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ALL TIGHT DESCRIPTIONS OF 3-PATHS CENTERED AT 2-VERTICES IN PLANE GRAPHS WITH GIRTH AT LEAST 6

O.V. BORODIN, A.O. IVANOVA

ABSTRACT. Lebesgue (1940) proved that every plane graph with minimum degree δ at least 3 and girth g (the length of a shortest cycle) at least 5 has a path on three vertices (3-path) of degree 3 each. A description is tight if no its parameter can be strengthened, and no triplet dropped.

Borodin et al. (2013) gave a tight description of 3-paths in plane graphs with $\delta \geq 3$ and $g \geq 3$, and another tight description was given by Borodin, Ivanova and Kostochka in 2017.

In 2015, we gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of tight descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we characterized (2018) all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$.

Recently, eleven tight descriptions of 3-paths were obtained for plane graphs with $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták, four of which descriptions are for $g \geq 9$. In 2018, Aksenov, Borodin and Ivanova proved ten new tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$ and showed that no other tight descriptions exist.

In this paper we give a complete list of tight descriptions of 3-paths centered at a 2-vertex in the plane graphs with $\delta = 2$ and $g \geq 6$.

Keywords: plane graph, structure properties, tight description, 3-path, minimum degree, girth.

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1. INTRODUCTION

Throughout the paper, G is a plane graph. Let $\delta(G)$ be the minimum vertex degree of G . We will drop the argument when G is clear from context. A *girth*, g , is the length of a shortest cycle in G . The degree of a vertex v or a face f , that is the number of edges incident with v or f , is denoted by $d(v)$ or $d(f)$, respectively. A k -*vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k , respectively. Hence, a k^+ -vertex v satisfies $d(v) \geq k$, etc. An edge uv is an (i, j) -*edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i, j, k) or (i, j, k) -*path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [21] proved that every G with $\delta = 5$ has a $(5, 6)$ -edge, and Franklin [14] strengthened this to the existence of at least two 6^- -neighbors of a 5^- -vertex. Recently, we proved [6] that there is also a $(5, 6, 6)$ -path, which is tight, and there are no tight descriptions of 3-paths for $\delta = 5$ other than $\{(6, 5, 6)\}$ [14] and $\{(5, 6, 6)\}$ [6].

A description of paths is *tight* if no its parameter can be strengthened and no term dropped. Recently, we found [7] the first tight description of 4-paths for $\delta = 5$, and then Batueva, Borodin and Ivanova [2] found all remaining nine such tight descriptions. In 1989, Borodin [3] proved that every G with $\delta \geq 3$ contains a $(3, 10)$ -, or $(4, 7)$ -, or $(5, 6)$ -edge, which description is tight. In 1997, Jendrol' [15] gave an approximate description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(10, 3, 10)$, $(7, 4, 7)$, $(6, 5, 6)$, $(3, 4, 15)$, $(3, 6, 11)$, $(3, 8, 5)$, $(3, 10, 3)$, $(4, 4, 11)$, $(4, 5, 7)$, or $(4, 7, 5)$.

In 2013, Borodin et al. [11] gave the first tight description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, $(6, 5, 6)$. Another similar tight description for $\delta \geq 3$ and $g \geq 3$ was given by Borodin, Ivanova, and Kostochka [12].

In 2015, we [8] gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we characterized [10] all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$. Other results on k -paths with $k \geq 3$ and $\delta \geq 3$ can be found in surveys Borodin–Ivanova [9], Cranston–West [13] and Jendrol'–Voss [20].

Recently, eleven tight descriptions of 3-paths were obtained for $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták [16–19], four of which descriptions are for $g \geq 9$ (for details, see Theorem 2 below).

In 2018, Aksenov, Borodin, and Ivanova [1] gave the following list of all tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$.

