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ASYMPTOTIC SOLUTIONS OF INTEGRO-DIFFERENTIAL
EQUATIONS WITH PARTIAL DERIVATIVES AND WITH
RAPIDLY VARYING KERNEL

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ABSTRACT. In the paper, ideas of the Lomov regularization method are generalized to the Cauchy problem for a singularly perturbed partial integro-differential equation in the case when the integral term contains a rapidly varying kernel. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

Keywords: singularly perturbed, partial integro differential equation, regularization of an integral, solvability of iterative problems.

1. INTRODUCTION

In the paper, we consider the Cauchy problem for the partial integro-differential equation

$$\varepsilon \frac{\partial y(x, t, \varepsilon)}{\partial x} = a(x)y(x, t, \varepsilon) + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s)y(s, t, \varepsilon) ds + h(x, t), \quad (1)$$

$$y(0, t, \varepsilon) = y^0(t) \quad ((x, t) \in [0, 1] \times [0, 1])$$

with the rapidly varying kernel. The problem of constructing a regularized asymptotic solution of the problem (1) [1] is posed. Earlier, in [3-7], systems for ordinary integro-differential equations were mainly considered. In contrast to these works, in this paper we consider an partial integro-differential equation. Construction of asymptotical solutions for partial integro-differential equations with rapidly varying kernels from the position of the regularization method are considered in [8-9]. However, in [9], the case of a purely imaginary spectral value $\mu(x)$ of the kernel

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of an integral operator was considered. This article allows purely imaginary $\mu(x) = \pm i\omega(x)$ ($\omega(x) > 0$).

2. Regularization of the problem. We suppose that the following conditions are satisfied:

- (1) functions $a(x) \in C^\infty([0, T], C^1)$, $\mu(x) \in C^\infty([0, T], C^1)$,
 $h(x, t) \in C^\infty([0, 1] \times [0, 1], C^1)$, the kernel $K(x, t, s)$ belongs to the space
 $K(x, t, s) \in C^\infty(\{0 < s < x < 1, 0 \leq x, t \leq 1\}, C^1)$;
- (2) $a(x), \mu(x) \neq 0$, $a(x) \neq \mu(x)$ ($\forall x \in [0, 1]$);
- (3) $Re\ a(x) \leq 0$, $Re\ \mu(x) \leq 0$, ($\forall x \in [0, 1]$);

Introduce the regularized variables:

$$\tau_1 = \frac{1}{\varepsilon} \int_0^x a(\theta) d\theta \equiv \frac{\varphi_1(x)}{\varepsilon}, \quad \tau_2 = \frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta \equiv \frac{\varphi_2(x)}{\varepsilon}. \quad (2)$$

We set the following problem for the function $\tilde{y}(x, t, \tau, \varepsilon)$:

$$\begin{aligned} \varepsilon \frac{\partial \tilde{y}(x, t, \tau, \varepsilon)}{\partial x} + a(x) \frac{\partial \tilde{y}}{\partial \tau_1} + \mu(x) \frac{\partial \tilde{y}}{\partial \tau_2} - a(x) \tilde{y} - \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) \times \\ \times \tilde{y}(s, t, \frac{\varphi(s)}{\varepsilon}, \varepsilon) ds + h(x, t), \quad \tilde{y}(0, t, 0, \varepsilon) = y^0(t). \end{aligned} \quad (3)$$

The connection between the problem (3) and the original problem (1) is as follows: if $\tilde{y} = \tilde{y}(x, t, \tau, \varepsilon)$ is the solution of the problem (3), then its restriction $y(x, t, \tau) \equiv \tilde{y}(x, t, \frac{\varphi(x)}{\varepsilon}, \varepsilon)$ on the regularizing functions (2) will be obviously the exact solution of the original problem (1). However, the problem (3) can not be considered as completely regularized since the integral operator

$$J\tilde{y} = \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) \tilde{y}(s, t, \frac{\varphi(s)}{\varepsilon}, \varepsilon) ds$$

is not regularized in it.

