Math-Net.Ru

All Russian mathematical portal

Yuri V. Lytkin, On finite groups isospectral to the simple group $S_4(3)$, *Sib. Èlektron. Mat. Izv.*, 2019, Volume 16, 1561–1566

DOI: <https://doi.org/10.33048/semi.2019.16.107>

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use

<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 3.93.64.190

April 26, 2023, 11:36:23

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 16, стр. 1561–1566 (2019)

DOI 10.33048/semi.2019.16.107

УДК 512.5

MSC 20D99

ON FINITE GROUPS ISOSPECTRAL TO THE SIMPLE GROUP

 $S_4(3)$

YURI V. LYTKIN

ABSTRACT. The *spectrum* of a finite group is the set of its element orders. A finite group G is called *critical* with respect to a subset ω of natural numbers if ω coincides with the spectrum of G and does not coincide with the spectra of proper sections of G . We study the structure of finite groups with the same spectrum as the simple symplectic group $PSp(4, 3)$. In particular, we describe groups critical with respect to the spectrum of $PSp(4, 3)$.

Keywords: finite group, spectrum, critical group, nonabelian simple group.

1. INTRODUCTION

In this work we consider only finite groups. Let G be a group. Denote by $\omega(G)$ the *spectrum* of G , i. e. the set of all element orders of G . We call groups *isospectral* if their spectra coincide.

We say that G is *recognizable* (by spectrum in the class of finite groups) if up to isomorphism G is the only group with the spectrum $\omega(G)$. It is *almost recognizable* if there exist only finitely many pairwise nonisomorphic groups isospectral to G . Otherwise it is called *unrecognizable*. The necessary and sufficient condition for a group to be unrecognizable by spectrum was established in [1]. A group is unrecognizable if and only if it is isospectral to a group containing a nontrivial solvable normal subgroup.

As a result of almost thirty years of research it was established that, as a general rule, finite simple groups are almost recognizable. More precisely, all nonabelian

LYTKIN, Y.V., ON FINITE GROUPS ISOSPECTRAL TO THE SIMPLE GROUP $S_4(3)$.

© 2019 LYTKIN Y.V.

The reported study was funded by Russian Science Foundation according to the research project No. 18-71-10007.

Received March, 3, 2019, published October, 31, 2019.

simple alternating groups of degree $n \neq 6, 10$ are recognizable [2] and so are all simple sporadic groups except J_2 [3]. All exceptional groups of Lie type except ${}^3D_4(2)$ are almost recognizable [4]. All simple classical groups of Lie type of a sufficiently large dimension are almost recognizable [5].

Thus in order to solve the problem of recognizability by spectrum for simple groups it is relevant to study the groups isospectral to unrecognizable simple groups, which is the purpose of this research. The list of all currently known unrecognizable nonabelian simple groups can be found in [6].

It turns out that when studying the structure of groups isospectral to unrecognizable groups a very handy notion is one of a *critical* group that was introduced in [1]. Let ω be a finite subset of the natural numbers. A group G is said to be *critical* with respect to ω (or ω -critical) if ω coincides with the spectrum of G and does not coincide with the spectra of proper *sections* of G (which are homomorphic images of subgroups of G).

Critical groups isospectral to unrecognizable simple groups have been studied before in [7, 8, 9]. Described in these works are all groups critical with respect to the spectra of finite simple alternating and sporadic groups as well as the exceptional groups of Lie type. Also in [7] there is a complete description of critical groups isospectral to the linear group $L_3(3)$, and in [9] there is a description of groups (critical, in particular) isospectral to the unitary group $U_3(3)$.

The structure of groups isospectral to the simple symplectic groups $S_4(q) = PSp(4, q)$ has already been studied in [6], albeit only in the case $q > 3$. (Note that in the abstract and the main theorem in [6] it is mistakenly stated that the case $q = 5$ is not considered. In fact the theorem is proved for all $q > 3$.)