Theorem 1 ([1]). *There exist precisely these tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth g at least 9:*

- (A) $g \geq 16$:
 $\{(2, 2, 2)\}$ (*folklore*);
- (B) $11 \leq g \leq 15$:
 $\{(2, 2, 3)\}$ (Jendrol' et al. [17]) and
 $\{(2, 3, 2)\}$;
- (C) $g = 10$:
 $\{(2, 2, 3), (2, 3, 2)\}$ (Jendrol' and Maceková [16], the tightness shown in Jendrol' et al. [17]);

- $\{(2, 4, 2)\}$ ([17]);
 $\{(2, 3, 3)\}$, $\{(2, 2, 4), (3, 2, 3)\}$; and
 $\{(3, 2, 4)\}$;
 (D) $g = 9$:
 $\{(2, 2, 5), (2, 3, 2)\}$ (Jendrol' et al. [18]);
 $\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 2, 5), (3, 2, 3)\}$, $\{(2, 5, 3)\}$, $\{(2, 3, 5)\}$; and
 $\{(3, 2, 5)\}$.

The problem of tightly describing 3-paths in plane graphs with $g \leq 8$ and $\delta = 2$ is more difficult; all what is known so far can be summarized as follows.

Theorem 2. *The following tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth g at most 8 hold:*

- (i) $g = 8$:
 $\{(2, 2, 5), (2, 3, 2)\}$ [17];
 (ii) $g = 7$:
 $\{(2, 2, 6), (2, 3, 3), (2, 4, 2)\}$ ([16], with the tightness confirmed in [17]) and
 $\{(2, 3, 3), (2, 5, 2)\}$ ([17]);
 (iii) $g = 6$:
 $\{(2, 2, \infty), (2, 3, 4), (2, 5, 2)\}$ ([18]);
 (iv) $g = 5$:
 $\{(2, \infty, 2), (3, 3, 3), (2, 5, 3), (3, 2, 4)\}$ ([18]) and
 $\{(2, \infty, 2), (3, 3, 3), (2, 3, 5), (2, 4, 3)\}$ ([17]);
 (v) $g = 4$:
 $\{(2, \infty, 2), (2, 7, 3), (3, 5, 3), (4, 2, 5), (4, 3, 4)\}$ (Jendrol', Maceková, Montassier, and Soták [19]).

In the present paper we completely describe tight descriptions of 3-paths centered at 2-vertices (called *minor 3-paths*) for $g \geq 6$.

Theorem 3. *There exist precisely these tight descriptions of minor 3-paths in plane graphs with minimum degree 2 and girth g at least 6:*

- (A) $g \geq 16$:
 $\{(2, 2, 2)\}$ (folklore);
 (B) $11 \leq g \leq 15$:
 $\{(2, 2, 3)\}$ ([17]);
 (C) $g = 10$:
 $\{(2, 2, 4), (3, 2, 3)\}$, and $\{(3, 2, 4)\}$ (both in [1]);
 (D) $g = 9$:
 $\{(2, 2, 5), (3, 2, 3)\}$ and
 $\{(3, 2, 5)\}$ (both in [1]);
 (E) $g = 8$:
 $\{(3, 2, 5)\}$;
 (F) $g = 7$:
 $\{(2, 2, 10), (3, 2, 5)\}$ and $\{(3, 2, 10)\}$;
 (G) $g = 6$:
 $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$,
 $\{(3, 2, \infty), (5, 2, 7)\}$, $\{(4, 2, \infty), (5, 2, 7)\}$, and
 $\{(5, 2, \infty)\}$.

Clearly, the items (A–D) in Theorem 3 follow immediately from Theorem 1, so we have to prove only Theorem 3(E–G). To do so, we need the following refinement

of the already mentioned Borodin's tight description [4,5] of 2-paths in plane graphs with $\delta \geq 3$: $\{(3, 10), (4, 7), (5, 6)\}$.

Theorem 4. *Every plane graph with minimum degree at least 3 has at least one of the following:*

- (a) a 3-face incident with a $(3, 10)$ -, or $(4, 7)$ -, or $(5, 6)$ -edge;
- (b) a 4-face incident either with two 3-vertices and another 5^- -vertex or with a 3-vertex, two 4-vertices and the fourth vertex of degree at most 5;
- (c) a 5-face incident with four 3-vertices and the fifth vertex of degree at most 5, where all parameters are best possible.

