATIVE CONFORMAL MODULI OF BOUNDARY CONDENSERS

In this section, we prove an analog of [1, Theorem 1.1] in the case of $n = 3$ (Theorem 2.1 below), which was announced in [5] and becomes possible due to a criterion obtained by V. V. Aseev (in [6]) for the equality of convex polyhedral angles in the case of $n = 3$ (see Lemma 2.1 below).

Let us list some features of the case $n = 3$. In the proof of [1, Theorem 1.1] (see, for example, [3, Theorem 8.1]), one of the most important points is to prove that if $f : \text{fr } V \rightarrow \text{fr } U$ is a homeomorphism preserving the relative conformal moduli of boundary condensers, then on every $(n - 1)$ -dimensional face s of $\text{fr } V$ there exists a connected subset σ open with respect to s such that $f(\sigma)$ is an open subset in some $(n - 1)$ -dimensional face of $\text{fr } U$; moreover, $f|_\sigma$ is an $(n - 1)$ -dimensional conformal mapping. Another important point is to prove the identity $\sigma = \text{Int } s$. Here, an essential role is played by the following property of spatial conformal mappings: any conformal mapping $g : \lambda \rightarrow \mathbb{R}^m$ of a domain λ in $\mathbb{R}^m$, $m \geq 3$, is the restriction $G|_\lambda$ of some Möbius transformation $G : \bar{\mathbb{R}}^m \rightarrow \bar{\mathbb{R}}^m$ (see, for example, [7]). However, in the case $m = 2$ ($n = 3$) we need to use plane conformal mappings whose properties are cardinally different from those of spatial conformal mappings.

So, our main aim (in this section) is to prove the following assertion:

Theorem 2.1. *Let $n = 3$. Then every bounded convex polyhedral domain $U \subset \mathbb{R}^3$ is uniquely determined by the relative conformal moduli of its boundary condensers in the class $\mathcal{P}$ of all bounded convex polyhedral domains $V \subset \mathbb{R}^3$. Furthermore, U is determined in $\mathcal{P}$ up to an additional similarity transformation $P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.*

The proof of Theorem 2.1 consists of several steps.

The first step is the following assertion:

Theorem 2.2. *Let a homeomorphism $f : \text{fr } U \rightarrow \text{fr } V$ of the boundaries of bounded convex polyhedral domains U and V in $\mathbb{R}^3$ satisfy the following conditions:*

- (1) *f and f^{-1} send edges to edges;*
- (2) *the interior dihedral angles between pairs of adjacent faces remain equal under the map f ;*
- (3) *f is a plane conformal mapping on each face of U .*

Then the domains U and V are similar and f extends to a similarity transformation of $\mathbb{R}^3$.

For proving Theorem 2.2, we need some auxiliary assertions. Let W be a given convex solid polyhedral angle in $\mathbb{R}^3$ with vertex O and edges $w_1, \dots, w_m$ enumerated in the order of circumvention around O along the exterior boundary $\text{fr } W$ in the positive direction (counterclockwise). The face with sides w_j, w_{j+1} is denoted by $\sigma_{j,j+1}$ (the operations in the subscripts are understood modulo m). Following [8], we construct the polar polyhedral angle W_* with vertex O whose edges are rays directed along the outward normals to the faces $\sigma_{j,j+1}$ ($j = 1, \dots, m$). We note that W_* is also a convex polyhedral angle (see [8]).

Lemma 2.1. Assume that for convex polyhedral angles $W, W' \subset \mathbb{R}^3$ there exists a bijection f between the sets of all edges of W and W' preserving the order of circumvention of edges around the vertex along the exterior boundaries of these angles. Let α_j be the angle between the edges w_j and w_{j+1} ($j = 1, \dots, m$), and let β_j be the angle between the edges $f(w_j)$ and $f(w_{j+1})$. If

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \dots = \frac{\alpha_m}{\beta_m}$$

and the interior dihedral angles at the edges w_j and $f(w_j)$ are equal, then W and W' are isometric in $\mathbb{R}^3$.

