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LATTICES IN GENERATIVE CLASSES

Y. KIOUVREKIS, P. STEFANEAS, S. V. SUDOPLATOV

ABSTRACT. We define and study lattices in generative classes associated with generic structures. It is shown that these lattices can be non-distributive and, moreover, arbitrary enough. Heights and weights of the lattices are described. A model-theoretic criterion for the linear ordering is proved and these linear orders are described. Boolean algebras generated by the considered lattices are also described.

Keywords: lattice, generative class, generic structure, linear order, Boolean algebra.

Considering syntactic approach to generic constructions and their limits [1, 2, 3, 4, 5, 6, 7] we study lattices in generative classes associated with generic structures. We show that these lattices can be non-distributive. Heights and weights of the lattices are described. A model-theoretic criterion for the linear ordering is proved and these linear orders are described. Boolean algebras generated by the considered lattices are described.

Standard notions on the lattice theory can be found in [8, 9, 10].

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1. PRELIMINARIES

We remind notions, notations and assertions of [1, 2, 3, 4, 5, 6, 7] that will be used in the next section.

We consider collections of sentence and formulas in first order logic over a language Σ . Thus, as usual, $\vdash$ means proof from no hypotheses deducing $\vdash \varphi$ for a formula φ of language Σ , which may contain function symbols and constants. If deducing φ , hypotheses in a set Φ of formulas can be used, we write $\Phi \vdash \varphi$. Usually Σ will be fixed in context and not mentioned explicitly.

Below we write $X, Y, Z, \dots$ for finite sets of variables, and denote by $A, B, C, \dots$ finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In model theory's literature, if $\mathcal{M}$ is a structure in a language Σ , then the collection of quantifier-free $\Sigma(\mathcal{M})$ -sentences true in $\mathcal{M}$ is called the *diagram* of $\mathcal{M}$ and the collection of all $\Sigma(\mathcal{M})$ -sentences true in $\mathcal{M}$ is called the *elementary* or *complete* diagram of $\mathcal{M}$.

Here, in diagrams, $A, B, C, \dots$ denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A), \Psi(B), X(C)$ stand for Σ -*diagrams* (of sets A, B, C), that is, *consistent sets of $\Sigma(A)$ -, $\Sigma(B)$ -, $\Sigma(C)$ -sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1, a_2 are distinct elements in A then $\neg(a_1 \approx a_2) \in \Phi(A)$. This means that if c is a constant symbol in Σ , then there is at most one element $a \in A$ such that $(a \approx c) \in \Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a}) \in \Phi(A) \mid \bar{a} \in B\}$. Similarly, for a language Σ , we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ .

Definition [1, 2, 3, 4, 5, 6, 7]. We denote by $[\Phi(A)]_B^A$ the diagram $\Phi(B)$ obtained by replacing a subset $A' \subseteq A$ by a set $B' \subseteq B$ of constants disjoint from Σ and with $|A'| = |B'|$, where $A \setminus A' = B \setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]_X^A$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A . In this case, we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]_A^X$.

Remark 1.1. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A \cup K)$, where K is the set of the constant symbols in Σ .

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition [1, 2, 3, 4, 5, 6, 7]. Let Σ be a vocabulary. We say that $(\mathbf{D}_0; \leq)$ (or $\mathbf{D}_0$) is *generic*, or *generative*, if $\mathbf{D}_0$ is a class of Σ -diagrams of finite sets so that $\mathbf{D}_0$ is partially ordered by a binary relation $\leq$ such that $\leq$ is preserved by bijective

substitutions, i. e., if $\Phi(A) \leq \Psi(B)$, and $A' \subseteq B'$ such that $[\Phi(A)]_{A'}^A = \Phi(A')$ and $[\Psi(B)]_{B'}^B = \Psi(B')$ are defined, then $[\Phi(A)]_{A'}^A, [\Psi(B)]_{B'}^B$ are in $\mathbf{D}_0$ and $[\Phi(A)]_{A'}^A \leq [\Psi(B)]_{B'}^B$.¹ Furthermore:

(i) if $\Phi(A) \in \mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in A$ either $\varphi(\bar{a}) \in \Phi(A)$ or $\neg\varphi(\bar{a}) \in \Phi(A)$;