2. PROVING THEOREM 4

2.1. Proving the tightness of Theorem 4. (a) A triangulation H_1 (see Fig. 1) is obtained from the icosahedron by putting a 3-vertex into each face and joining it to the boundary vertices of this face. Note that every 3-vertex in H_1 is surrounded by 10-vertices.

To construct a triangulation H_2 , we start from the $(3, 4, 4, 4)$ Archimedean solid, in which every vertex is incident with a 3-face and three 4-faces, and put a 4-vertex into each 4-face, followed by joining it to the four boundary vertices. As a result, each 4-vertex of H_2 has four 7-neighbors.

A triangulation H_3 is obtained from the dodecahedron by putting a 5-vertex into each face and joining it to the five boundary vertices. So each 5-vertex of H_3 is surrounded by 6-vertices.

(b) To obtain a quadrangulation H_4 , we delete all edges joining 10-vertices to 10-vertices from an H_1 . Thus H_4 is bipartite, with every edge joining a 3-vertex to a 5-vertex,

To construct a quadrangulation H_5 , we put a vertex inside each face of the dodecahedron and join it with the middles of the boundary edges. As a result, the boundary of each face of H_5 has a 3-vertex, two non-consecutive 4-vertices, and a 5-vertex.

(c) In the $(3, 3, 3, 3, 5)$ Archimedean solid H_6 , each face is a pentagon incident with four 3-vertices and a 5-vertex.

It is not hard to see that no parameter in Theorem 4 can be lowered due to the graphs H_1 – H_6 .

2.2. Proving the existence of a suitable 5^- -face in Theorem 4. Let G avoid 5^- -faces listed in Theorem 4. Without loss of generality, we can assume that G is connected. Let V , E , and F be the sets of vertices, edges and faces of G , respectively. Euler's formula $|V| - |E| + |F| = 2$ for G may be rewritten as

$$(1) \quad \sum_{x \in V \cup F} (d(x) - 4) = -8.$$

Every vertex and face $x \in V \cup F$ contributes the *charge* $\mu(x) = d(x) - 4$ to (1), so only the charges of 3^- -vertices and 3^- -faces are negative. We define a local redistribution of μ 's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -8 .

Рис. 1. Faces of graphs H_1 – H_6 in Theorem 4 and of some their derivatives used in proving Theorem 3.

We apply the following rules of discharging.

R1. Every face f gives $\frac{1}{3}$ to each incident 3-vertex.

R2. Every 5-vertex gives $\frac{1}{5}$ to each incident face.

R3. Every 6-vertex gives $\frac{1}{3}$ to each incident face.

R4. Every 3-face incident with a 5^- -vertex v_1 receives:

- (a) $\frac{2}{3}$ from each of the 11^+ -vertices v_2 and v_3 if $d(v_1) = 3$,
- (b) $\frac{1}{2}$ from each of the 8^+ -vertices v_2 and v_3 if $d(v_1) = 4$, and
- (c) $\frac{2}{5}$ from each of the 7^+ -vertices v_2 and v_3 if $d(v_1) = 5$.

R5. Every 7^+ -vertex gives $\frac{1}{3}$ to each incident face f if $4 \leq d(f) \leq 5$.

We now check $\mu'(x) \geq 0$ whenever $x \in V \cup F$.

CASE 1. $v \in V$.

If $d(v) = 3$ then $\mu'(v) \geq 3 - 4 + 3 \times \frac{1}{3} = 0$ in view of R1.

If $d(v) = 4$ then $\mu'(v) = \mu(v) = 0$.

By R2, R3, R4c, we have $\mu'(v) = 1 - 5 \times \frac{1}{5} = 0$ for $d(v) = 5$, $\mu'(v) = 2 - 6 \times \frac{1}{3} = 0$ for $d(v) = 6$, and $\mu'(v) = 3 - 7 \times \frac{2}{5} > 0$ for $d(v) = 7$.

If $8 \leq d(v) \leq 10$ then $\mu'(v) \geq d(v) - 4 - d(v) \times \frac{1}{2} = \frac{d(v)-8}{2} \geq 0$ by R4b, R4c, R5.