To regularize the integral term $J\tilde{y}$ of the problem (3), we need to introduce the space M_ε asymptotically invariant with respect to the operator J . This is done in this way. It is introduced the class U of solutions of iterative problems (see below):

$$U = \{y(x, t, \tau) : y = y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} + y_0(x, t), \quad y_j(x, t) \in C^\infty([0, 1] \times [0, 1])\},$$

and then the restriction of this class is taken at $\tau = \varphi(x)/\varepsilon$. This will be the space M_ε . To justify this fact, we need to show that the image $Jy(x, t, \tau)$ of the integral operator J on the element of the space U can be represented in the form of a power series

$$\sum_{k=0}^{\infty} \varepsilon^k \left(y_1^{(k)}(x, t)e^{\tau_1} + y_2^{(k)}(x, t)e^{\tau_2} + y_0^{(k)}(x, t) \right)$$

convergent asymptotically for $\varepsilon \rightarrow +0$ (uniformly with respect to $(x, t) \in [0, 1] \times [0, 1]$). Deal with this procedure.

The image of the integral operator J on an arbitrary element $y(x, t, \tau)$ of the space U has the form

$$\begin{aligned} Jy(x, t, \varepsilon) = \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_1(s, t) e^{\frac{\varphi_1(s)}{\varepsilon}} ds + \\ + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_2(s, t) e^{\frac{\varphi_2(s)}{\varepsilon}} ds + \\ + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_0(s, t) ds. \end{aligned} \quad (4)$$

Apply the operation of integration by parts to the each term.

$$\begin{aligned}
 & \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_1(s, t) e^{\frac{\varphi_1(s)}{\varepsilon}} ds = \\
 & = e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \int_0^x K(x, t, s) y_1(s, t) e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} ds = \\
 & = \varepsilon e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \int_0^x \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)} d \left(e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} \right) = \\
 & = \varepsilon e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \left[\frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)} e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} \right]_{s=0}^{s=x} - \\
 & \quad - \int_0^x \left(\frac{\partial}{\partial s} \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)} \right) e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} ds = \\
 & = \varepsilon \left[\frac{K(x, t, x) y_1(x, t)}{a(x) - \mu(x)} e^{\frac{1}{\varepsilon} \int_0^x a(\theta) d\theta} - \frac{K(x, t, 0) y_1(0, t)}{a(0) - \mu(0)} e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \right] - \\
 & \quad - \varepsilon e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \int_0^x \left(\frac{\partial}{\partial s} \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)} \right) e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} ds.
 \end{aligned}$$

Introduce the notation $I_1^0(K(x, t, s) y_1(s, t)) \equiv \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)}$. Then the previous result of transformations can be written in the form

$$\begin{aligned}
 & \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_1(s, t) e^{\frac{\varphi_1(s)}{\varepsilon}} ds = \\
 & = \varepsilon [I_1^0(K(x, t, s) y_1(s, t))_{s=x} e^{\tau_1} - I_1^0(K(x, t, s) y_1(s, t))_{s=0} e^{\tau_2}] - \\
 & \quad - \varepsilon e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \int_0^x \frac{\partial}{\partial s} (I_1^0(K(x, t, s) y_1(s, t))) e^{\frac{1}{\varepsilon} \int_0^s [a(\theta) - \mu(\theta)] d\theta} ds
 \end{aligned}$$

where $\tau_1 = \varphi_1(x, \varepsilon)$. Continuing this process, we obtain the series

$$\begin{aligned}
 & \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_1(s, t) e^{\frac{\varphi_1(s)}{\varepsilon}} ds = \sum_{k=0}^{\infty} (-1)^k \varepsilon^{k+1} \\
 & \quad \times [(I_1^k(K(x, t, s) y_1(s, t)))_{s=x} e^{\tau_1} - (I_1^k(K(x, t, s) y_1(s, t)))_{s=0} e^{\tau_2}]
 \end{aligned} \quad (5)$$

where $\tau_1 = \varphi_1(x, \varepsilon)$, and the operators I_1^k have the form:

$$\begin{aligned}
 & I_1^0(K(x, t, s) y_1(s, t)) \equiv \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)}, \\
 & I_1^1(K(x, t, s) y_1(s, t)) \equiv \frac{1}{a(s) - \mu(s)} \frac{\partial}{\partial s} I_1^0(K(x, t, s) y_1(s, t)), \dots, \\
 & I_1^m(K(x, t, s) y_1(s, t)) \equiv \frac{1}{a(s) - \mu(s)} \frac{\partial}{\partial s} I_1^{m-1}(K(x, t, s) y_1(s, t)), m \geq 1.
 \end{aligned} \quad (6)$$