Доказательство. For the polyhedral angles W and W' we construct the polar convex polyhedral angles W_* and W'_* . A bijection f between the edges of W and W_* generates a bijection between the edges of W' and W'_* . By duality, in W' and W'_* , the plane angles in the corresponding faces are equal, and, for the differences of interior dihedral angles $\pi - \alpha_j$ and $\pi - \beta_j$ at the corresponding edges, we have $(\pi - \alpha_j) - (\pi - \beta_j) = (t - 1)\alpha_j$, where $t = \alpha_j/\beta_j$ is independent of j . Assume that $t \neq 1$. Then these differences must change the sign at least four times in accordance with [9, Lemma 2a]. However, in our situation, they preserve the sign for all j . The so-obtained contradiction shows that $t = 1$, and hence $\alpha_j = \beta_j$ for all j . Hence, in the convex polyhedral angles W and W' , the corresponding plane and bihedral angles are equal. By the Cauchy lemma (see, for example, [9]), these polyhedral angles are equal. $\square$

Lemma 2.2. Assume that plane bounded polygons P (with vertices $p_1, \dots, p_k$) and Q (with vertices $q_1, \dots, q_k$) are such that their interior angles $\pi\delta_j$ at the corresponding vertices p_j and q_j are equal. If a conformal mapping $f : P \rightarrow Q$ takes p_j to q_j , $j = 1, \dots, k$, then f is a similarity transformation.

Доказательство. By the Schwarz–Christoffel theorem (see, for example, [10]), there is a collection of finite points $a_1, \dots, a_k$ on the real axis and a conformal mapping g_1 from the upper half-plane $\mathbb{C}_+$ to the polygon P such that $g_1(a_j) = p_j$. For $z_0 \in \mathbb{C}_+$ this mapping can be defined by

$$g_1(z) = C_1 \int_{z_0}^z (z - a_1)^{\delta_1 - 1} \cdot \dots \cdot (z - a_k)^{\delta_k - 1} dz + C_2,$$

where for every factor $(z - a_j)^{\delta_j - 1} = \exp\{(\delta_j - 1) \operatorname{Log}(z - a_j)\}$ the single branch of the function $\operatorname{Log}(z - a_j)$ in the upper half-plane $\mathbb{C}_+$ is defined by the condition $0 \leq \operatorname{Im}(\operatorname{Log}(z - a_j)) \leq \pi$. Moreover, the constants C_1 and C_2 are uniquely determined by the point z_0 . Applying the Schwarz–Christoffel theorem to an (analogous) conformal mapping $g_2 : \mathbb{C}_+ \rightarrow Q$ and taking into account that $f(g_1(a_j)) = q_j$, we get

$$\zeta = g_2(z) = f(g_1(z)) = C_1^* \int_{z_0}^z (z - a_1)^{\delta_1 - 1} \cdot \dots \cdot (z - a_k)^{\delta_k - 1} dz + C_2^*.$$

Consequently, for any $w \in P$ we have the equality $(\zeta - C_2^*)/C_1^* = (w - C_2)/C_1$ which means that f is a similarity transformation: $\zeta = C_2^* + (C_1^*/C_1)(w - C_2)$. $\square$

Lemma 2.3. If, under the assumptions of Theorem 2.2, f is a similarity transformation of an open subset u of a face σ of the domain U , then f is extended to a similarity transformation of the whole space $\mathbb{R}^3$.

Доказательство. Let $f|_u$ coincide with the restriction $F|_u$ of some similarity transformation $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then the composition $g = F^{-1} \circ f$ satisfies all the assumptions of Theorem 2.2 for convex polyhedral domains U and $F(V)$ and is identical on the open set u . By the interior uniqueness theorem for analytic functions, the mapping g is identical on the whole face σ . Let the faces σ_1 and σ have a common edge L . In the plane, we consider the involute Σ of $\text{fr } U$, where the polygons σ and σ_1 are glued together along their common side L , and the corresponding involute Σ' of $\text{fr } F(V)$, where the polygons $F(\sigma') = g(\sigma)$ and $F(\sigma'_1) = g(\sigma_1)$ are glued together along their common side $L' = g(L)$ (see [9, Chapter 1]). The orientation preserving homeomorphism g from the plane domain $\sigma \cup L \cup \sigma_1$ to the plane domain $F(\sigma') \cup L' \cup F(\sigma'_1)$ is a conformal mapping on each of the faces σ and σ_1 . By the continuous extension principle (see [10, Theorem 1]), g is conformal in the entire domain $\sigma \cup L \cup \sigma_1$ and, since it is identical on σ , it is identical everywhere in $\sigma \cup L \cup \sigma_1$, in particular, on σ_1 . Thus, passing from face to face, we verify that the homeomorphism g on each face of the domain U is the identity mapping. Hence the mapping $g : \text{fr } U \rightarrow \text{fr } F(V)$ is an isometry on each face. By the known result (see [8, Chapter 3]), the mapping g is extended to an isometry $\tilde{g}$ of the whole space $\mathbb{R}^3$. Therefore, the similarity transformation $\tilde{f} = F^{-1}(\tilde{g})$ serves as an extension of the homeomorphism f to the whole space $\mathbb{R}^3$. $\square$