(ii) if $\Phi \leq \Psi$ then $\Phi \subseteq \Psi$;²

(iii) if $\Phi \leq X, \Psi \in \mathbf{D}_0$, and $\Phi \subseteq \Psi \subseteq X$, then $\Phi \leq \Psi$;

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0; \leq)$, and $\mathbf{D}_0 \setminus \{\Phi_0(\emptyset)\}$ is nonempty;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A), \Psi(B), X(C) \in \mathbf{D}_0$, if there exist injections $f_0: A \rightarrow B$ and $g_0: A \rightarrow C$ with $[\Phi(A)]_{f_0(A)}^A \leq \Psi(B)$ and $[\Phi(A)]_{g_0(A)}^A \leq X(C)$, then there are a diagram $\Theta(D) \in \mathbf{D}_0$ and injections $f_1: B \rightarrow D$ and $g_1: C \rightarrow D$ for which $[\Psi(B)]_{f_1(B)}^B \leq \Theta(D)$, $[X(C)]_{g_1(C)}^C \leq \Theta(D)$ and $f_0 \circ f_1 = g_0 \circ g_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $X(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0, g_0, f_1, g_1) (see Figure 1);

Fig. 1

(vi) (the *local realizability property*) if $\Phi(A) \in \mathbf{D}_0$ and $\Phi(A) \vdash \exists x \varphi(x)$, then there are a diagram $\Psi(B) \in \mathbf{D}_0$, $\Phi(A) \leq \Psi(B)$, and an element $b \in B$ for which $\Psi(B) \vdash \varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ if $A \subseteq B$ and the set $\Phi(A) \cup \Psi(B)$ is consistent then $\Phi(A) = \{\varphi(\bar{b}) \in \Psi(B) \mid \bar{b} \in A\}$.

Example 1.2. Consider a vocabulary $\Sigma = \{<, f\}$ where $<$ is a linear order and f is a unary function. For $A = \{a\}$ and assuming that $\forall x(f(x) > x)$ and $a = a$ belongs to a diagram $\Phi(A)$, we get that the diagram consists of the formulas $f^{(n+1)}(a) > f^{(n)}(a)$ for every $n \in \omega$. Thus the finite set A generates an infinite structure.

Example 1.3. Consider a vocabulary $\Sigma = \{<, f\}$ where $<$ is a dense linear order and f is a binary function. For $B = \{a, b\}$ with $a < b$ and assuming that $\forall x \forall y(x < f(x, y) < y)$ and $a = a, b = b$ belong to a diagram $\Phi(B)$, we obtain

¹Note that $\mathbf{D}_0$ is closed under bijective substitutions since $\leq$ is preserved by bijective substitutions and $\leq$ is reflexive.

²Note that $\Phi(A) \leq \Psi(B)$ implies $A \subseteq B$, since if $a \in A$ then $(a \approx a) \in \Phi(A)$, so $\Phi(A) \leq \Psi(B)$ implies $\Phi(A) \subseteq \Psi(B)$ and we have $(a \approx a) \in \Psi(B)$, whence $a \in B$.

that the diagram contains the formulas $a < f(a, b) < b$, $a < f(a, f(a, b)) < b$, $a < f(a, f(a, f(a, b))) < b$, etc. Thus the finite set B generates an infinite structure.

Definition [1, 2, 3, 4, 5, 6, 7]. A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi \leq \Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f: A \rightarrow B$ such that $[\Phi(A)]_{f(A)}^A \subseteq \Psi(B)$ ($[\Phi(A)]_{f(A)}^A \leq \Psi(B)$). The injection f , in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f: \Phi(A) \rightarrow \Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure $\mathcal{M}$ if $\Phi(A)$ is (*strongly*) embeddable in some diagram $\Psi(B)$, where $\mathcal{M} \models \Psi(B)$. The corresponding embedding $f: \Phi(A) \rightarrow \Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure $\mathcal{M}$ and is denoted by $f: \Phi(A) \rightarrow \mathcal{M}$.