The second summand in (4) is transformed as follows:

$$\begin{aligned}
 & \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_2(s, t) e^{\frac{\varphi_2(s)}{\varepsilon}} ds \equiv \\
 & \equiv \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_2(s, t) e^{\frac{1}{\varepsilon} \int_0^s \mu(\theta) d\theta} ds = \\
 & = e^{\frac{1}{\varepsilon} \int_0^x \mu(\theta) d\theta} \int_0^x K(x, t, s) y_2(s, t) ds = \left(\int_0^x K(x, t, s) y_2(s, t) ds \right) e^{\tau_2}, \\
 & \tau_2 = \frac{\varphi_2(t)}{\varepsilon}.
 \end{aligned} \quad (7)$$

And, finally, we have for the last summand in (4)

$$\begin{aligned}
 & \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_0(s, t) ds = \varepsilon \int_0^x \frac{K(x, t, s) y_0(s, t)}{-\mu(s)} d e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} = \\
 & = \varepsilon \frac{K(x, t, s) y_0(s, t)}{-\mu(s)} e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} \Big|_{s=0}^{s=x} - \varepsilon \int_0^x \frac{\partial}{\partial s} \left(\frac{K(x, t, s) y_0(s, t)}{-\mu(s)} \right) e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} ds = \\
 & = \varepsilon \left[\frac{K(x, t, x) y_0(x, t)}{-\mu(x)} - \frac{K(x, t, 0) y_0(0, t)}{-\mu(0)} e^{\tau_2} \right] + \\
 & + \varepsilon \int_0^x \left(\frac{\partial}{\partial s} \frac{K(x, t, s) y_0(s, t)}{\mu(s)} \right) e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} ds = \\
 & = \sum_{k=0}^{\infty} \varepsilon^{k+1} \left[(I_0^k(K(x, t, s) y_0(s, t)))_{s=0} e^{\tau_2} - I_0^k(K(x, t, s) y_0(s, t))_{s=x} \right]
 \end{aligned} \tag{8}$$

where the following operators are introduced:

$$\begin{aligned}
 I_0^0(K(x, t, s) y_0(s, t)) & \equiv \frac{K(x, t, s) y_0(s, t)}{-\mu(s)}, \\
 I_0^1(K(x, t, s) y_0(s, t)) & \equiv \frac{1}{-\mu(s)} \frac{\partial}{\partial s} I_0^0(K(x, t, s) y_0(s, t)), \dots, \\
 I_0^m(K(x, t, s) y_0(s, t)) & \equiv \frac{1}{-\mu(s)} \frac{\partial}{\partial s} I_0^{m-1}(K(x, t, s) y_0(s, t)), m \geq 1.
 \end{aligned} \tag{9}$$

The asymptotic convergence of the series (5) and (8) is proved in the same way as the analogous statement in [1]. Let now $\tilde{y}(x, t, \tau, \varepsilon)$ be an arbitrary function continuous with respect to $(x, t, \tau) \in [0, 1] \times [0, 1] \times \{Re a(x)\}$ and having the asymptotic expansion

$$\tilde{y}(x, t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(x, t, \tau), \quad y_k(x, t, \tau) \in U, \tag{10}$$

convergent at $\varepsilon \rightarrow +0$ (uniformly with respect to $(x, t, \tau) \in [0, 1] \times [0, 1] \times \{Re a(x)\}$). Introduce the operators $R_m : U \rightarrow U$ acting on the each element $y(x, t, \tau)$ of the space U according the law:

$$\begin{aligned}
 Ry(x, t, \tau) & \equiv R_0(y_1(x, t) e^{\tau_1} + y_2(x, t) e^{\tau_2} + y_0(x, t)) = e^{\tau_2} \int_0^x K(x, t, s) y_2(s, t) ds, \\
 R_{k+1}y(x, t, \tau) & = \\
 & = (-1)^k \left[(I_1^k(K(x, t, s) y_1(s, t)))_{s=x} e^{\tau_1} - (I_1^k(K(x, t, s) y_1(s, t)))_{s=0} e^{\tau_2} \right] + \\
 & + \left[(I_0^k(K(x, t, s) y_0(s, t)))_{s=0} e^{\tau_2} - I_0^k(K(x, t, s) y_0(s, t))_{s=x} \right]
 \end{aligned} \tag{11}$$

where the operators I_1^k have the form (6), and operators I_0^k have the form (9), $k \geq 0$. Operators R_m are called the operators of order (with respect to ε , since, applying to the function $y(x, t, \tau)$, these operators separate terms of the order ε^m). It is natural to define the extended operator for the integral operator I in the following way.