Proof of Theorem 2.2. In the polyhedral domain U , we consider a vertex O and the adjacent faces $\sigma_1, \dots, \sigma_m$ enumerated in the order of circumvention around O along $\text{fr } U$ in the positive direction. Denote by α_j ($j = 1, \dots, m$) the interior angle at the vertex O in the polygon σ_j and by $L_j = \sigma_j \cap \sigma_{j+1}$ ($j = 1, \dots, m; m+1 \equiv 1$) the edges outgoing from O . We set

$$\Sigma = \bigcup_{j=1}^m \sigma_j, \quad \Phi = \bigcup_{j=1}^m \alpha_j.$$

In the polyhedral domain V , we consider the vertex $O' = f(O)$, the faces $\sigma'_j = f(\sigma_j)$, the edges $L'_j = f(L_j) = \sigma'_j \cap \sigma'_{j+1}$, and the neighborhood

$$\Sigma' = f(\Sigma) = \bigcup_{j=1}^m \sigma'_j$$

of the point O' on $\text{fr } V$. We set

$$\Psi = \bigcup_{j=1}^m \beta_j,$$

where β_j ($j = 1, \dots, m$) is an interior angle at the vertex O' in the polygon σ'_j .

We construct an orientation preserving homeomorphism $H : \Sigma \rightarrow D \subset \mathbb{C}$ from Σ onto the closed neighborhood $D = H(\Sigma)$ of the point 0. The homeomorphism H is uniquely determined by the following conditions:

(1) H takes line segments with origin O and endpoints on $\text{fr } D$ to line segments with origin 0 and endpoints on $\text{fr } D$; moreover, the edge L_m goes to the line segment $H(L_m) \subset \{z \in \mathbb{C} : \text{Im } z = 0, \text{Re } z \geq 0\}$,

(2) $|H(x)| = |x - O|^{2\pi/\Phi}$ for all $x \in \Sigma$,

(3) for any pair of line segments with origin O and endpoints on $\text{fr } \Sigma$ that lie on the same face, the angle φ between these segments is connected with the angle $\tilde{\varphi}$ between their images by the equality $\tilde{\varphi} = (2\pi/\Phi)\varphi$.

Then, for any face σ_j , the restriction $H|_{\sigma_j}$ maps $\text{Int } \sigma_j$ conformally onto

$$H(\text{Int } \sigma_j) \subset \mathbb{C}.$$

For the polyhedral domain V with vertex O' , we construct an orientation preserving homeomorphism $H' : \Sigma' (= f(\Sigma)) \rightarrow D' \subset \mathbb{C}$ which is uniquely determined by the following three conditions:

(1') H' takes line segments with origin O' and endpoints on $\text{fr } \Sigma'$ to line segments with origin 0 and endpoints on $\text{fr } D'$; moreover, the edge L'_m goes to the line segment $H'(L'_m) \subset \{z \in \mathbb{C} : \text{Im } z = 0, \text{Re } z \geq 0\}$;

(2') $|H'(x)| = |x - O'|^{2\pi/\Psi}$ for all $x \in \Sigma'$,

(3') for any pair of line segments with origin O and endpoints on $\text{fr } \Sigma'$ that lie on the same face, the angle ψ between these segments is connected with the angle $\hat{\psi}$ between their images by the equality $\hat{\psi} = (2\pi/\Psi)\psi$.

Then for any face σ'_j the restriction $H'|_{\sigma'_j}$ maps $\text{Int } \sigma'_j$ conformally onto

$$H'(\text{Int } \sigma'_j) \subset \mathbb{C}.$$

Let us verify that the homeomorphism $g = H' \circ f \circ H^{-1} : D \rightarrow D'$ is a conformal mapping in a neighborhood of any interior point $z_0 \in \text{Int } D$.

If $H^{-1}(z_0)$ is an interior point of some face σ_j , then g is conformal in a neighborhood of this point because H^{-1} is conformal in a neighborhood of z_0 , $f|_{\sigma_j}$ is conformal in a neighborhood of $H^{-1}(z_0)$, and H' is conformal in a neighborhood of the point $f(H^{-1}(z_0))$ which is an interior point of the face σ'_j .