In this work we study the structure of groups isospectral to the group $S_4(3)$. The reason this group was excluded in [6] is that it possesses some unique properties among the groups $S_4(q)$ (and finite simple groups in general). For instance, $S_4(3)$ is one of the two nonabelian simple groups (along with $U_3(3)$) that can be isospectral to a 2-Frobenius group ([10]). Note that in both cases there exist such 2-Frobenius groups (see [11, 12]).

Theorem. *Let G be a finite group isospectral to $S_4(3)$. Then G is either $S_4(3)$, or a 2-Frobenius group, or an extension of a nilpotent group K by the alternating group A_5 or the symmetric group S_5 .*

Moreover, the following holds.

1. *If G is a 2-Frobenius group then $G = ABC$, where A is a group of order 3^n , where n is divisible by 4, and BC is a Frobenius group of order 20.*
2. *If $G/K \simeq A_5$, then K is a direct product of a 2-group A and a 3-group B , and both of these groups are nontrivial. Every G -chief factor of A as an A_5 -module is isomorphic to an irreducible A_5 -module V over $GF(2)$, and the G -chief factors of B as A_5 -modules are isomorphic to an absolutely irreducible A_5 -module W over $GF(3)$.*

Let $H = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$. Then $H \simeq A_5$, and the action of H on V , given an appropriate base, can be defined by matrices

$$[a] = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

and the action of H on W , given an appropriate base, can be defined by matrices

$$[a] = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad [b] = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}.$$

There are exactly two critical groups of this type. Both are extensions of $V \times W$ by H , but one of them is split while the other is not.

3. If $G/K \simeq S_5$ then K is a 3-group and G is a split extension of K by S_5 . Every G -chief factor of K as an S_5 -module is isomorphic to an absolutely irreducible S_5 -module V over $GF(3)$.

Let $H = \langle a, b \mid a^2 = b^4 = (ab)^5 = [a, b]^3 = 1 \rangle$. Then $H \simeq S_5$, and the action of H on V , given an appropriate base, can be defined by the matrices

$$[a] = \begin{pmatrix} -1 & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & -1 \end{pmatrix}, \quad [b] = \begin{pmatrix} -1 & \cdot & \cdot & -1 \\ 1 & 1 & 1 & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The group $V \rtimes H$ is up to isomorphism the only critical group of this type.

Note that there is an example of a 2-Frobenius group isospectral to $S_4(3)$ that can be found in [11]. It can be shown that this group is critical, but the exact number of pairwise nonisomorphic critical groups of this type is unknown.

2. PRELIMINARY RESULTS

The spectrum $\omega(G)$ of a group G defines the *Gruenberg-Kegel graph* (or the *prime graph*) $GK(G)$ of G . The vertices of this graph are the elements of $\pi(G)$, i. e. the set of all prime divisors of the order of G . Distinct vertices p and q are adjacent if $pq \in \omega(G)$. We denote by $s(G)$ the number of connected components of $GK(G)$, $\pi_i(G)$ denotes the i -th connected component, $i = 1, \dots, s$. If the order of G is even, we suppose that $2 \in \pi_1(G)$. Let $\omega_i(G)$ be the set of all $n \in \omega(G)$ such that every prime divisor of n is contained in $\pi_i(G)$.

Lemma 1 (Gruenberg, Kegel, [13]). *If G is a finite group with a disconnected graph $GK(G)$ then one of the following holds.*

1. $s(G) = 2$ and G is a Frobenius group, i. e. G has a nontrivial normal nilpotent Hall subgroup A and $C_G(a) \leq A$ for every nonidentity $a \in A$.
2. $s(G) = 2$ and G is a 2-Frobenius group, i. e. $G = ABC$, where the groups A and AB are normal in G and AB and BC are Frobenius groups with respective kernels A and B and complements B and C .
3. There exists a nonabelian simple group P such that

$$P \leq \overline{G} = G/K \leq \text{Aut } P$$

for some nilpotent normal $\pi_1(G)$ -subgroup K of G and $\overline{G}/P$ is a $\pi_1(G)$ -group. Moreover, the graph $GK(P)$ is disconnected, $s(P) \geq s(G)$, and for every integer i , $2 \leq i \leq s(G)$, there exists j , $2 \leq j \leq s(P)$ such that $\omega_i(G) = \omega_j(P)$.