Next suppose $d(v) = 11$. If v donates at most nine times by $\frac{2}{3}$ according to R4a, then v can donate at most $\frac{1}{2}$ to each other incident face by R4b, R4c, which implies that $\mu'(v) \geq 7 - 9 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$ by R4, R5.

We note that if v makes ten donations of $\frac{2}{3}$ by R4a, then v can participate neither in R4b, nor in R4c, and so $\mu'(v) \geq 7 - 10 \times \frac{2}{3} - \frac{1}{3} = 0$ in view of R5.

Finally, it remains to observe that v cannot give $\frac{2}{3}$ to all incident edges due to the oddness of 11.

CASE 2. $f \in F$. Note that f gives at most $\frac{1}{3}$ to each incident vertex; namely, to a 3-vertex by R1. If $d(f) \geq 6$, this implies $\mu'(f) \geq d(f) - 4 - d(f) \times \frac{1}{3} = \frac{2(d(f)-6)}{3} \geq 0$, as desired.

If $d(f) = 5$ then f is incident with at most four 3-vertices since G is a counterexample to Theorem 4(c). Furthermore, if f has four 3-vertices in the boundary, then f receives $\frac{1}{3}$ by R3 or R5, which implies $\mu'(f) \geq 1 - 4 \times \frac{1}{3} + \frac{1}{3} = 0$ in view of R1. Finally, if there are at most three 3-vertices in $\partial(f)$ (the boundary of a face f), then $\mu'(f) \geq 1 - 3 \times \frac{1}{3} = 0$.

Next suppose $d(f) = 4$. We know by Theorem 4(b) that f is incident either with two 3-vertices and two 6^+ -vertices, in which case $\mu'(f) = 0 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0$ by R1, R3, or with one 3-vertex and at least two 5^+ -vertices, in which case $\mu'(f) = -\frac{1}{3} + 2 \times \frac{1}{5} > 0$ by R1, R2, or with one 3-vertex, two 4-vertices and a 6^+ -vertex, in which case $\mu'(f) = -\frac{1}{3} + \frac{1}{3} = 0$ by R1, R3, or else has no incident 3-vertices, in which case $\mu'(f) = \mu(f) = 0$.

Finally, suppose $d(f) = 3$ and v_1 has the smallest degree in $\partial(f)$. If $d(v_1) \geq 6$, then $\mu'(f) = -1 + 3 \times \frac{1}{3} = 0$ by R3, R5. If $d(v_1) = 5$, then $d(v_2) \geq 7$ and $d(v_3) \geq 7$ due to the absence of (5,6)-edge in our counterexample G (see Theorem 4(a)), which implies $\mu'(f) = -1 + 2 \times \frac{2}{5} + \frac{1}{5} = 0$ by R2, R4c. If $d(v_1) = 4$, then we have $\mu'(f) = -1 + 2 \times \frac{1}{2} = 0$ due to the absence of (4,7)-edge in $\partial(f)$ combined with R4b. For $d(v_1) = 3$ we deduce from Theorem 4(a), R1 and R4a that $\mu'(f) = -1 - \frac{1}{3} + 2 \times \frac{2}{3} = 0$, as desired.

3. PROVING THEOREM 3(E)

The graph H_4^* (see Fig. 1) is obtained by putting a 2-vertex on every edge of H_4 .

3.1. Proving that $\{(3, 2, 5)\}$ is a description. Suppose on the contrary that G has no $(3, 2, 5)$ -paths and contract all 2-vertices in G to obtain a graph G^* with $\delta(G^*) \geq 3$. As follows from Theorem 4, G^* has a face f^* that is either a 3-face incident with a 5^- -vertex or a 4-face of one of the types $(3, 3, 5, \infty)$, $(3, 4, 4, 5)$, or else a $(3, 3, 3, 3, 5)$ -face.

The pre-image f of f^* under contraction is an 8^+ -face in G by assumption on girth.