Definition 1. The operator $\tilde{I}$ acting on each function $\tilde{y}(x, t, \tau, \varepsilon)$ of the form (10) by the law

$$\tilde{I}\tilde{y} \equiv \tilde{J} \left(\sum_{k=0}^{\infty} \varepsilon^k y_k(x, t, \tau) \right) = \sum_{r=0}^{\infty} \varepsilon^r \left(\sum_{k=0}^r R_{r-k} y_k(x, t, \tau) \right) \tag{12}$$

is said to be a formal extension of the operator I .

Now one can write the completely regularized problem (with respect to the original problem (1)):

$$\begin{aligned}
 L_\varepsilon \tilde{y}(x, t, \tau, \varepsilon) & \equiv \varepsilon \frac{\partial \tilde{y}}{\partial x} + a(x) \frac{\partial \tilde{y}}{\partial \tau_1} + \mu(x) \frac{\partial \tilde{y}}{\partial \tau_2} - \\
 & - a(x) \tilde{y} - \tilde{J}\tilde{y} = h(x, t), \quad \tilde{y}(0, t, 0, \varepsilon) = y^0(t).
 \end{aligned} \tag{13}$$

where $\tilde{y}(x, t, \tau, \varepsilon)$ is the series (10).

3. Solvability of iterative problems. Substituting the series (10) into (13) and equating the coefficients of the same powers of ε , we obtain the following iterative problems:

$$Ly_0(x, t, \tau) \equiv a(x) \frac{\partial y_0}{\partial \tau_1} + \mu(x) \frac{\partial y_0}{\partial \tau_2} - a(x)y_0 - R_0 y_0 = h(x, t), \quad y_0(0, t, 0) = y^0(t); \quad (14_0)$$

$$Ly_1(x, t, \tau) = -\frac{\partial y_0}{\partial x} + R_1 y_0, \quad y_1(0, t, 0) = 0; \quad (14_1)$$

.....

$$Ly_k(x, t, \tau) = -\frac{\partial y_{k-1}}{\partial x} + R_1 y_{k-1} + \dots + R_k y_0, \quad y_k(0, t, 0) = 0, \quad k \geq 1. \quad (14_k)$$

Each iterative problem (14_k) has the form

$$Ly(x, t, \tau) \equiv a(x) \frac{\partial y}{\partial \tau_1} + \mu(x) \frac{\partial y}{\partial \tau_2} - a(x)y - R_0 y = H(x, t, \tau), \quad y(0, t, 0) = y_*(t), \quad (15)$$

where $H(x, t, \tau) = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U$, $y_*(x) \in C^\infty[0, 1]$ are known functions, and $R_0 y$ is the operator:

$$R_0 y(x, t, \tau) \equiv R_0(y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} + y_0(x, t)) = e^{\tau_2} \int_0^x K(x, t, s)y_2(s, t)ds.$$

Let's try to solve the problem (15). Substituting the element

$$y(x, t, \tau) = y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} + y_0(x, t)$$

of the space U into (15), we obtain

$$a(x)y_1(x, t)e^{\tau_1} + \mu(x)y_2(x, t)e^{\tau_2} - a(x)y_1(x, t)e^{\tau_1} - \mu(x)y_2(x, t)e^{\tau_2} + a(x)y_0(x, t)) - e^{\tau_2} \int_0^x K(x, t, s)y_2(s, t)ds = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t).$$

Equating separately the free terms and coefficients for identical exponentials, we obtain equations:

$$\begin{aligned} -a(x)y_0(x, t) &= h_0(x, t), \\ [a(x) - a(x)]y_1(x, t) &= h_1(x, t), \\ [\mu(x) - a(x)]y_2(x, t) - \int_0^x K(x, t, s)y_2(s, t)ds &= h_2(x, t). \end{aligned} \quad (16)$$