If $H^{-1}(z_0)$ is an interior point of some edge L_j , then z_0 is an interior point of the line segment $H(L_j)$. Then, in the disk $B(z_0, r)$ of a sufficiently small radius, the function g is analytic in $B(z_0, r) \setminus H(L_j)$ and continuous in $B(z_0, r)$. By the continuous extension principle (see [10, Theorem 1]), g is analytic in the whole disk $B(z_0, r)$. Thus, the function g is analytic in a punctured neighborhood of the point 0. Since $\lim_{z \rightarrow z_0} g(z) = g(0) = 0$ exists, 0 is a removable singular point of g and, consequently, g is analytic in a neighborhood of 0 and the homeomorphism $g : \text{Int } D \rightarrow \text{Int } D'$ is a conformal mapping. Then g preserves the angles between the directions at the point 0. Therefore, taking into account (3) and (3'), for any pair of edges L_j and L_{j+1} we have $(2\pi/\Phi)\alpha_j = (2\pi/\Psi)\beta_j$ which shows that the ratio $\alpha_j/\beta_j = \Phi/\Psi$ is independent of $j = 1, \dots, m$. Extending the edges L_j and L'_j to rays w_j and w'_j outgoing from the points O and O' respectively, we find polyhedral angles W and W' satisfying all the assumptions of Lemma 2.1. Then for each face σ_j ($j = 1, \dots, m$) we get the equality $\alpha_j = \beta_j$ for interior angles in σ_j and $\sigma'_j = f(\sigma_j)$ at the vertices O and O' respectively. Since the vertex O is chosen arbitrarily, in the polyhedral domain U , all the corresponding plane angles in the polygons σ_1 and $\sigma'_1 = f(\sigma_1)$ are equal. By Lemma 2.2, $f|_{\sigma_1}$ is a similarity transformation. Applying Lemma 2.3, we obtain the desired result. $\square$

Proof of Theorem 2.1. By definition, for $V \in \mathcal{P}(3)$ there exists a homeomorphism $f : \text{fr } U \rightarrow \text{fr } V$ preserving the relative conformal moduli of boundary condensers. Our goal is to prove that the domain V can be conformally mapped onto the domain U . For this purpose, we first consider the unions T_1 and T'_1 of all edges of $\text{fr } U$ and $\text{fr } V$ respectively. The sets $\Sigma = \text{fr } U \setminus \{T_1 \cup f^{-1}(T'_1)\}$ and $\Sigma' = \text{fr } V \setminus \{f(T_1) \cup T'_1\} = f(\Sigma)$ are everywhere dense in $\text{fr } U$ and $\text{fr } V$ (respectively) and are open in the topologies of these boundaries induced by the Euclidean metric in $\mathbb{R}^3$. The

restriction $f|_{\Sigma}$ ($f^{-1}|_{\Sigma'}$) of f (f^{-1}) on Σ (Σ') is a 2-dimensional conformal mapping, which can be proved in the same way as Theorem 8.1 in [3].

Let us prove that $T'_1 = f(T_1)$. Assume that $f(T_1)$ is not a subset of T'_1 . Since f is continuous, there exists an edge L of U and a point $a \in \text{Int } L$ such that $f(a) \in \text{Int } s'$, where s' is a face of V . Since $f(a)$ does not belong to T'_1 and f is a homeomorphism, there exists an (open) three-dimensional ball $B_3(a, R)$ with center a and radius R possessing the following properties: R is less than the distance from the point a to the set $f^{-1}(T'_1)$, $f(B_3(a, R) \cap \text{fr } U) \subset s'$, and the intersection of $B_3(a, R)$ with each of the planes τ_1 and τ_2 containing one of the neighboring faces s_1 and s_2 with the common edge L is a subset of the face s_1 (respectively s_2). In particular, this means that the intersection of $B_3(a, R)$ with the line l containing the edge L is a subset of L . Rotating h around the line l , we develop s_1 in such a way that the planes of the faces s_1 and s_2 coincide. Then, without loss of generality, the mapping $\tilde{f}: B_2(a, R) \rightarrow \tau'$ from the disk $B_2(a, R)$ in the plane τ_2 into the plane τ' containing the face s' of the domain V such that $\tilde{f}(x) = f(h^{-1}(x))$ for $x \in h(B_2(a, R) \cap s_1)$ and $\tilde{f}(x) = f(x)$ for $x \in B_2(a, R) \cap s_2$ can be regarded as a mapping from $B_2(a, R) \subset \mathbb{C}$ into $\mathbb{C}$. By construction and the relation $\text{Int}(B_3(a, R) \cap s_j) \subset \Sigma$, $j = 1, 2$, we conclude that $\tilde{f}$ is a continuous bijection from $B_2(a, R)$ into $f(B_3(a, R) \cap \partial U)$; moreover, $\tilde{f}$ is a plane conformal mapping in the half-disks $\text{Int } h(B_3(a, R) \cap s_1)$ and $\text{Int}(B_3(a, R) \cap s_2)$. Hence, L is ACL-removable for $\tilde{f}$. Consequently, $\tilde{f}$ is conformal in $B_2(a, R)$. Therefore, $\frac{d\tilde{f}}{dz}(a) \neq 0$, and $f|_{B_3(a, R) \cap L}$ is a smooth mapping.