If $f^* = v_1 v_2 v_3$ with $d(v_1) \leq 5$, then at most one 2-vertex can be put on each of the edges $v_1 v_2$, $v_1 v_3$ when going back from G^* to G since G has no $(2, 2, 5)$ -paths, and at most two 2-vertices can be put between v_2 and v_3 in G^* due to the absence of $(2, 2, 2)$ -paths in G . Hence $d(f) \leq 3 + 2 \times 2 < 8$, a contradiction.

Next suppose $d(f^*) = 4$. Note that no 2-vertex can be put on each of at least two $(3, 5)$ -edges in the boundary $\partial(f^*)$ of f^* due to the absence of $(3, 2, 5)$ -paths in G . Furthermore, each edge in $\partial(f^*)$ is incident with at most one 5^- -vertex. So, we have $d(f) \leq 2 \times 1 + 2 \times 2 < 8$, a contradiction.

Finally, if $d(f^*) = 5$ then $d(f) = 5 < 8$ due to the absence of $(3, 2, 5)$ -paths in G , a contradiction.

3.2. Completing the proof Theorem 3(E). The tightness of the description $\{(3, 2, 5)\}$ follows from the graph H_4^* .

The non-existence of other tight descriptions is trivial. Indeed, if $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$, $k \geq 1$, is a tight description of minor 3-paths in plane graphs with $\delta = 2$ and $g = 8$, then to be able to describe the graph H_4^* , our D must have a term, (x_i, y_i, z_i) with $x_i \geq 3$, $y_i \geq 2$ and $z_i \geq 5$. On the other hand, since $\{(3, 2, 5)\}$ is known to be tight, we have $D = \{(3, 2, 5)\}$, as desired.

4. PROVING THEOREM 3(F)

The graph H_1^{**} is obtained from H_1 (see Fig. 1) by putting two 2-vertices on every edge joining 10-vertices to 10-vertices and one 2-vertex on every other edge (that is, joining 3-vertices to 10-vertices).

We first prove that $\{(2, 2, 10), (3, 2, 5)\}$ is a description of minor 3-paths when $g = 7$. This implies that $\{(3, 2, 10)\}$ is also a description. Indeed, if G has a $(2, 2, 10)$ -path, then G also has a $(3, 2, 10)$ -path by definition. Similarly, a $(3, 2, 5)$ -path is also a $(3, 2, 10)$ -path.

Then we prove that $\{(2, 2, 10), (3, 2, 5)\}$ and $\{(3, 2, 10)\}$ are tight.

Finally, we will show that there are no tight descriptions for $g = 7$ other than those two in Theorem 3(F).

4.1. Proving that $\{(2, 2, 10), (3, 2, 5)\}$ is a description. Suppose on the contrary that G does not obey this description and consider the contracted graph G^* with $\delta(G^*) \geq 3$, as in Section 3, and its 5^- -faces f^* implied by Theorem 4.

First suppose $f^* = v_1 v_2 v_3$ with $d(v_1) \leq 5$ and $d(v_2) \leq 10$. Note that at most one 2-vertex can be put on each of the edges incident with f^* when going back from G^* to G since G has no $(2, 2, 10)$ -paths. Hence $d(f) \leq 3 + 3 \times 1 < 7$, a contradiction.

Next suppose $d(f^*) = 4$. Note that no 2-vertex can be put on each of at least two $(3, 5)$ -edges in the boundary of f^* due to the absence of $(3, 2, 5)$ -paths in G .

Furthermore, there are at most two edges joining a 5^- -vertex with a 4^- -vertex, and at most one 2-vertex can be put on each of such edges due to the absence of $(2, 2, 5)$ -paths in G . So $d(f) \leq 4 + 2 \times 1 < 7$, a contradiction.

Finally, if $d(f^*) = 5$ then $d(f) = 5 < 7$ due to the absence of $(3, 2, 5)$ -paths in G , a contradiction.