The first equation of (16) has the unique solution $y_0(x, t) = -\frac{h_0(x, t)}{a(x)}$. For solvability of the second system in (16), it is necessary and sufficient the fulfillment of the condition

$$h_1(x, t) \equiv 0, \quad (\forall(x, t) \in [0, 1] \times [0, 1]).$$

The third equation of (16) is a Volterra equation of the second kind with the smooth kernel $G(x, t, s) = [\mu(x) - a(x)]^{-1}K(x, t, s)$, that's why it has the unique solution in the space $C^\infty([0, 1] \times [0, 1])$. Thus, we prove the following

Theorem 1. *Let the right-hand side in (15)*

$$H(x, t, \tau) = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U,$$

and conditions (i)–(ii) be satisfied. Then the equation (15) is solvable in the space U if and only if

$$h_1(x, t) \equiv 0, \quad (\forall(x, t) \in [0, 1] \times [0, 1]). \quad (17)$$

Remark. If (17) is satisfied, then (15) has the following solution in U :

$$y(x, t, \tau) = \alpha(x, t)e^{\tau_1} - \frac{h_0(x, t)}{a(x)} + \left[\int_0^x R(x, t, s)[\mu(s) - a(s)]^{-1} h_2(s, t) ds + [\mu(x) - a(x)]^{-1} h_2(x, t) \right] e^{\tau_2}, \quad (18)$$

where $R(x, t, s)$ —is the resolvent of the kernel $G(x, t, s) = [\mu(x) - a(x)]^{-1} K(x, t, s)$, $\alpha(x, t) \in C^\infty([0, 1] \times [0, 1])$ is an arbitrary function.

Subject the solution (18) to the initial condition $y(0, t, 0) = y_*(t)$. Then we get

$$\alpha(0, t) - \frac{h_0(0, t)}{a(0)} = y_*(t) \Leftrightarrow \alpha(0, t) = y_*(t) + \frac{h_0(0, t)}{a(0)}. \quad (19)$$

However, the functions $\alpha_j(x, t)$ were not found completely. An additional requirement is required to solve problem (15). Such a requirement is dictated by iterative problems (14_k), from which it can be seen that the natural additional constraint is the condition

$$-\frac{\partial y}{\partial x} + R_1 y + P(x, t, \tau) \equiv 0, \quad (\forall(x, t) \in [0, 1] \times [0, 1]), \quad (20)$$

where $P(x, t, \tau) = P_1(x, t)e^{\tau_1} + P_2(x, t)e^{\tau_2} + P_0(x, t) \in U$ is the known vector-function. We show that the problem (15) has the unique solution in the space U if (20) is satisfied

Theorem 2. *Let the conditions (i)–(ii) take place and the right-hand side $H(x, t) \equiv h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U$ satisfy the condition (17). Then the problem (15) is uniquely solvable in the space U under the additional condition (20).*

Proof. To use the condition (20), we calculate the expression $-\frac{\partial y}{\partial x} - R_1 y$. Since

$$\begin{aligned} R_1 y(x, t, \tau) &= - \left[(I_1^k(K(x, t, s)y_1(s, t)))_{s=x} e^{\tau_1} - (I_1^k(K(x, t, s)y_1(s, t)))_{s=0} e^{\tau_2} \right] + \\ &+ \left[(I_0^k(K(x, t, s)y_0(s, t)))_{s=0} e^{\tau_2} - I_0^k(K(x, t, s)y_0(s, t))_{s=x} \right], \\ y_1(s, t) &= \alpha(s, t), \quad y_0(s, t) = -\frac{h_0(x, t)}{a(x)}, \\ I_1^0(K(x, t, s)y_1(s, t)) &\equiv \frac{K(x, t, s)y_1(s, t)}{a(s) - \mu(s)}, \end{aligned}$$

