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RECTANGLE AS A GENERALIZED ANGLE

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ABSTRACT. In order to extend the notion of quasimöbius mapping to non-injective case the concept of generalized angle $\Psi = (A_1, A_2; B_1, B_2)$ with sides A_1, A_2 and vertices B_1, B_2 (the sets in a Ptolemaic space) has been employed. The value of a generalizes angle is defined through Ptolimaic characteristic of tetrads and is not easy to by calculated in general case. Here we present the geometric way of calculation in the case where the general angle Ψ is a rectangle.

Keywords: quasimöbius mapping, quasiregular mapping, Ptolemaic space, generalized angle, mapping of bounded angular distortion, set-valued mapping.

1. BASIC NOTIONS

Semimetric ρ on X is a real-valued function $\rho : X \times X \rightarrow [0, +\infty)$ such that $\rho(a, b) = 0$ iff $a = b$, and $\rho(a, b) = \rho(b, a)$ for all $a, b \in X$. Then the pair (X, ρ) is called semimetric space.

In a semimetric space (X, ρ) , we consider the numerical characteristic

$$\alpha(a_1, a_2; b_1, b_2) := \frac{\rho(a_1, a_2)\rho(b_1, b_2)}{\rho(a_1, b_1)\rho(a_2, b_2) + \rho(a_1, b_2)\rho(a_2, b_1)} \quad (1.1)$$

of a *tetrad* (i.e. a four of distinct points) $T = (a_1, a_2; b_1, b_2)$ in addition to well-known notion of it's *absolute cross-ratio*

$$R(T) = R(a_1, a_2; b_1, b_2) := \frac{\rho(a_1, a_2)\rho(b_1, b_2)}{\rho(a_1, b_1)\rho(a_2, b_2)}.$$

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Recall that an injective mapping $f : (X, \rho) \rightarrow (Y, \sigma)$ of semimetric spaces is called *Möbius map* iff the equality $R(f(T)) = R(T)$ holds for every tetrad T in X . The both characteristics are invariant under Möbius mappings. The semimetric space (X, ρ) is *Ptolemaic* iff the Ptolemy's inequality

$$\rho(a_1, b_1)\rho(a_2, b_2) + \rho(a_1, b_2)\rho(a_2, b_1) \geq \rho(a_1, a_2)\rho(b_1, b_2)$$

holds for all a_1, a_2, b_1, b_2 in X . In a Ptolemaic space (X, ρ) , (1.1) is well-defined for any quadruple $T = \{a_1, a_2, b_1, b_2\}$ provided it does not contain a triple of coinciding points, and $\alpha(T) \leq 1$.

In order to obtain a suitable analog of quasiconformality in general metric spaces the concept of quasimöbius mapping has been introduced in [1]. That is

Definition 1. ([1, p. 219]) *Given a homeomorphism $\eta : [0, +\infty) \rightarrow [0, +\infty)$ (the control function) the injective mapping $f : (X, \rho) \rightarrow (Y, \sigma)$ in semimetric spaces is said to be η -quasomöbius if the estimate $R(f(T)) \leq \eta(R(T))$ holds for all tetrads T in (X, ρ) .*

The theory of quasimöbius mappings has been developed in close connections with theory of quasiconformal mappings in $\overline{R}^n$. The condition of injectivity seems to be the only obstacle that hinder us from extending the notion of quasimöbiusness to non-injective mappings such as, for example, quasiregular mappings in $\overline{R}^n$. The attempts to overcome this obstacle were undertaken in [2] and [3], but they cannot be admitted as satisfactory. Another way to deal with non-injective mappings in general concept of Möbius geometry has been presented in [4] and obtained further development in [5]-[9].

Definition 2. ([5, §4]) *A quadruple of sets $\Psi = (A_1, A_2; B_1, B_2)$ such that $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ and $B_1 \cup B_2$ has at least two distinct points is considered as generalized angle with sides A_1, A_2 and vertices B_1, B_2 . In a Ptolemaic space (X, ρ) , a value of generalized angle Ψ is defined as*

$$\alpha(\Psi) := \inf_{a_1 \in A_1, a_2 \in A_2} \sup_{b_1 \in B_1, b_2 \in B_2} \alpha(a_1, a_2; b_1, b_2),$$

if $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$, and $\alpha(\Psi) = 1$ in the case where $A_1 = \emptyset$ or $A_2 = \emptyset$.

The properties of generalized angles were listed in [7, §3]. In particular,

- (1) if $A_1 = \emptyset$ or $A_2 = \emptyset$ then $\alpha(A_1, A_2; B_1, B_2) = 1$;
- (2) $\alpha(A_1, A_2; B_1, B_2) = \alpha(A_1 \setminus (B_1 \cup B_2), A_2 \setminus (B_1 \cup B_2); B_1, B_2)$.

Definition 3. ([7, §4]) *Let (X, ρ) , (Y, σ) be Ptolemaic spaces, and $F : Y \rightarrow 2^X$ be a set-valued mapping such that $y_1 \neq y_2$ is equivalent to $F(y_1) \cap F(y_2) = \emptyset$ (the hyperinjectivity). Given an increasing function $\omega : [0, 1] \rightarrow [0, \infty)$ (the control function), the mapping F is said to have ω -BAD property (Bounded Angular Distortion) if $\alpha(F(\Psi)) \geq \omega(\alpha(\Psi))$ for all generalised angles Ψ in Y .*

Some basic properties of set-valued mappings with BAD property as well as their connections with quasiregular mappings may be found in [7], [9], [10].