4.2. Proving the tightness of $\{(2, 2, 10), (3, 2, 5)\}$ and $\{(3, 2, 10)\}$. To see that neither of the weakest strengthenings of $\{(2, 2, 10), (3, 2, 5)\}$, which are $\{(2, 2, 9), (3, 2, 5)\}$, $\{(2, 2, 10), (2, 2, 5)\}$ and $\{(2, 2, 10), (3, 2, 4)\}$ is a description, it suffices to consider the graphs H_1^{**} and H_4^* .

We note that H_1^{**} has neither $(2, 2, 9)$ -paths nor $(3, 2, 5)$ -paths, which shows that $\{(2, 2, 9), (3, 2, 5)\}$ is not a description.

The graph H_4^* has neither $(2, 2, 10)$ -paths nor $(3, 2, 4)$, and so forbids $\{(2, 2, 10), (2, 2, 5)\}$ and $\{(2, 2, 10), (3, 2, 4)\}$ to be descriptions.

Checking the tightness of $\{(3, 2, 10)\}$ is even simpler: its minimal strengthening $\{(2, 2, 10)\}$ is rejected by H_4^* , while $\{(3, 2, 9)\}$ is not a description for H_1^{**} .

4.3. Proving the non-existence of tight descriptions other than $\{(2, 2, 10), (3, 2, 5)\}$ and $\{(3, 2, 10)\}$. Suppose $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g = 7$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$. It follows from the graph H_1^{**} above that D must have a z_i , say z_1 , not smaller than 10.

CASE 1. $x_1 \geq 3$. Now $D = \{(3, 2, 10)\}$ since $\{(3, 2, 10)\}$ is known to be a tight description.

CASE 2. $x_1 = 2$. So $(x_1, y_1, z_1) = (2, 2^+, 10^+)$. It follows from H_4^* that D must have an entry, say (x_2, y_2, z_2) such that $x_2 \geq 3$ and $z_2 \geq 5$. This implies that D coincides with the known tight description $\{(2, 2, 10), (3, 2, 5)\}$, as desired.

5. PROVING THEOREM 3(G)

The graph H_7^* is obtained from the $(3, 3, 3, N)$ Archimedean solid as shown in Fig. 2.

The graph H_2^* is obtained from H_2 by putting a 2-vertex on every edge.

To obtain the graph H_3^* from H_3 , we put a 2-vertex on every edge.

We first prove that $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$ is a description of minor 3-paths when $g = 6$. This implies that each of $\{(3, 2, \infty), (5, 2, 7)\}$, $\{(4, 2, \infty), (5, 2, 7)\}$, and $\{(5, 2, \infty)\}$ is also a description. Indeed, if G has a $(3, 2, \infty)$ -path, then G also has a $(4, 2, \infty)$ -path and, moreover, a $(5, 2, \infty)$ -path. Similarly, both a $(4, 2, 7)$ -path and $(5, 2, 6)$ -path are $(5, 2, 7)$ -paths.

Then we prove the tightness of all descriptions in Theorem 3(G).

Finally, we will show that there are no tight descriptions for $g = 6$ other than those four listed in Theorem 3(G).

5.1. Proving that $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$ is a description. Suppose G does not obey $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$ and consider the contracted graph G^* , as in Section 4, and its 5^- -faces f^* implied by Theorem 4.

First suppose $f^* = v_1 v_2 v_3$ with $d(v_1) \leq 5$. If $d(v_1) = 3$ and $d(v_2) \leq 10$, then no 2-vertex can be put on the edges $v_1 v_2$ and $v_1 v_3$ when going back from G^* to G since

G has no $(3, 2, \infty)$ -paths, and at most one 2-vertex can be put on the edge v_2v_3 since G has no $(2, 2, \infty)$ -paths, which implies $d(f) \leq 3 + 1 < 6$, a contradiction.

If $d(v_1) = 4$ and $d(v_2) \leq 7$, or if $d(v_1) = 5$ and $d(v_2) \leq 6$, then we similarly deduce $d(f) \leq 3 + 2 \times 1 < 6$, a contradiction.

Next suppose $d(f^*) = 4$. Note that no 2-vertex can be put on each of at least three edges in the boundary of f^* due to the absence of $(3, 2, \infty)$ -paths and $(4, 2, 7)$ -paths in G , and at most one 2-vertex can appear on the forth edge due to the absence of $(2, 2, \infty)$ -paths. So $d(f) \leq 4 + 1 < 6$, a contradiction.