we have

$$\begin{aligned} -\frac{\partial y}{\partial x} + R_1 y + P(x, t, \tau) &= -\frac{\partial(\alpha(x, t))}{\partial x} e^{\tau_1} - \\ &- \frac{\partial}{\partial x} \left[\int_0^x R(x, t, s)[\mu(s) - a(s)]^{-1} h_2(s, t) ds + [\mu(x) - a(x)]^{-1} h_2(x, t) \right] e^{\tau_2} + \\ &+ \frac{\partial}{\partial x} \left(\frac{h_0(x, t)}{a(x)} \right) - \left[(I_1^k(K(x, t, s)y_1(s, t)))_{s=x} e^{\tau_1} - (I_1^k(K(x, t, s)y_1(s, t)))_{s=0} e^{\tau_2} \right] + \\ &+ \left[(I_0^k(K(x, t, s)y_0(s, t)))_{s=0} e^{\tau_2} - I_0^k(K(x, t, s)y_0(s, t))_{s=x} \right] + \\ &+ P_1(x, t)e^{\tau_1} + P_2(x, t)e^{\tau_2} + P_0(x, t), \end{aligned}$$

therefore (20) takes the form

$$\frac{\partial(\alpha(x, t))}{\partial x} + \frac{K(x, t, x)y_1(x, t)}{a(x) - \mu(x)} \alpha(x, t) + P_1(x, t)e^{\tau_1} + P_2(x, t)e^{\tau_2} + P_0(x, t) = 0.$$

Taking into account the initial condition (19), this equation has the unique solution

$$\alpha(x, t) = e^{q(x, t)} \left[\alpha(0, t) + \int_0^x P_i(s, t) e^{-q(s, t)} ds \right],$$

where $q(x, t) = \int_0^x \frac{K(s, t, s)}{a(s) - \mu(s)} ds$. Hence, under the conditions of Theorem 2, the solution (18) in the space U is uniquely determined. Theorem 2 is proved.

4. Justification of the asymptotic convergence of approximate solutions to the exact. Applying Theorems 1 and 2 to the iteration problems (14_k), we construct a series (10) with coefficients from the class U . Let be the restriction of the $y_{\varepsilon N}(x, t) = \sum_{k=0}^N \varepsilon^k y_k(x, t, \frac{\varphi(x)}{\varepsilon})$ N -th partial sum of this series at $\tau = \frac{\varphi(t)}{\varepsilon}$. Just as in [2], it is easy to prove the following statement.

Lemma. Let conditions (i) – (iii) be fulfilled. Then the partial sum $y_{\varepsilon N}(x, t)$ satisfies the problem (1) up to terms containing ε^{N+1} , i.e.

$$\begin{aligned} \varepsilon \frac{\partial y_{\varepsilon N}(x, t)}{\partial x} &\equiv a(x) y_{\varepsilon N}(x, t) + h(x, t) + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y_{\varepsilon N}(s, t) ds + \\ &+ \varepsilon^{N+1} R_N(x, t, \varepsilon), \\ y_{\varepsilon N}(0, t) &= y^0(t), \forall (x, t) \in [0, 1] \times [0, 1], \end{aligned} \quad (21)$$

where $\|R_N(x, t, \varepsilon)\|_{C[0, T]} \leq \bar{R}_N$ for all $(x, t) \in [0, 1] \times [0, 1]$ and all $\varepsilon \in (0, \varepsilon_N]$ (here $\bar{R}_N > 0$ – some constant that does not depend on $\varepsilon \in (0, \varepsilon_N]$, $\varepsilon_N > 0$ is sufficiently small).

Now consider the following problem:

$$\begin{aligned} \varepsilon \frac{\partial y(x, t, \varepsilon)}{\partial x} &\equiv a(x) y(x, t, \varepsilon) + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y(s, t, \varepsilon) ds + \\ &+ \Phi(x, t, \varepsilon), \quad y(0, t, \varepsilon) = 0, \quad (x, t) \in [0, 1] \times [0, 1]. \end{aligned} \quad (22)$$

Let us prove its correct solvability in space $C^1([0, 1] \times [0, 1])$.

Theorem 3. Let conditions (i) – (iii) be fulfilled. Then, for sufficiently small $\varepsilon \in (0, \varepsilon_0]$, problem (22) for any right-hand side $\Phi(x, t, \varepsilon) \in C([0, 1] \times [0, 1])$ has a unique solution $y(x, t, \varepsilon)$ in space $C^1([0, 1] \times [0, 1])$, and the estimate

$$\|y(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} \leq \frac{\nu_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])}, \quad (23)$$

is holds, where $\nu_0 > 0$ is a constant independent of $\varepsilon \in (0, \varepsilon_0]$.