Let $r_0 > 0$ be such that $r_0 < R$ and $B_3(a, r_0) \cap \widetilde{V}_\alpha \subset U$; moreover, $\widetilde{V}_\alpha = \text{contg}_U a = \text{cl}(P(V_\alpha))$ ($P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a similarity transformation) and

$$V_\alpha = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_j \in \mathbb{R}, j = 1, 2, 3, x_2 = \lambda \cos \theta, \\ x_3 = \lambda \sin \theta, 0 < \lambda < \infty, 0 < \theta < \alpha\},$$

where $0 < \alpha < \pi$ (we note that additional conditions on the number r_0 will be imposed below). Let $\mu \geq 2$ be a natural number, and let $t > 0$ be a real number such that $2\mu t \leq r_0$. Without loss of generality, we can assume that l is the abscissa axis (in $\mathbb{R}^3$), $\widetilde{V}_\alpha = V_\alpha$, and $a = 0$. We consider the boundary condenser $A_\mu = \{F_1^\mu, F_2^\mu\}$ of both domains U and V_α with components

$$F_1^\mu = \{x \in \mathbb{R}^3 : -t \leq x_1 \leq 0, x_2 = x_3 = 0\}, \\ F_2^\mu = \{x \in \mathbb{R}^3 : t \leq x_1 \leq \mu t, x_2 = x_3 = 0\}.$$

Since $B_3(0, r_0) \cap V_\alpha \subset U$, we have

$$\Gamma_{F_1^\mu, F_2^\mu, U} \subset \Gamma_{F_1^\mu, F_2^\mu, B_3(0, 2\mu t) \cap V_\alpha} \cup \Gamma^\mu \subset \Gamma_{F_1^\mu, F_2^\mu, V_\alpha} \cup \Gamma^\mu,$$

where $\Gamma_{F_1^\mu, F_2^\mu, B_3(0, 2\mu t) \cap V_\alpha}$ is a family of paths γ in $\Gamma_{F_1^\mu, F_2^\mu, U}$ whose images $\text{Im } \gamma$ are contained in the ball $B_3(0, 2\mu t)$ and Γ^μ is a family of paths γ_0 in $\Gamma_{F_1^\mu, F_2^\mu, U}$ such that $\{\text{Im } \gamma_0\} \cap \{\mathbb{R}^3 \setminus B_3(0, 2\mu t)\} \neq \emptyset$; moreover, Γ^μ is minorized by the family $\Gamma_{S_1, S_2, \bar{A}_\mu}$ of paths joining the boundary spheres $S_1 = \{x \in \mathbb{R}^3 : |x| = t\}$ and $S_2 = \{x \in \mathbb{R}^3 : |x| = 2\mu t\}$ of the spherical shell $\bar{A}_\mu = \{x \in \mathbb{R}^3 : t < |x| < 2\mu t\}$. By [11, Theorems 6.2, 6.4, and 7.5],

$$M^U(A_\mu) \leq M^{V_\alpha}(A_\mu) + 4\pi\{\log(2\mu)\}^{-2} \leq M^{V_\alpha}(A) + 4\pi(\log \mu)^{-2}. \quad (2.1)$$

We note that estimate (2.1) is independent of the parameter t provided that its values are sufficiently small, for example, $t \leq r_0/(2\mu)$. By the conformal invariance

of the modulus (3-modulus) of boundary condensers, we can assume that

$$f(0) = 0, \quad \frac{d\bar{f}}{dz}(0) = C > 0, \quad \tau' = \{x \in \mathbb{R}^3 : x_3 = 0\}, \quad V \subset \mathbb{R}_+^3 = \{x \in \mathbb{R}^3 : x_3 > 0\}.$$