Let $\mathbf{D}_0$ be a class of diagrams, $\mathbf{P}_0$ be a class of structures of some language, and $\mathcal{M}$ be a structure in $\mathbf{P}_0$. The class $\mathbf{D}_0$ is *cofinal* in the structure $\mathcal{M}$ if for each finite set $A \subseteq M$, there are a finite set B , $A \subseteq B \subseteq M$, and a diagram $\Phi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Phi(B)$. The class $\mathbf{D}_0$ is *cofinal* in $\mathbf{P}_0$ if $\mathbf{D}_0$ is cofinal in every structure of $\mathbf{P}_0$. We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures $\mathcal{M}$ with the condition that $\mathbf{D}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi \in \mathbf{D}_0$ is true in some structure in $\mathbf{P}$.

Now we extend the relation $\leq$ from the generative class $(\mathbf{D}_0; \leq)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let $\mathcal{M}$ be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in $\mathcal{M}$ with $A \subseteq B$. We call A a *strong subset* of the set B (in the structure $\mathcal{M}$), and write $A \leq B$, if there exist diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, for which $\Phi(A) \leq \Psi(B)$ and $\mathcal{M} \models \Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0 \subseteq M$ (in the structure $\mathcal{M}$), where $A \subseteq M_0$, if $A \leq B$ for any finite set B such that $A \subseteq B \subseteq M_0$ and $\Phi(A) \subseteq \Psi(B)$ for some diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$ with $\mathcal{M} \models \Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A \leq M_0$. If $A \leq M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Notice that, by the d -uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A) \in \mathbf{D}_0$, corresponding to a self-sufficient set A in $\mathcal{M}$, is said to be a *self-sufficient diagram* (in $\mathcal{M}$).

Definition [1, 2, 3, 4, 5, 6, 7]. A class $(\mathbf{D}_0; \leq)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_0$, there is a diagram $X(C) \in \mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $X(C)$.

Clearly, every generative class has JEP since JEP means the d -amalgamation property over the empty set.

Definition [1, 2, 3, 4, 5, 6, 7]. A structure $\mathcal{M} \in \mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leq)$, or is *finitely generated over Σ* , if any finite set $A \subseteq M$ is contained in some finite self-sufficient set in $\mathcal{M}$, i. e., there is a finite set B with $A \subseteq B \subseteq M$ and $\Psi(B) \in \mathbf{D}_0$ such that $\mathcal{M} \models \Psi(B)$ and $\Psi(B) \leq X(C)$ for any $X(C) \in \mathbf{D}_0$ with $\mathcal{M} \models X(C)$ and $\Psi(B) \subseteq X(C)$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0; \leq)$, or is *finitely generated over Σ* , if each structure in $\mathbf{P}$ has finite closures (with respect to $(\mathbf{D}_0; \leq)$).

Clearly, an at most countable structure $\mathcal{M}$ has finite closures with respect to $(\mathbf{D}_0; \leq)$ if and only if $M = \bigcup_{i \in \omega} A_i$ for some self-sufficient sets A_i with $A_i \leq A_{i+1}$, $i \in \omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\text{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure $\mathcal{M}$ with respect to $(\mathbf{D}_0; \leq)$ only and B can be both a universe of a substructure of $\mathcal{M}$ or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0; \subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in $\mathbf{D}_0$ has only infinite models.

Definition [1, 2, 3, 4, 5, 6, 7]. A structure $\mathcal{M} \in \mathbf{K}(\mathbf{D}_0)$ is $(\mathbf{D}_0; \leq)$ -generic, or a *generic limit for the class* $(\mathbf{D}_0; \leq)$ and denoted by $\text{glim}(\mathbf{D}_0; \leq)$, if it satisfies the following conditions:

- (a) $\mathcal{M}$ has finite closures with respect to $\mathbf{D}_0$;
- (b) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{D}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leq \Psi(B)$, then there exists a set $B' \leq M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Theorem 1.4 [1, 2, 3, 6, 7]. *For any generative class $(\mathbf{D}_0; \leq)$ with at most countably many diagrams whose copies form $\mathbf{D}_0$, there exists a $(\mathbf{D}_0; \leq)$ -generic structure.*

Theorem 1.5 [4, 6, 7]. *Every ω -homogeneous structure $\mathcal{M}$ is $(\mathbf{D}_0; \leq)$ -generic for some generative class $(\mathbf{D}_0; \leq)$.*

Thus any first-order theory has a generic model and therefore can be represented by it.