Finally, if $d(f^*) = 5$ then $d(f) = 5 < 6$ due to the absence of $(3, 2, \infty)$ -paths, a contradiction.

5.2. Proving the tightness of $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$ and the three weaker descriptions in Theorem 3(F). To see that none of the weakest strengthenings of $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$, which are

- (d1) $\{(2, 2, \infty), (4, 2, 7), (5, 2, 6)\}$,
- (d2) $\{(3, 2, C), (4, 2, 7), (5, 2, 6)\}$ with $C \neq \infty$,
- (d3) $\{(3, 2, \infty), (3, 2, 7), (5, 2, 6)\}$,
- (d4) $\{(3, 2, \infty), (4, 2, 6), (5, 2, 6)\}$,
- (d5) $\{(3, 2, \infty), (4, 2, 7), (4, 2, 6)\}$, and
- (d6) $\{(3, 2, \infty), (4, 2, 7), (5, 2, 5)\}$

is a description, it suffices to consider the graphs H_2^* , H_3^* , and H_7^* .

Рис. 2. The graph H_7^* .

Note that neither (d1) nor (d2) describes H_7^* , while neither (d3) nor (d4) describes H_2^* , and neither (d5) nor (d6) describes H_3^* . This confirms the tightness of $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$.

Checking the tightness of the other three descriptions is similarly based on the graphs H_2^* , H_3^* and H_7^* . For example, the parameter 5 in $\{(5, 2, \infty)\}$ is best possible due to H_3^* , while ∞ is everywhere confirmed by H_7^* . The rest of checking is left to the reader.

5.3. Proving the non-existence of tight descriptions other than $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$, $\{(3, 2, \infty), (5, 2, 7)\}$, $\{(4, 2, \infty), (5, 2, 7)\}$, and $\{(5, 2, \infty)\}$. Suppose $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g = 6$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$.

It follows from the graph H_7^* that D must have an entry, say the first, with $x_1 \geq 3$ and $z_1 = \infty$.

CASE 1. $x_1 \geq 5$. Now $D = \{(5, 2, \infty)\}$ since $\{(5, 2, \infty)\}$ is known to be a tight description.

CASE 2. $x_1 = 4$. It follows from H_2^* , which has no $(4, 2, \infty)$ -paths, that D has an entry, say the second, such that $x_2 \geq 5$ and $z_2 \geq 6$. However, then D coincides with the known tight description $\{(4, 2, \infty), (5, 2, 6)\}$, and we are done.

CASE 3. $x_1 = 3$. It follows from H_3^* , which has no $(3, 2, \infty)$ -paths, that D has an entry, say the second, such that $x_2 \geq 4$ and $z_2 \geq 7$.

SUBCASE 3.1. $x_2 \geq 5$. Here, D is the known tight description $\{(3, 2, \infty), (5, 2, 7)\}$.

SUBCASE 3.2. $x_2 = 4$. It follows from H_3^+ , which has neither $(3, 2, \infty)$ - nor $(4, 2, 7^+)$ -paths, that D has an entry, say the third, such that $x_3 \geq 5$ and $z_3 \geq 6$. This time, our D is the known tight description $\{(3, 2, \infty), (4, 2, 7), (5, 2, 6)\}$, as desired.