Proof. We introduce an additional unknown function

$$z(x, t, \varepsilon) = \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} K(x, t, s) y(s, t, \varepsilon) ds.$$

Differentiating it by x , we will have

$$\begin{aligned} \varepsilon \frac{\partial z(x, t, \varepsilon)}{\partial x} &= \mu(x) z(x, t, \varepsilon) + \varepsilon K(x, t, x) y(x, t, \varepsilon) + \\ &+ \varepsilon \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} \frac{\partial K(x, t, s)}{\partial x} y(s, t, \varepsilon) ds. \end{aligned}$$

For the vector function $w = \{y, z\}$, we get the system

$$\begin{aligned} \varepsilon \frac{\partial w(x, t, \varepsilon)}{\partial x} &= \begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix} w(x, t, \varepsilon) + \\ &+ \varepsilon \begin{pmatrix} 0 \\ K(x, t, x) z(x, t, \varepsilon) + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta) d\theta} \frac{\partial K(x, t, s)}{\partial x} y(s, t, \varepsilon) ds \end{pmatrix} + \\ &+ \begin{pmatrix} \Phi(x, t, \varepsilon) \\ 0 \end{pmatrix}, \quad w(0, t, \varepsilon) = 0. \end{aligned} \quad (24)$$

Let $G(x, \eta, \varepsilon)$ be the fundamental Cauchy matrix of the differential system

$$\varepsilon \frac{dG(x, \eta, \varepsilon)}{dx} = \begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix} G(x, \eta, \varepsilon), \quad G(x, x, \varepsilon) = I, \quad 0 \leq \eta \leq x \leq 1.$$

Since the matrix $\begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix}$ is a matrix of simple structure and its eigenvalues $a(x)$ and $\mu(x)$ lie in the half-plane $\operatorname{Re} \lambda \leq 0$, the Cauchy matrix $G(x, \eta, \varepsilon)$ is uniformly bounded, i.e.

$$\|G(x, \eta, \varepsilon)\| \leq c_0 \forall (x, \eta, \varepsilon) : 0 \leq \eta \leq x \leq 1, \varepsilon > 0$$

(see, for example, [10], pp. 119-120). We now write an integral system equivalent to system (24):

$$\begin{aligned} w(x, t, \varepsilon) &= \\ &= \int_0^x G(x, \eta, \varepsilon) \left(K(\eta, t, \eta) z(\eta, t, \varepsilon) + \int_0^\eta e^{\frac{1}{\varepsilon} \int_s^\eta \mu(\theta) d\theta} \frac{\partial K(\eta, t, s)}{\partial \eta} y(s, t, \varepsilon) ds \right) d\eta \\ &+ \frac{1}{\varepsilon} \int_0^x G(x, \eta, \varepsilon) \begin{pmatrix} \Phi(\eta, t, \varepsilon) \\ 0 \end{pmatrix} d\eta. \end{aligned}$$

This Volterra integral system has for each $\varepsilon > 0$ a unique solution $w(x, t, \varepsilon)$ in the class $C([0, 1] \times [0, 1])$. Substituting it into the indicated system, and passing to the norms in the resulting identity, we will have

$$\begin{aligned} \|w(x, t, \varepsilon)\| &\leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\| + c_1 \int_0^x \|z(\eta, t, \varepsilon)\| d\eta + c_2 \int_0^x \left(\int_0^\eta \|y(s, t, \varepsilon)\| ds \right) d\eta \leq \\ &\leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\| + c_1 \int_0^x \|w(\eta, t, \varepsilon)\| d\eta + c_2 \int_0^x \left(\int_0^\eta \|w(s, t, \varepsilon)\| ds \right) d\eta \leq \\ &\leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\| + c_1 \int_0^x \|w(\eta, t, \varepsilon)\| d\eta + c_2 \int_0^x \left(\int_0^\eta \|w(s, t, \varepsilon)\| ds \right) d\eta = \\ &= \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\| + c_1 \int_0^x \|w(\eta, t, \varepsilon)\| d\eta + c_2 \int_0^x \|w(s, t, \varepsilon)\| ds \cdot \int_0^x d\eta \leq \\ &\leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} + c_1 \int_0^x \|w(\eta, t, \varepsilon)\| d\eta + c_2 \int_0^x \|w(s, t, \varepsilon)\| ds = \\ &= \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} + (c_1 + c_2) \int_0^x \|w(\eta, t, \varepsilon)\| d\eta \end{aligned}$$

