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ELEMENTARY FORMULAS FOR KIRCHHOFF INDEX OF MÖBIUS LADDER AND PRISM GRAPHS

G.A. BAIGONAKOVA, A.D. MEDNYKH

ABSTRACT. Let G be a finite connected graph on n vertices with Laplacian spectrum $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$. The Kirchhoff index of G is defined by the formula

$$Kf(G) = n \sum_{j=2}^n \frac{1}{\lambda_j}$$

The aim of this paper is to find an explicit analytical formula for the Kirchhoff index of Möbius ladder graph $M_n = C_{2n}(1, n)$ and Prism graph $Pr_n = C_n \times P_2$. The obtained formulas provide a simple asymptotical behavior of both invariants as n is going to the infinity.

Keywords: Laplacian matrix, circulant graph, Kirchhoff index, Wiener index, Chebyshev polynomial.

1. INTRODUCTION

Let G be a finite connected graph on n vertices. Denote by $D(G)$ the diagonal matrix formed by degrees of vertices and by $A(G)$ the adjacency matrix of the graph G . The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G . It is well known [17] that $L(G) \geq 0$ and $\det(L(G)) = 0$. For a connected graph G , all eigenvalues of $L(G)$ except one are strictly positive. Hence, the Laplacian spectrum of G can be represented in the form $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$. The Kirchhoff index of G originally was defined by Klein and Randić [12] as a new distance function named resistance distance framed in terms of electrical network theory. More precisely, let vertices of the graph G are labeled by $1, 2, \dots, n$. Then

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the resistance distance between vertices i and j , denoted by $r_{ij} = r_{ij}(G)$ is defined to be the effective resistance between them when unit resistors are placed on every edge of G . Following [12] define

$$Kf(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n r_{ij}$$

to be the Kirchhoff index of G . The motivation for such a definition was a famous Wiener index $W(G)$, which counts the sum of distances between pairs of vertices in G , that is

$$W(G) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n d_{ij},$$

where d_{ij} is the distance between vertices i and j . The Wiener index is named after Harry Wiener, who introduced it in [24]. See also [13] for more advance properties. Klein and Randić [12] proved that $Kf(G) \leq W(G)$ with equality, if and only if G is a tree. Closed-form formulae for the Kirchhoff index have been given for some classes of graphs, such as cycles [12, 15], complete graphs [15], distance-transitive graphs [15], circulant graphs [25], ladder-like chains [4] and others [1, 14, 10, 18, 19]. Numerical values of the Kirchhoff index have been computed for platonic solids [15, 16] and some fullerene graphs [1, 2, 3, 8, 10]. In [26] the Kirchhoff index of join, corona and cluster of graphs have been investigated.

There is a nice relationship [9] between the Laplacian spectrum and the Kirchhoff index given by the formula

$$(1) \quad Kf(G) = n \sum_{j=2}^n \frac{1}{\lambda_j}.$$

2. AN EXPLICIT FORMULA FOR THE KIRCHHOFF INDEX

Consider the Möbius ladder $M_n = C_{2n}(1, n)$ and the Prism graph $Pr_n = C_n \times P_2$. Here $C_{2n}(1, n)$ is a 3-valent circulant graphs on $2n$ vertices $v_1, v_2, \dots, v_{2n}$ with edges $v_i v_{i+1}$ and $v_i v_{i+n}, i \bmod 2n$, C_n is the cycle graph on n vertices and P_2 is the path graph on two vertices. For $n = 6$ the graphs are shown in Fig. 1 and Fig. 2 respectively. The main results of this paper are the following two theorems.

Theorem 1. *The Kirchhoff index of Möbius ladder $M_n = C_{2n}(1, n)$ is given by the formula*

$$Kf(M_n) = \frac{n^3 - n}{6} + \frac{n^2 \tanh(\frac{n}{2} \operatorname{arccosh} 2)}{\sqrt{3}}.$$

Theorem 2. *The Kirchhoff index of Prism graph $Pr_n = C_n \times P_2$ is given by the formula*

$$Kf(Pr_n) = \frac{n^3 - n}{6} + \frac{n^2 \coth(\frac{n}{2} \operatorname{arccosh} 2)}{\sqrt{3}}.$$

FIG. 1. Möbius ladder M_6 FIG. 2. Prism graph Pr_6

Remark. By Besot's theorem, the Kirkhoff indices in Theorem 1 and Theorem 2 are rational numbers. Their explicit values up to $n = 30$ are given in Table 1.