Definition [1, 2, 3, 4, 5, 6, 7]. A generative class $(\mathbf{D}_0; \leq)$ is *self-sufficient* if the following *axiom of self-sufficiency* holds:

- (viii) if $\Phi, \Psi, X \in \mathbf{D}_0$, $\Phi \leq \Psi$, and $X \subseteq \Psi$, then $\Phi \cap X \leq X$.

Theorem 1.6 [1, 2, 3, 6, 7]. *Let $(\mathbf{D}_0; \leq)$ be a self-sufficient class, $\mathcal{M}$ be at most countable $(\mathbf{D}_0; \leq)$ -generic structure, and $\mathbf{K}$ be the class of all models of $T = \text{Th}(\mathcal{M})$ which has finite closures. Then the generic structure $\mathcal{M}$ is homogeneous.*

Thus, since any ω -homogeneous structure can be considered as generic with respect to a generic class with complete diagrams, a countable structure $\mathcal{M}$ is homogeneous if and only if it is generic for an appropriate self-sufficient generative class $(\mathbf{D}_0; \leq)$.

Considering an ω -homogeneous structure $\mathcal{M}$, being $(\mathbf{D}_0; \leq)$ -generic, and its elementary ω -homogeneous substructure $\mathcal{N}$ we generate the *restriction* $(\mathbf{D}'_0; \leq')$ of $(\mathbf{D}_0; \leq)$ taking, for any diagram $\Phi(A) \in \mathbf{D}_0$ satisfying $\mathcal{M} \models \Phi(A)$, the subdiagram $\Phi'(A \cap N) = \Phi(A) \upharpoonright N$ and forming $(\mathbf{D}'_0; \leq')$ by copies of $\Phi'(A \cap N)$ with $\Phi'(A \cap N) \leq' \Psi'(B \cap N) \Leftrightarrow \Phi(A) \leq \Psi(B)$. These subdiagrams, satisfied in $\mathcal{N}$, indeed exist since elementary substructures preserve the satisfaction of formulas.

Fig. 2

Fig. 3

Clearly, $(\mathbf{D}'_0; \leq')$ is a generative class, moreover, if $(\mathbf{D}_0; \leq)$ is self-sufficient then $(\mathbf{D}'_0; \leq')$ is self-sufficient, too. Besides, $\mathcal{N}$ is $(\mathbf{D}'_0; \leq')$ -generic. Thus we obtain the following:

Proposition 1.7. *For any ω -homogeneous structures $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{N} \prec \mathcal{M}$, any (self-sufficient) generative class $(\mathbf{D}_0; \leq)$, such that $\mathcal{M}$ is $(\mathbf{D}_0; \leq)$ -generic, has a (self-sufficient) restriction $(\mathbf{D}'_0; \leq')$ such that $\mathcal{N}$ is $(\mathbf{D}'_0; \leq')$ -generic.*

Definition. The restriction $(\mathbf{D}'_0; \leq')$ of $(\mathbf{D}_0; \leq)$ is called *conservative* if $\mathbf{D}'_0 \subseteq \mathbf{D}_0$.

The following proposition is obvious.

Proposition 1.8. *If $(\mathbf{D}_0; \leq)$ is formed by copies of all quantifier-free (respectively, complete) diagrams for $\mathcal{M}$ then any restriction $(\mathbf{D}'_0; \leq')$ of $(\mathbf{D}_0; \leq)$ (by complete diagrams), for a (elementary) substructure $\mathcal{N}$ of $\mathcal{M}$, is conservative.*

2. LATTICES ASSOCIATED WITH GENERIC STRUCTURES

Note that if $(\mathbf{D}_0; \leq)$ be a self-sufficient class then for $\Phi, \Psi \in \mathbf{D}_0$ with consistent $\Phi \cup \Psi$, we have $\Phi \cap \Psi \in \mathbf{D}_0$ and there is the least amalgam $\Theta \in \mathbf{D}_0$ containing $\Phi \cup \Psi$. We denote $\Phi \cap \Psi$ by $\Phi \wedge \Psi$ and Θ by $\Phi \vee \Psi$. Hence, for a $(\mathbf{D}_0; \leq)$ -generic structure $\mathcal{M}$, the set $L(\mathcal{M}, \mathbf{D}_0, \leq)$ of all diagrams $\Phi \in \mathbf{D}_0$ with $\mathcal{M} \models \Phi$ form a lattice $\mathcal{L} = \langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$. The structure $\mathcal{L}$ is called the *lattice associated with the generic structure $\mathcal{M}$* .