In particular, G possesses at most one unsolvable composition factor.

Lemma 2 ([14, Lemma 3]). *If G is a 2-Frobenius group ABC , then B is cyclic of odd order, C is cyclic and G contains an element of order*

$$|C| \cdot \prod_{r \in \pi(A)} r.$$

Lemma 3 ([15, Lemma 9]). *Let H be an extension of a nontrivial 3-group V by the symmetric group S_6 . Then H contains an element of order 15 or 18.*

Lemma 4 ([16, Lemma 1]). *Let G be a finite group, N a normal subgroup of G and let G/N be a Frobenius group with a kernel F and a cyclic complement H . If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$ then G contains an element of order $p|H|$ for some $p \in \pi(N)$.*

Proposition 1 ([6, Proposition 1]). *Let G be a finite group with a disconnected graph $GK(G)$ and suppose that for some nontrivial nilpotent normal $\pi_1(G)$ -subgroup K of G the quotient group G/K is simple. Then for every subgroup H of G the group $KC_H(K)/K$ is trivial.*

Lemma 5 (Gaschütz, [17, p. 121]). *Let G be a finite group and A an abelian p -subgroup of G . Assume that A is complemented in a Sylow p -subgroup of G . Then A is complemented in G .*

3. PROOF OF THE THEOREM

Denote by ω the spectrum of $S_4(3)$. It follows from [18] that ω consists of all divisors of numbers 5, 9, and 12.

Let G be a group isospectral to $S_4(3)$. By Lemma 1, G is either a Frobenius group, or a 2-Frobenius group, or an extension by a nilpotent $\{2, 3\}$ -group K of some group that lies between P and $\text{Aut } P$ for some nonabelian simple group P , i. e. $P \leq G/K \leq \text{Aut } P$.

It follows from [10] that G cannot be a Frobenius group. Suppose that G is a 2-Frobenius group ABC . We are going to use Lemma 2. Since there are no elements of order $5 \cdot t$ in G for $t > 1$, it follows that 5 cannot divide the order of A or C , so $|B| = 5$. As BC is a Frobenius group, the order of C must divide $|B| - 1$, so C is of order 2 or 4.

If $|C| = 2$, then since G has an element of order 12, there must be an element of order 2 in A . In any case, there is also an element of order 9 in A , and since A is nilpotent, we conclude that there is an element of order 18 in G , which is a contradiction. Thus, C is of order 4 and A is a 3-group.

Finally, since AB is also a Frobenius group, the order of A must be congruent to 1 modulo 5, so $|A| = 3^n$, where n is divisible by 4.

We note that such group exists and is described in [11]. It can be shown that the group in this work is critical, but the exact number of pairwise nonisomorphic ω -critical groups of this type is unknown.

Consider the last case, i. e. $P \leq G/K \leq \text{Aut } P$ for some simple group P and a nilpotent normal $\{2, 3\}$ -subgroup K of G . It follows from [19] that P is one of the groups A_5 , A_6 or $S_4(3)$.

First, suppose that $P \simeq A_6$. The group A_6 contains a Frobenius subgroup of order $9 \cdot 4$, so by Lemma 4 the set $\pi(K)$ cannot contain 2 and therefore K is a 3-group. Also, note that every nontrivial automorphic extension of A_6 apart from S_6 contains an element of order 8, so the group G/K cannot be one of those. Finally, it follows from Lemma 3 that G/K cannot be S_6 .