Proof of Theorems 1 and 2. Consider an order n polynomial $P(w) = \prod_{k=1}^n (w - w_k)$. Then we have

$$(2) \quad \sum_{k=1}^n \frac{1}{w - w_k} = \frac{P'(w)}{P(w)},$$

where the both sides of the equation are considered as meromorphic functions of w . Let $T_n(w)$ and $U_{n-1}(w)$ be the Chebyshev polynomials of the first and the second kind respectively. We take $P(w) = T_n(w + a) - 1$. Then $P'(w) = nU_{n-1}(w + a)$. Since $T_n(w) = \cos(n \arccos w)$, all the roots of $P(w)$ are given by the formulas $w_k = -a + \cos \frac{2k\pi}{n}$, $n = 1, 2, \dots, n$. Hence

$$(3) \quad \sum_{k=1}^n \frac{1}{w + a - \cos \frac{2k\pi}{n}} = \frac{nU_{n-1}(w + a)}{T_n(w + a) - 1}.$$

Taking $w = 0$ in (3) and dividing by 2 we obtain

$$(4) \quad \sum_{k=1}^n \frac{1}{2a - 2 \cos \frac{2k\pi}{n}} = \frac{n U_{n-1}(a)}{2(T_n(a) - 1)}.$$

As a result, we have the following two elementary identities

$$(5) \quad \sum_{k=1}^n \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} = \frac{n U_{n-1}(2)}{2(T_n(2) - 1)}$$

and

$$(6) \quad \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} = \lim_{a \rightarrow 1} \left(\frac{n U_{n-1}(a)}{2(T_n(a) - 1)} - \frac{1}{2a - 2} \right) = \frac{n^2 - 1}{12}.$$

Now, we are going to find the Kirchhoff index for Möbius ladder M_n . Recall that by ([7], p. 71) the non-zero Laplacian eigenvalues of M_n are given by the formulas $\mu_j = 3 - 2 \cos \frac{j\pi}{n} - (-1)^j$, $j = 1, 2, \dots, 2n - 1$. Hence,

$$\begin{aligned} Kf(M_n) &= 2n \sum_{j=1}^{2n-1} \frac{1}{\mu_j} = 2n \left(\sum_{k=1}^n \frac{1}{4 - 2 \cos \frac{(2k-1)\pi}{n}} + \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} \right) \\ &= 2n \left(\sum_{k=1}^{2n} \frac{1}{4 - 2 \cos \frac{2k\pi}{2n}} - \sum_{k=1}^n \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} + \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} \right). \end{aligned}$$

By making use of identities (5) and (6) we get

$$(7) \quad Kf(M_n) = 2n \left(\frac{2n U_{2n-1}(2)}{2(T_{2n}(2) - 1)} - \frac{n U_{n-1}(2)}{2(T_n(2) - 1)} + \frac{n^2 - 1}{12} \right).$$

We set $\theta = \operatorname{arccosh} 2$ and note that

$$T_n(2) = \cosh(n\theta), \quad U_{n-1}(2) = \frac{\sinh(n\theta)}{\sinh(\theta)}.$$

Observing that $\coth(n\theta) - \frac{1}{2} \coth(\frac{n}{2}\theta) = \frac{1}{2} \tanh(\frac{n}{2}\theta)$, we have

$$(8) \quad \frac{2n U_{2n-1}(2)}{2(T_{2n}(2) - 1)} - \frac{n U_{n-1}(2)}{2(T_n(2) - 1)} = \frac{n \tanh(\frac{n}{2}\theta)}{2 \sinh(\theta)} = \frac{n \tanh(\frac{n}{2} \operatorname{arccosh} 2)}{2\sqrt{3}}.$$

Putting the last equation (8) into (7), we obtain the statement of Theorem 1:

$$Kf(M_n) = \frac{n^3 - n}{6} + \frac{n^2 \tanh(\frac{n}{2} \operatorname{arccosh} 2)}{\sqrt{3}}.$$

The Kirchhoff index for prism graph Pr_n can be find by similar arguments. Indeed, by ([17], Theorem 3.5) the Laplacian eigenvalues of Pr_n are given the following list $4 - 2 \cos \frac{2k\pi}{n}$, $2 - 2 \cos \frac{2k\pi}{n}$, $k = 1, 2, \dots, n$. Hence,

$$Kf(Pr_n) = 2n \left(\sum_{k=1}^n \frac{1}{4 - 2 \cos \frac{2k\pi}{n}} + \sum_{k=1}^{n-1} \frac{1}{2 - 2 \cos \frac{2k\pi}{n}} \right).$$

Again, by (5) and (6) we have

$$Kf(Pr_n) = 2n \left(\frac{n U_{n-1}(2)}{2(T_n(2) - 1)} + \frac{n^2 - 1}{12} \right).$$

Since

$$\frac{U_{n-1}(2)}{2(T_n(2) - 1)} = \frac{\coth(\frac{n}{2}\theta)}{2\sinh(\theta)} = \frac{\coth(\frac{n}{2}\operatorname{arccosh} 2)}{2\sqrt{3}},$$

the statement of Theorem 2 follows.

3. ASYMPTOTICAL BEHAVIOR AND ESTIMATES

The asymptotic formulas for Kirchhoff index of the above graphs can be derived from the following theorem.

Theorem 3. *Let $K(n) = \frac{1}{6}(n^3 + 2\sqrt{3}n^2 - n)$. Then*

- (i) $Kf(M_n) < K(n) < Kf(Pr_n)$;
- (ii) $0 < Kf(Pr_n) - Kf(M_n) \leq \frac{2n^2}{3^n}$.