Note that the lattice $\mathcal{L}$ can be non-distributive admitting both the lattice M_3 and the lattice P_5 (see Figures 2 and 3, respectively).

Indeed, having always $(\Phi \wedge \Psi) \vee (\Phi \wedge X) \leq \Phi \wedge (\Psi \vee X)$ and $\Phi \vee (\Psi \wedge X) \leq (\Phi \vee \Psi) \wedge (\Phi \vee X)$, we can consider a 3-element structure $\mathcal{M}$ with constants a, b, c and diagrams $\Phi = \Phi(\{a\})$, $\Psi = \Psi(\{b\})$, $X = X(\{c\})$, $\Theta = \Theta(\{a, b, c\})$ such that $\Phi \vee \Psi = \Phi \vee X = \Psi \vee X = \Theta$. We have $(\Phi \wedge \Psi) \vee (\Phi \wedge X) = \Phi_0 \vee \Phi_0 = \Phi_0$ whereas $\Phi \wedge (\Psi \vee X) = \Phi \wedge \Theta = \Phi$, and similarly $\Phi \vee (\Psi \wedge X) = \Phi \vee \Phi_0 = \Phi$ whereas $(\Phi \vee \Psi) \wedge (\Phi \vee X) = \Theta \wedge \Theta = \Theta$, realizing the lattice M_3 (see Figure 4).

Considering a 4-element structure $\mathcal{M}$ with constants a, b, c, d and diagrams $\Phi = \Phi(\{a\})$, $\Psi = \Psi(\{b\})$, $X = X(\{b, c\})$, $\Theta = \Theta(\{a, b, c, d\})$ such that a and b are separated by a formula $\varphi(x)$ with $\varphi(a) \in \Phi$, $\neg\varphi(b) \in \Psi$, $\Phi \vee \Psi = \Phi \vee X = \Theta$ we realize the lattice P_5 (see Figure 5).

Thus we have the following:

Proposition 2.1. *For any self-sufficient class $(\mathbf{D}_0; \leq)$ and a $(\mathbf{D}_0; \leq)$ -generic structure $\mathcal{M}$, the structure $\langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$ is a lattice which can be non-distributive.*

Note that in the way above arbitrary finite lattices and their superatomic limits can be constructed.

Studying the lattices $\mathcal{L}$ we recall that the *height* (respectively, *width*) of $\mathcal{L}$ is the supremum of cardinalities of (anti)chains in $\mathcal{L}$. The height of $\mathcal{L}$ is denoted by $h(\mathcal{L})$ and the width is denoted by $w(\mathcal{L})$.

Since each element $\Phi(A)$ in $\mathcal{L}$ corresponds to a finite set $A \subseteq M$, the height of $\mathcal{L}$ is at most countable and $h(\mathcal{L}) < \omega$ if and only if $\mathcal{M}$ is finite. Finite $h(\mathcal{L})$ can vary from 2 to $|M| + 1$ (counting the least diagram $\Phi_0(\emptyset)$ and the greatest diagram $\Psi(M)$). Indeed, considering a structure $\mathcal{N}$ consisting of n distinct constants $c_1, \dots, c_n$ we can take, for any positive $k \leq n$ the following chain of diagrams:

$$\Phi_0(\emptyset) \subset \Phi_1(\{a_1, \dots, a_{n-k+1}\}) \subset \Phi_2(\{a_1, \dots, a_{n-k+2}\}) \subset \dots \subset \Phi_k(\{a_1, \dots, a_n\}),$$

containing the formulas $(a_i \approx c_i)$. The lattice $\mathcal{L}$ for the structure $\mathcal{N}$ with the described set of diagrams is linearly ordered with $h(\mathcal{L}) = k + 1$.