Thus, the only possible case here is $G/K \simeq A_6$. Since there are no elements of order 15 in G , the elements of order 5 from A_6 must act on K without nontrivial fixed points. It can be checked using [20, 21] that there is only one irreducible representation of A_6 in characteristic 3 that satisfies this property. In this case the elements of order 4 from A_6 also act without nontrivial fixed points on K , thus

there is no element of order 12 in G ; a contradiction. Therefore, the case $P \simeq A_6$ is impossible.

Since the group $S_4(3)$ contains a subgroup isomorphic to S_6 , the case $P \simeq S_4(3)$ is also not possible, except for trivial K . Thus, either $G \simeq S_4(3)$, or G/K is one of the groups A_5 or S_5 .

Suppose that $G/K \simeq S_5$. The group S_5 contains a Frobenius subgroup of order $5 \cdot 4$, so again by Lemma 4 the subgroup K is a 3-group. Also, the elements of order 5 from S_5 act without nontrivial fixed points on K .

Let a be an element of order 2 from G such that $\bar{a} = Ka$ is a transposition from S_5 . Let $\bar{x} = Kx$ be an element of order 3 from S_5 that commutes with $\bar{a}$. Then the coset Kx is a -invariant.

Since $|Kx|$ is odd, there exists an element $c \in Kx$ such that $c^a = c$. If c is of order 9, then there is an element of order 18 in G , which is impossible. Thus, $|c| = 3$ and therefore by Lemma 5 the subgroup K is complemented in G by a subgroup $S \simeq S_5$.

We can assume that $a = (12)$ and $c = (345)$. Let $s = ac$. Since there are no elements of order 18 in G , for every element $v \in K$ the equality $(vs)^6 = 1$ must hold. There exists only one irreducible S_5 -module V in characteristic 3 such that $1 + [s] + \dots + [s]^5 = 0$, where $[s]$ is the matrix of s . Put $b = (2345)$, such that $S_5 \simeq \langle a, b \mid a^2 = b^4 = (ab)^5 = [a, b]^3 = 1 \rangle$. It can be checked using [20, 21] that the action of S_5 on V , given a suitable base, can be defined by the matrices

$$[a] = \begin{pmatrix} -1 & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & -1 \end{pmatrix}, \quad [b] = \begin{pmatrix} -1 & \cdot & \cdot & -1 \\ 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix}.$$

It can also be checked directly that $\omega(V \rtimes S_5) = \omega$. Since V is the only irreducible S_5 -module that satisfies all the necessary properties, it follows that the group $V \rtimes S_5$ is critical and, furthermore, up to isomorphism it is the only ω -critical group of this type.

Finally, assume that $G/K \simeq A_5$. Since $\mu(A_5) = \{2, 3, 5\}$, K cannot be a p -group in this case. Let P and R be the Sylow 2- and 3-subgroups of K , respectively. Since K is nilpotent, these are both normal in K and $K = PR$. It is clear that the group G/R must contain an element of order 4 and the group G/P must contain an element of order 9. Also, the elements of order 3 from A_5 cannot fix any nontrivial points in P , since otherwise we multiply this fixed point by an element of order 9 from G and get an element of order 18. Finally, obviously, the elements of order 5 from A_5 cannot fix any nontrivial points in P or R .

It can be checked using [20, 21] that there is only one irreducible A_5 -module V over the field of order 2 and only one irreducible A_5 -module W over the field of order 3 that satisfy all the necessary conditions. Put $a = (12)(34)$, $b = (135)$, such that $A_5 \simeq \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$. The action of A_5 on V is then defined by matrices

$$[a] = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix},$$

and the action of A_5 on W is defined by matrices

$$[a] = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad [b] = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

It only remains to check that an extension of $V \times W$ by A_5 contains an element of order 12. To see this, note that the elements of order 2 from A_5 act on W with

fixed points. Let s be an element of order 2 from A_5 , and let $v \in V$ and $w \in W$ be such elements that vs is of order 4 and ws is of order 6. Then vws is of order 12. Therefore, an extension of $V \times W$ by A_5 is isospectral to $S_4(3)$.