REFERENCES

- [1] V.A. Aksenov, O.V. Borodin, A.O. Ivanova, *All tight descriptions of 3-paths in plane graphs with girth at least 9*, Sib. Elektron. Mat. Izv., **15** (2018), 1174–1181. Zbl 1411.05064
- [2] Ts.Ch-D. Batueva, O.V. Borodin, A.O. Ivanova, *All tight descriptions of 4-paths in 3-polytopes with minimum degree 5*, Graphs and Combinatorics, **33**:1 (2017), 53–62. Zbl 1365.05056
- [3] O.V. Borodin, *On the total coloring of planar graphs*, J. Reine Angew. Math., **394** (1989), 180–185. Zbl 0653.05029
- [4] O.V. Borodin, *Joint extension of two Kotzig's theorems on 3-polytopes*, Combinatorica, **13**:1 (1992), 121–125.
- [5] O.V. Borodin, *Structure of neighborhoods of an edge in planar graphs and the simultaneous coloring of vertices, edges and faces*, Math. Notes, **53**:5–6 (1993), 483–489. Zbl 0795.05048
- [6] O.V. Borodin, A.O. Ivanova, *An analogue of Franklin's Theorem*, Discrete Math., **339**:10 (2016), 2553–2556. Zbl 1339.05067
- [7] O.V. Borodin, A.O. Ivanova, *Describing 4-paths in 3-polytopes with minimum degree 5*, Siberian Math. J., **57**:5 (2016), 764–768. Zbl 1355.05091
- [8] O.V. Borodin, A.O. Ivanova, *Describing tight descriptions of 3-paths in triangle-free normal plane maps*, Discrete Math., **338**:11 (2015), 1947–1952. Zbl 1314.05042
- [9] O.V. Borodin, A.O. Ivanova, *New results about the structure of plane graphs: a survey*, AIP Conference Proceedings, **1907**:1 (2017), 030051.
- [10] O.V. Borodin, A.O. Ivanova, *All one-term tight descriptions of 3-paths in normal plane maps without $K_4 - e$* , Discrete Math., **341**:12 (2018), 3425–3433.
- [11] O.V. Borodin, A.O. Ivanova, T.R. Jensen, A.V. Kostochka, M.P. Yancey, *Describing 3-paths in normal plane maps*, Discrete Math., **313**:23 (2013), 2702–2711. Zbl 1280.05026
- [12] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, *Tight descriptions of 3-paths in normal plane maps*, J. Graph Theory, **85**:1 (2017), 115–132. Zbl 1365.05150
- [13] D.W. Cranston, D.B. West, *An introduction to the discharging method via graph coloring*, Discrete Math., **340**:4 (2017), 766–793. Zbl 1355.05104
- [14] Ph. Franklin, *The four-color problem*, Amer. J. Math., **44** (1922), 225–236. JFM 48.0664.02
- [15] S. Jendrol', *A structural property of convex 3-polytopes*, Geom. Dedicata, **68**:1 (1997), 91–99. Zbl 0893.52007
- [16] S. Jendrol', M. Maceková, *Describing short paths in plane graphs of girth at least 5*, Discrete Math., **338**:2 (2015), 149–158. Zbl 1302.05040

- [17] S. Jendrol', M. Maceková, M. Montassier, R. Soták, *Optimal unavoidable sets of types of 3-paths for planar graphs of given girth*, Discrete Math., **339**:2 (2016), 780–789. Zbl 1327.05081
- [18] S. Jendrol', M. Maceková, M. Montassier, R. Soták, *3-paths in graphs with bounded average degree*, Discuss. Math. Graph Theory, **36**:2 (2016), 339–353. Zbl 1338.05054
- [19] S. Jendrol', M. Maceková, R. Soták, *Note on 3-paths in plane graphs of girth 4*, Discrete Math., **338**:9 (2015), 1643–1648. Zbl 1311.05042
- [20] S. Jendrol', H.-J. Voss, *Light subgraphs of graphs embedded in the plane — a survey*, Discrete Math., **313**:4 (2013), 406–421. Zbl 1259.05045
- [21] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann., **58** (1904), 413–426. JFM 35.0511.01

OLEG VENIAMINOVICH BORODIN
SOBOLEV INSTITUTE OF MATHEMATICS,
4, KOPTYUGA AVE.,
NOVOSIBIRSK, 630090, RUSSIA
E-mail address: brdnoleg@math.nsc.ru

ANNA OLEGOVNA IVANOVA
AMMOSOV NORTH-EASTERN FEDERAL UNIVERSITY,
48, KULAKOVSKOGO STR.,
YAKUTSK, 677000, RUSSIA
E-mail address: shmgnanna@mail.ru