At the same time, for finite $\mathcal{M}$, $w(\mathcal{L})$ can vary from 1 to $|M|$ taking, for instance, diagrams $\Phi_1(\{a_1, \dots, a_{n-k+1}\}), \Phi'_2(\{a_{n-k+1}\}), \dots, \Phi'_k(\{a_n\})$ with $(a_i \approx c_i)$ for the n -element example $\langle M; c_1, \dots, c_n \rangle$ as above. Continuing the process adding new constants, we obtain $1 \leq w(\mathcal{L}) \leq \omega$ for countable $\mathcal{M}$.

Since $h(\mathcal{L}) \leq \omega$ we have $w(\mathcal{L}) = |M|$ if $\mathcal{M}$ is uncountable.

Summarizing the arguments we have the following:

Theorem 2.2. *For any self-sufficient class $(\mathbf{D}_0; \leq)$ and a $(\mathbf{D}_0; \leq)$ -generic structure $\mathcal{M}$, the lattice $\mathcal{L} = \langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$ has the following characteristics:*

- (1) $1 < h(\mathcal{L}) \leq |M| + 1$ if $\mathcal{M}$ is finite, and $h(\mathcal{L}) = \omega$ if $\mathcal{M}$ is infinite;
- (2) $1 \leq w(\mathcal{L}) \leq |M|$ if $\mathcal{M}$ is at most countable, and $w(\mathcal{L}) = |M|$ if $\mathcal{M}$ is uncountable.

All values in the described intervals can be realized in appropriate generic structures.

By Theorem 2.2 having a linearly ordered $\mathcal{L}$ we obtain $|\mathcal{L}| \leq \omega$ and diagrams $\Phi(A)$ in $\mathcal{L}$ corresponds to finite sets A forming at most countable well-ordered set.

Fig. 4

Fig. 5

Therefore, $\mathcal{L}$ is well-ordered, too, and if $\mathcal{L}$ is infinite, i. e., $\mathcal{M}$ is countable, then $\mathcal{L}$ is ordered by the type ω . Thus we have the following:

Corollary 2.3. *If the lattice $\mathcal{L} = \langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$ is linearly ordered then $\mathcal{L}$ is at most countable and well-ordered, being finite for finite $\mathcal{M}$ and having the type ω for countable $\mathcal{M}$.*

Definition. A structure $\mathcal{N}$ is *almost rigid* if for any $\bar{a} \in N$ the type $\text{tp}(\bar{a})$ has finitely many realizations in $\mathcal{N}$.

Theorem 2.4. *For a generic structure $\mathcal{M}$ with a class $\mathbf{K}$ of all models of $T = \text{Th}(\mathcal{M})$ which has finite closures, the following conditions are equivalent:*

- (1) $\mathcal{M}$ has a linearly ordered lattice $\mathcal{L}$;
- (2) $\mathcal{M}$ has a linearly ordered lattice $\mathcal{L}$ modulo finitely many incomparable elements;
- (3) $\mathcal{M}$ is almost rigid and at most countable.

Proof. (1) $\Leftrightarrow$ (2) holds since finitely many incomparable elements can be replaced by their amalgam.

(1) $\Rightarrow$ (3). If $\mathcal{M}$ has a linearly ordered lattice $\mathcal{L}$ then $\mathcal{M}$ can not be uncountable in view of Corollary 2.3. So by Theorem 1.6, $\mathcal{M}$ is homogeneous. Now if $\mathcal{M}$ is not almost rigid, then for any generic class $(\mathbf{D}_0; \leq)$ of diagrams whose copies are satisfied in $\mathcal{M}$ there is a diagram $\Phi(A) \in \mathbf{D}_0$ with $\mathcal{M} \models \Phi(A)$, and such that $\text{tp}(A)$ has infinitely many realizations in $\mathcal{M}$. Therefore there is a copy $\Phi(A')$ of $\Phi(A)$ such that A' is incomparable with A . Thus, $\langle L(\mathcal{M}, \mathbf{D}_0, \leq) \rangle$ is not linearly ordered.