Furthermore, if r_0 is sufficiently small, then, because of the smoothness of $f|_{B_3(0,R) \cap L}$, we have $\frac{df}{dt}(r) \neq 0$ for all $r \in [-r_0, r_0]$ and the function $r \mapsto |f(r)|$ is (strictly) decreasing on $[-r_0, 0]$ and (strictly) increasing on $[0, r_0]$. We can also assume that r_0 satisfies the condition

$$B_3(0, \max_{|z| \leq r_0, z \in \mathbb{C}} |\tilde{f}(z)|) \subset V.$$

Two cases are possible:

Case 1: 0 lies in the closure of the set of $t \in]0, r_0/2\mu[$ such that $|f(-t)| \geq |f(t)|$,

Case 2: 0 lies in the closure of the set of $t \in]0, r_0/2\mu[$ such that $|f(-t)| < |f(t)|$.

Case 1. Assume that r_0 is small enough so that for any $t \in]0, r_0/2\mu[$ the ray χ outgoing from the point $f(2\mu t)$ in the opposite direction to the direction of the ray that is the contingency of the set $f([0, 2\mu t])$ at the point $2\mu t$ and the vector e_1 (e_1, e_2, e_3 is the canonical basis for $\mathbb{R}^3$) form an angle less than $\pi/2$. For any such t we extend the curve $f([0, 2\mu t])$ by this ray (the obtained curve is denoted by E_t) and consider the boundary condensers

$$\tilde{A}_\mu = \{\tilde{F}_1^\mu, \tilde{F}_2^\mu\}, \quad \tilde{F}_1^\mu = f(F_1^\mu) = f([-t, 0]), \quad \tilde{F}_2^\mu = f(F_2^\mu)$$

of the domain V and, simultaneously, the half-space $\mathbb{R}_+^3$ and

$$\tilde{A}_\mu^* = \{\tilde{F}_1^\mu, \tilde{F}_2^{\mu*}\}, \quad \tilde{A} = \{\tilde{F}_1, \tilde{F}_2\}, \quad \tilde{F}_2^{\mu*} = E_t \setminus f([0, \mu t]), \quad \tilde{F}_1 = \tilde{F}_1^\mu, \quad \tilde{F}_2 = E_t \setminus f([0, t])$$

of the half-space $\mathbb{R}_+^3$. For these condensers

$$M^{V_\pi}(\tilde{A}) \leq M^V(\tilde{A}_\mu) + M_3(\tilde{\Gamma}_\mu) + M_3(\tilde{A}_\mu^*, \mathbb{R}^3) \quad (2.2)$$

(see [3], formula (8.32)), where $\tilde{\Gamma}_\mu$ is the family of paths γ joining $\tilde{F}_1$ with $\tilde{F}_2$ in $\mathbb{R}^3 \setminus \{\tilde{F}_1 \cup \tilde{F}_2\}$ and such that $\{\text{Im } \gamma\} \cap \{\mathbb{R}^3 \setminus (B_3(0, f(2\mu t)))\} \neq \emptyset$; moreover,

$$M_3(\tilde{A}_\mu, \mathbb{R}^3) = M_3(\Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^{\mu*}, \mathbb{R}^3 \setminus \{\tilde{F}_1^\mu \cup \tilde{F}_2^{\mu*}\}}).$$

Indeed,

$$\Gamma_{\tilde{F}_1, \tilde{F}_2, V_\pi} \subset \Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^\mu, V_\pi} \cup \Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^{\mu*}, \mathbb{R}^3 \setminus \{\tilde{F}_1^\mu \cup \tilde{F}_2^{\mu*}\}}, \quad \Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^\mu, V_\pi} \subset \Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^\mu, V} \cup \tilde{\Gamma}_\mu.$$