Finally, note that there is a unique split extension of V by A_5 , and there is a nonsplit extension of W by A_5 as well as a split one. Thus, there are exactly two nonisomorphic ω -critical groups of this type.

REFERENCES

- [1] V.D. Mazurov, W.J. Shi, *A criterion of unrecognizability by spectrum for finite groups*, Algebra Logic, **51**:2 (2012), 160–162. MR2986582
- [2] I.B. Gorshkov, *Recognizability of alternating groups by spectrum*, Algebra Logic, **52**:1 (2013), 41–45. MR3113477
- [3] V.D. Mazurov, W.J. Shi, *A note to the characterization of sporadic simple groups*, Algebra Colloq., **5**:3 (1998), 285–288. MR1679564
- [4] A.V. Vasil'ev, A.M. Staroletov, *Almost recognizability by spectrum of simple exceptional groups of Lie type*, Algebra Logic, **53**:6 (2014), 433–449. MR3408302
- [5] M.A. Grechkoseeva, A.V. Vasil'ev, *On the structure of finite groups isospectral to finite simple groups*, J. Group Theory, **18**:5 (2015), 741–759. MR3393413
- [6] Y.V. Lytkin, *On finite groups isospectral to the simple groups $S_4(q)$* , Siberian electronic mathematical reports, **15** (2018), 570–584. MR3814161
- [7] Y.V. Lytkin, *On groups critical with respect to a set of natural numbers*, Siberian electronic mathematical reports, **10** (2013), 666–675. MR3262316
- [8] Y.V. Lytkin, *Groups critical with respect to the spectra of alternating and sporadic groups*, Sib. Math. J., **56**:1 (2015), 101–106. MR3407944
- [9] Y.V. Lytkin, *On finite groups isospectral to $U_3(3)$* , Sib. Math. J., **58**:4 (2017), 633–643. MR3762357
- [10] M.R. Aleeva, *On finite simple groups with the set of element orders as in a Frobenius group or a double Frobenius group*, Math. Notes, **73**:3 (2003), 299–313. MR1992593
- [11] A.V. Zavarnitsine, *A solvable group isospectral to $S_4(3)$* , Sib. Math. J., **51**:1 (2010), 20–24. MR2654518
- [12] V.D. Mazurov, *2-Frobenius groups isospectral to the simple group $U_3(3)$* , Sib. Math. J., **56**:6 (2015), 1108–1113. MR3492914
- [13] J.S. Williams, *Prime graph components of finite groups*, J. Algebra, **69**:2 (1981), 487–513. MR617092
- [14] V.D. Mazurov, M.C. Xu, H.P. Cao, *Recognition of the finite simple groups $L_3(2^m)$ and $U_3(2^m)$ from the orders of their elements*, Algebra Logic, **39**:5 (2000), 324–334. MR1805756
- [15] V.D. Mazurov, *Recognition of finite simple groups $S_4(q)$ by their element orders*, Algebra Logic, **41**:2 (2002), 93–110. MR1922988
- [16] V.D. Mazurov, *Characterizations of finite groups by sets of orders of their elements*, **36**:1 (1997), 23–32. MR1454690
- [17] B. Huppert, *Endliche Gruppen*, Springer Verlag, 1979. Zbl 0412.20002
- [18] B. Srinivasan, *The characters of the finite symplectic group $Sp(4, q)$* , Trans. Amer. Math. Soc., **131** (1968), 488–525. MR220845
- [19] A.V. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Siberian electronic mathematical reports, **6** (2009), 1–12. MR2586673
- [20] GAP: Groups, algorithms, and programming (<http://www/gap-system.org>).
- [21] ATLAS of Finite Group Representations (<http://brauer.maths.qmul.ac.uk/Atlas/v3/>).

YURI VSEVOLODOVICH LYTKIN
 SOBOLEV INSTITUTE OF MATHEMATICS,
 4, KOPTYUGA AVE.,
 NOVOSIBIRSK, 630090, RUSSIA
 E-mail address: jurasicus@gmail.com