(3) $\Rightarrow$ (1). Clearly, finite structures has two-element lattices $\mathcal{L}$ which are linearly ordered. Now for the countable almost rigid structure $\mathcal{M}$ we construct a required linearly ordered lattice $\mathcal{L}$ and a corresponding self-sufficient generic class, starting with the diagram $\Phi_0(\emptyset) = \text{Th}(\mathcal{M})$. Let linearly ordered diagrams $\Phi_0(\emptyset) \subset \Phi_1(A_1) \subset \dots \subset \Phi_n(A_n)$ be already constructed with $\mathcal{M} \models \Phi_n(A_n)$ and without copies satisfied in $\mathcal{M}$. Choose an element $b_n \in M \setminus A_n$ and consider the (finite) set B_n of all elements $b \in M$ such that $\text{tp}(b/A_n) = \text{tp}(b_n/A)$. Now we set $A_{n+1} = A_n \cup B_n$ $\Phi_{n+1}(A_{n+1}) = [\text{tp}_{X_{n+1}}(A_n \cup B_n)]_{A_n \cup B_n}^{X_{n+1}}$. Clearly, $\mathcal{M} \models \Phi_{n+1}(A_{n+1})$, $\Phi_n(A_n) \subset \Phi_{n+1}(A_{n+1})$ and $\Phi_{n+1}(A_{n+1})$ does not have copies satisfied in $\mathcal{M}$.

Extending the set $\{\Phi_n(A_n) \mid n \in \omega\}$ by all possible copies we form a generative class $(\mathbf{D}_0; \leq)$ with the linearly ordered lattice $\langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$. $\square$

Since ω -saturated (infinite) structures are not almost rigid we obtain the following:

Corollary 2.5. *There are no infinite ω -saturated structures $\mathcal{M}$ with linearly ordered lattices $\mathcal{L}$.*

By the definition the lattice $\mathcal{L}$ has relative complements if and only if for each $\Phi(A) \leq X(B)$ there is a diagram $\Psi(C) \leq X(B)$ (being the relative complement of $\Phi(A)$) such that $C \subseteq B \setminus A$ and $X(B) = \Phi(A) \vee \Psi(C)$. Since unions $\Phi(A) \vee \Psi(C)$ are always diagrams over finite sets, the distributive lattice $\mathcal{L}$ with relative complements is a Boolean algebra if and only if $\mathcal{L}$ is finite, i. e., $\mathcal{M}$ is finite.

Summarizing the arguments we have the following:

Proposition 2.6. *The distributive lattice $\mathcal{L}$ with relative complements is a Boolean algebra if and only if $\mathcal{M}$ is finite.*

Clearly, any finite structure $\mathcal{M}$ has only finite lattices $\mathcal{L}$ such that being Boolean algebras their cardinalities $|\mathcal{L}|$ can vary from 2 to $2^{|M|}$.

Considering the dynamics of the lattices $\mathcal{L}$ and applying Proposition 1.7 we note the following:

Proposition 2.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be any ω -homogeneous structures with $\mathcal{N} \prec \mathcal{M}$; $(\mathbf{D}_0; \leq)$ and $(\mathbf{D}'_0; \leq')$ be self-sufficient classes such that $\mathcal{M}$ is $(\mathbf{D}_0; \leq)$ -generic, $\mathcal{N}$ is $(\mathbf{D}'_0; \leq')$ -generic, and $(\mathbf{D}'_0; \leq')$ is a restriction of $(\mathbf{D}_0; \leq)$ coordinated with $\mathcal{N} \prec \mathcal{M}$; $\mathcal{L} = \langle L(\mathcal{M}, \mathbf{D}_0, \leq); \wedge, \vee \rangle$, $\mathcal{L}' = \langle L(\mathcal{N}, \mathbf{D}'_0, \leq'); \wedge, \vee \rangle$. Then $\mathcal{L}'$ is isomorphic to a quotient of $\mathcal{L}$. If, moreover, the restriction $(\mathbf{D}'_0; \leq')$ of $(\mathbf{D}_0; \leq)$ is conservative then $\mathcal{L}'$ is a sublattice of $\mathcal{L}$.*

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