Proof. The first statement of the theorem is a consequence of Theorems 1 and 2 and elementary inequalities $\tanh(\frac{n}{2}\theta) < 1 < \coth(\frac{n}{2}\theta)$, where $\theta = \operatorname{arccosh} 2$. To prove the second statement we note that by virtue of Theorems 1 and 2,

$$Kf(Pr_n) - Kf(M_n) = \frac{n^2(\coth(\frac{n}{2}\theta) - \tanh(\frac{n}{2}\theta))}{\sqrt{3}} = \frac{2n^2}{\sqrt{3}\sinh(n\theta)} \leq \frac{2n^2}{3^n}.$$

The latter inequality follows from the observation

$$\sqrt{3}\sinh(n\theta) = \frac{\sqrt{3}}{2}((2 + \sqrt{3})^n - (2 - \sqrt{3})^n) \geq 3^n \text{ for } n \geq 1.$$

One can easily check from Theorem 3 that for $n \geq 16$ the values $Kf(M_n)$, $Kf(Pr_n)$ and $K(n)$ are accurate to within 10^{-5} . The respective values are given in Table 2.

As an immediate consequence of the above theorem we have the following result.

Theorem 4. *Let $K(n) = \frac{1}{6}(n^3 + 2\sqrt{3}n^2 - n)$. Then*

$$Kf(M_n) = K(n) + O\left(\frac{n^2}{3^n}\right), \quad n \rightarrow \infty,$$

and

$$Kf(Pr_n) = K(n) + O\left(\frac{n^2}{3^n}\right), \quad n \rightarrow \infty.$$

Remark. When the paper was already prepared for publication, we discovered that the results equivalent to Theorems 1 and 2 were obtained by completely different methods in [5] and [6] respectively. A weak form of the asymptotics for Kirchhoff index in the Möbius ladder was done in [25].

4. THE TABLE OF KIRCHHOFF INDICES FOR GRAPHS M_n AND Pr_n TABLE 1. Kirchhoff indices for graphs M_n and Pr_n

n	$Kf(M_n)$	$Kf(Pr_n)$
3	9	47/5
4	134/7	58/3
5	1135/33	655/19
6	725/13	279/5
7	10367/123	5985/71
8	11732/97	2540/21
9	2835/17	44193/265
10	40315/181	139655/627
11	496529/1713	286671/989
12	498706/1351	23994/65
13	2950831/6393	1703663/3691
14	1432333/2521	4961747/8733
15	1828935/2651	380137/551
16	15576744/18817	1686232/2037
17	87516289/89043	50527553/51409
18	40592787/35113	5208057/4505
19	448098641/332313	258709871/191861
20	409102110/262087	177146410/113487
21	247299255/137801	1285004623/716035
22	1002789029/489061	3473763095/1694157
23	10781766143/4628523	6224855585/2672279
24	9609876860/3650401	231177708/87815
25	51145271425/17273883	29528736225/9973081
26	2258288523/6811741	78229420607/23596563
27	2942312769/795889	137598350607/37220045
28	4261146826/1037623	90411404314/22015893
29	1093633131439/240594153	631409382831/138907099
30	475763190685/94875313	7324853499/1460701

TABLE 2. Numerical values of $Kf(M_n)$, $K(n)$ and $Kf(Pr_n)$

n	$Kf(M_n)$	$K(n)$	$Kf(Pr_n)$
3	9.000000000	9.196152423	9.400000000
4	19.14285714	19.23760431	19.33333333
5	34.39393939	34.43375673	34.47368421
6	55.76923077	55.78460969	55.80000000
7	84.28455285	84.29016319	84.29577465
8	120.9484536	120.9504172	120.9523810
9	166.7647059	166.7653718	166.7660377
10	222.7348066	222.7350269	222.7352472
11	289.8593112	289.8593826	289.8594540
12	369.1384160	369.1384388	369.1384615
13	461.5721883	461.5721955	461.5722027
14	568.1606505	568.1606528	568.1606550
15	689.9038099	689.9038106	689.9038113
16	827.8016687	827.8016689	827.8016691
17	982.8542277	982.8542278	982.8542279
18	1156.061487	1156.061487	1156.061487
19	1348.423447	1348.423447	1348.423447
20	1560.940108	1560.940108	1560.940108
21	1794.611469	1794.611469	1794.611469
22	2050.437530	2050.437530	2050.437530
23	2329.418292	2329.418292	2329.418292
24	2632.553755	2632.553755	2632.553755

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GALYA AMANBOLDYNOVNA BAIGONAKOVA
 GORNO-ALTAYSK STATE UNIVERSITY,
 34, SOCIALISTICHESKAYA STR.,
 GORNO-ALTAYSK, 639000, RUSSIA
E-mail address: galyaab@mail.ru

ALEKSANDR DMITRIEVICH MEDNYKH
 SOBOLEV INSTITUTE OF MATHEMATICS,
 4, KOPTYUGA AVE.,
 NOVOSIBIRSK, 630090, RUSSIA
 NOVOSIBIRSK STATE UNIVERSITY,
 1, PIROGOVA STR.,
 NOVOSIBIRSK, 630090, RUSSIA
E-mail address: ilyamednykh@mail.ru