(here we used the continuity of functions $\Phi(x, t, \varepsilon)$, $K(\eta, t, \eta)$, $\frac{\partial K(\eta, t, s)}{\partial \eta}$ and the uniform boundedness of the matrix $G(x, \eta, \varepsilon)$). So, we obtain inequality

$$\|w(x, t, \varepsilon)\| \leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} + (c_1 + c_2) \int_0^x \|w(\eta, t, \varepsilon)\| d\eta.$$

Applying to this inequality the Gronwall-Bellman lemma, we arrive at the inequality

$$\begin{aligned} \|w(x, t, \varepsilon)\| &\leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \cdot e^{\int_0^x (c_1 + c_2) ds} \Rightarrow \\ &\Rightarrow \|w(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \leq \frac{\nu_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \Rightarrow \\ &\Rightarrow \|y(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \leq \frac{\nu_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0,1] \times [0,1])}, \end{aligned}$$

where $\nu_0 = c_0 e^{c_1 + c_2}$. The theorem is proved.

Theorem 4. Let the conditions (i)–(iii) be satisfied. Then for $\varepsilon \in (0, \varepsilon_N]$, where $\varepsilon_N > 0$ is sufficiently small, the problem (1) has the unique solution $y(x, t, \varepsilon) \in C([0, 1] \times [0, 1])$, and the estimate

$$\|y(x, t, \varepsilon) - y_{\varepsilon N}(x)\|_{C([0,1] \times [0,1])} \leq C_N \varepsilon^{N+1} \quad (N = 0, 1, 2, \dots)$$

takes place, where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_N]$.

Proof. By the lemma, the partial sum $y_{\varepsilon N}(x, t)$ satisfies the problem (21), therefore the remainder term $r_N(x, t, \varepsilon) \equiv y(x, t, \varepsilon) - y_{\varepsilon N}(x, t)$ satisfies the following problem:

$$\varepsilon \frac{r_N(x, t, \varepsilon)}{dx} = a(x) r_N(x, t, \varepsilon) + \int_0^x e^{\frac{1}{\varepsilon} \int_s^x a(\theta) d\theta} K(x, t, s) r_N(s, t, \varepsilon) ds - \varepsilon^{N+1} R_N(x, t, \varepsilon), \quad r_N(0, t, \varepsilon) = 0,$$

Here the role of the function $\Phi(x, t, \varepsilon)$ (see (22) and (23)) plays function $-\varepsilon^{N+1}R_N(x, t, \varepsilon)$. From here we get the estimate $\|r_N(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \leq \varepsilon^N \bar{R}_N$, fair for any $N = 0, 1, 2, \dots$ and all $\varepsilon \in (0, \varepsilon_N]$, which means that for a partial sum $y_{\varepsilon, N+1}(x, t) \equiv y_{\varepsilon N}(x, t) + \varepsilon^{N+1}y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon})$ the estimate

$$\|y(x, t, \varepsilon) - y_{\varepsilon, N+1}(x, t)\|_{C([0,1] \times [0,1])} \equiv \|(y(x, t, \varepsilon) - y_{\varepsilon N}(x, t)) - \varepsilon^{N+1}y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon})\|_{C([0,1] \times [0,1])} \leq \bar{C}_{N+1}\varepsilon^{N+1}.$$

is true. Using inequality $\|a - b\| \geq \||a\| - \|b\|\|$, we will have

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C([0,1] \times [0,1])} \leq \bar{C}_N \varepsilon^{N+1} + \varepsilon^{N+1} \left\| y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon}) \right\|_{C([0,1] \times [0,1])},$$

where we derive the usual estimate

$$\|y(t, \varepsilon) - y_{\varepsilon N}(t)\|_{C([0,1] \times [0,1])} \leq C_N \varepsilon^{N+1},$$

where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_N]$. From this inequality it follows that the restriction of the series (10) for is an asymptotic series for the exact solution of problem (1) for the Theorem is proved.

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