By [11, Theorem 6.2],

$$M^{V_\pi}(\tilde{A}) \leq M^{V_\pi}(\tilde{A}_\mu) + M_3(\tilde{A}_\mu^*, \mathbb{R}^3) \leq M^V(\tilde{A}_\mu) + M_3(\tilde{\Gamma}_\mu) + M_3(\tilde{A}_\mu^*, \mathbb{R}^3),$$

which implies (2.2). We assume that r_0 also satisfies the inequality $|f(r) - Cre_1| \leq Cr/4$ for all $r \in [-r_0, r_0]$. Taking into account that $\Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^{\mu*}, \mathbb{R}^3 \setminus \{\tilde{F}_1^\mu \cup \tilde{F}_2^{\mu*}\}}$ is minorized by the family $\Gamma_{\tilde{S}_1, \tilde{S}_2', \tilde{A}_\mu'}$ of paths joining the boundary spheres $\tilde{S}_1 = \{x \in \mathbb{R}^3 : |x| = |f(-t)|\}$ and $\tilde{S}_2' = \{x \in \mathbb{R}^3 : |x| = |f(\mu t)|\}$ of the spherical shell $\tilde{A}_\mu' = \{x \in \mathbb{R}^3 : |f(-t)| < |x| < |f(\mu t)|\}$ and $\tilde{\Gamma}_\mu$ is minorized by the family $\Gamma_{\tilde{S}_1, \tilde{S}_2'', \tilde{A}_\mu''}$ of paths joining the boundary spheres $\tilde{S}_1$ and $\tilde{S}_2'' = \{x \in \mathbb{R}^3 : |x| = |f(2\mu t)|\}$ of the spherical shell $\tilde{A}_\mu'' = \{x \in \mathbb{R}^3 : |f(-t)| < |x| < |f(2\mu t)|\}$ and using [11, Theorems 6.2, 6.4, and 7.5], we get

$$M_3(\tilde{A}_\mu^*, \mathbb{R}^3) \leq 4\pi \{\log(|f(\mu t)|/|f(-t)|)\}^{-2} \leq 4\pi \{\log(3\mu/5)\}^{-2}, \quad (2.3)$$

$$M_3(\tilde{\Gamma}_\mu) \leq 4\pi\{\log(|f(2\mu t)|/|f(-t)|)\}^{-2} \leq 4\pi\{\log(6\mu/5)\}^{-2} \leq 4\pi\{\log(3\mu/5)\}^{-2}. \quad (2.4)$$

In turn, inequalities (2.2)-(2.4) imply

$$M^{V_\pi}(\tilde{A}) \leq M^V(\tilde{A}_\mu) + 8\pi\{\log(3\mu/5)\}^{-2}. \quad (2.5)$$

On the other hand, by [12, Theorem 3.1],

$$M^{V_\pi}(\tilde{A}) \geq M^{V_\pi}(\tilde{A}_0), \quad (2.6)$$

where $\tilde{A}_0$ is the boundary condenser of V_π with components

$$F_1^0 = \{x \in \mathbb{R}^3 : -|f(-t)| \leq x_1 \leq 0, x_2 = x_3 = 0\}, \\ F_2^0 = \{x \in \mathbb{R}^3 : |f(t)| \leq x_1 \leq \infty, x_2 = x_3 = 0\}.$$

Further, the conformal invariance of the n -moduli of boundary condensers leads to the identity

$$M^{V_\pi}(\tilde{A}_\mu) = M(\{x \in \mathbb{R}^3 : -1 \leq x_1 \leq 0, x_2 = x_3 = 0\}, \\ \{x \in \mathbb{R}^3 : |f(t)|/|f(-t)| \leq x_1 \leq \infty, x_2 = x_3 = 0\}) = \psi(|f(t)|/|f(-t)|), \quad (2.7)$$

where ψ is the function introduced in [12] as follows. If $u > 0$, then ψ is the (conformal) modulus of the family of arcs joining the segment $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -1 \leq x_1 \leq 0, x_2 = x_3 = 0\}$ with the ray $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : u \leq x_1 \leq \infty, x_2 = x_3 = 0\}$ in the half-space $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -\infty < x_1, x_2 < \infty, x_3 > 0\}$. Taking into account that f leaves the conformal moduli of boundary condensers unchanged and using the relations (2.1) and (2.5)-(2.7), we find

$$\psi(|f(t)|/|f(-t)|) \leq M^{V_\pi}(\tilde{A}) \leq M^V(\tilde{A}_\mu) + 8\pi\{\log(3\mu/5)\}^{-2} \\ = M^U(A_\mu) + 8\pi\{\log(3\mu/5)\}^{-2} \\ \leq M^{V_\alpha}(A) + 8\pi\{\log(3\mu/5)\}^{-2} + 4\pi(\log \mu)^{-2} \\ = (\alpha/\pi)\psi(1) + 8\pi\{\log(3\mu/5)\}^{-2} + 4\pi(\log \mu)^{-2}, \quad (2.8)$$

where [12, Lemma 7.1] was used. Taking into account the continuity of $u \mapsto \psi(u)$ and letting $t \rightarrow 0$ and then $\mu \rightarrow \infty$ in (2.8), we arrive at the contradictory inequality

$$(0 <) \psi(1) \leq (\alpha/\pi)\psi(1). \quad (2.9)$$

Case 2. We argue as in Case 1, but, in this case, the minorizing family for $\Gamma_{\tilde{F}_1^\mu, \tilde{F}_2^{\mu*}, \mathbb{R}^3 \setminus \{\tilde{F}_1^\mu \cup \tilde{F}_2^{\mu*}\}}$ is the family $\Gamma_{\tilde{S}_1^0, \tilde{S}_2^0, \tilde{A}_\mu^0}$ of paths joining the boundary spheres $\tilde{S}_1^0 = \{x \in \mathbb{R}^3 : |x| = |f(t)|\}$ and $\tilde{S}_2^0 = \tilde{S}_2$, of the spherical shell $\tilde{A}_\mu^0 = \{x \in \mathbb{R}^3 : |f(t)| < |x| < |f(\mu t)|\}$ and the minorizing family for $\tilde{\Gamma}_\mu$ is the family $\Gamma_{\tilde{S}_1^{00}, \tilde{S}_2^{00}, \tilde{A}_\mu^{00}}$ of paths joining the boundary spheres $\tilde{S}_1^{00} = \tilde{S}_1^0$ and $\tilde{S}_2^{00} = \{x \in \mathbb{R}^3 : |x| = |f(2\mu t)|\}$ of the spherical shell $\tilde{A}_\mu^{00} = \{x \in \mathbb{R}^3 : |f(t)| < |x| < |f(2\mu t)|\}$; moreover, $\psi(|f(t)|/|f(-t)|)$ should be replaced with $\psi(|f(-t)|/|f(t)|)$ in (2.7) and (2.8). As a result, we arrive at the same contradictory inequality (2.9).

The contradiction obtained (in both cases) shows that $f(T_1) \subset T_1'$. Replacing f with its inverse f^{-1} , we get the inclusion $f^{-1}(T_1') \subset T_1$ which is equivalent to the inclusion $T_1' \subset f(T_1)$. Thus, $T_1' = f(T_1)$, which implies that f transforms the edges, vertices, and faces of the polyhedral domain U to the edges, vertices, and faces of the polyhedral domain V . But then the connected components of the set

Σ are exactly the interiors of the faces of U . Consequently, f is a plane conformal mapping in the interior of each face.

The next step of the proof is to show that the dihedral angles between any two faces of U with a common edge are equal to their images. This fact is proved by contradiction in the same way as the equality $T'_1 = f(T_1)$ (with slight modifications). As a result, we arrive at the contradictory relation

$$\psi(1) \leq (\alpha/\alpha')\psi(1),$$

where α is the value of the interior dihedral angle between the neighboring faces s_1 and s_2 in U (V), α' is the value of the dihedral angle between the faces $f(s_1)$ and $f(s_2)$ ($f^{-1}(s_1)$ and $f^{-1}(s_2)$) in V (U) and $\alpha < \alpha'$.

At the final step, we apply Theorem 2.2. $\square$

Theorem 2.3. *Suppose that $U_1 \subset \mathbb{R}^3$ and $U_2 \subset \mathbb{R}^3$ are bounded convex polyhedral domains for which there exist a homeomorphism $f : \text{fr } U_1 \rightarrow \text{fr } U_2$ between the boundaries of these domains and a number $p \in \{]1, 3[\cup]3, \infty[\}$ such that f preserves both the relative 3-moduli and the relative p -moduli of boundary ring-shaped condensers.*

Then there exists an isometry $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying the condition $H(U_1) = U_2$.

Remark 2.1. *In the case where $n = 3$ and U_1 and U_2 are bounded convex polyhedral domains, Theorem 2.3 corresponds to [1, Corollary 3.1]. It is very likely that, for $n = 3$, there are complete analogs of [1, Corollary 3.1] and [1, Theorem 2.1]. Unfortunately, at present, the author is not able to prove these assertions.*

The proof of Theorem 2.3 is based on Theorems 2.1 and the following

Lemma 2.4. *Assume that $U \subset \mathbb{R}^3$ is bounded convex polyhedral domain. Then there exists a ring-shaped boundary condenser F of U such that $0 < M_p^U(F) < \infty$ for every $p \in [1, \infty[$.*

The proof of this assertion is similar to that of [1, Corollary 3.1].

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