The value of generalised angle in Definition 2 is not easy for calculation in general case. In this paper, we present the geometric way of calculation in the case where the generalised angle Ψ is a rectangle in the plane.

2. THE LOWER BOUNDS

By Ptolemy's Theorem the equality $\alpha(a_1, a_2; b_1, b_2) = 1$ for distinct points in $\overline{R^n}$ has place if and only if they belong to a generalized circle $S \subset \overline{R^n}$ and pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$ do not divide each other on this circle. Thus, due to the property (2) we have

Proposition 1. *Let a generalized angle $\Psi = (A_1, A_2; B_1, B_2)$ in $\overline{R^n}$ be such that every pair $a_1 \in A_1 \setminus (B_1 \cup B_2)$, $a_2 \in A_2 \setminus (B_1 \cup B_2)$ (if exists) lie on some circle $S \subset \overline{R^n}$ such that there exists a pair of points $b_1 \in B_1 \cap S$, $b_2 \in B_2 \cap S$ dividing the pair a_1, a_2 on S . Then $\alpha(A_1, A_2; B_1, B_2) = 1$.*

Theorem 1. *Let the domain D in the extended complex plane $\bar{\mathbf{C}}$ be bounded by two circular arcs γ_1, γ_2 with common ends in distinct points b_1, b_2 (the case $\gamma_1 = \gamma_2$ is possible as well). If D has the inner angle $\varphi \in (0, 2\pi]$ at the boundary points b_1, b_2 then the generalized angle $\Psi = (\gamma_1, \gamma_2; \{b_1\}, \{b_2\})$ has the value $\alpha(\Psi) = \sin(\varphi/2)$. In particular, for each pair of points $a_1 \in \gamma_1$, $a_2 \in \gamma_2$ the lower estimate $\alpha(a_1, a_2; b_1, b_2) \geq \sin(\varphi/2)$ is valid. Here, the equality holds only in the case where a_1, a_2 lie on a generalized circle which is orthogonal to both γ_1 and γ_2 or in the case $\varphi = \pi$.*

Proof. Since a Möbius mapping does not change the values of generalized angles, we may assume that $b_1 = 0$, $b_2 = \infty$, $\gamma_1 = \{Im = 0, Re > 0\}$, and $\gamma_2 = \{re^{i\varphi} : 0 < r < +\infty\}$. Then

$$\alpha(\Psi) = \inf_{r_1 > 0, r_2 > 0} \frac{|r_2 - r_1 e^{i\varphi}|}{r_1 + r_2}.$$

Given $a_1 \in \gamma_1$, $a_2 \in \gamma_2$ we have the equality $\alpha(a_1, a_2; 0, \infty) = \alpha(1, re^{i\varphi}; 0, \infty)$ where $r = r_2/r_1$. Since $(t + 1/t)/2 \geq 1$ for all $t > 0$, the function

$$\left(\frac{|1 - re^{i\varphi}|}{1 + r} \right)^2 = 1 - \frac{1 + \cos \varphi}{2[(\sqrt{r} + 1/\sqrt{r})/2]^2}$$

takes the minimal value $1 - (1 + \cos \varphi)/2 = \sin^2(\varphi/2)$ when $r = 1$ or $\varphi = \pi$. Thus $\alpha(r_1 e^{i\varphi}, r_2 e^{i\varphi}; 0, \infty) \geq \sin(\varphi/2)$ with the equality sign in the cases $r_1 = r_2$ or $\varphi = \pi$. $\square$

Corollary 1. *Let the line $L \subset \mathbf{C}$ be the common boundary of half-planes P_1, P_2 in $\mathbf{C}$. For distinct points $v_1, v_2 \in L$ and for all $u_1 \in (P_1 \cup L) \setminus \{b_1, b_2\}$, $u_2 \in (P_2 \cup L) \setminus \{b_1, b_2\}$ the estimate*

$$\alpha(u_1, u_2; v_1, v_2) \geq \sin(\varphi/2) \quad (2.1)$$

holds with $\varphi = \angle v_1 u_1 v_2 + \angle v_1 u_2 v_2$. Let the line L' be orthofonal to L and pass through $(v_1 + v_2)/2$. If $u_1 \in L' \cap (P_1 \cup L)$ and $u_2 \in L' \cap (P_2 \cup L)$ then (2.1) holds as the equality.

Proof. Given u_1, u_2 consider the circular arc γ_j with ends v_1, v_2 passing through u_j ($j = 1, 2$). If $\gamma_1 = \gamma_2$ then $\varphi \in \{0, 2\pi\}$, $\sin(\varphi/2) = 0$, and (2.1) is true. If $\gamma_1 \cup \gamma_2 = L$ then $\varphi = \pi$, $\sin(\varphi/2) = 1$, and $\alpha(u_1, u_2; v_1, v_2) = 1$ by Proposition 1. So (2.1) holds as equality.

In other cases, $\gamma_1 \cup \gamma_2$ is a common boundary of two complementary domains D and D^* in $\bar{\mathbf{C}}$ with $\infty \in \overline{D^*}$.

For all $z \in \gamma_j \setminus \{v_1, v_2\}$ we have the equality $\angle v_1 z v_2 = \varphi_j = \angle v_1 u_j v_2$ ($j = 1, 2$). While $z \rightarrow v_2$ along γ_j , the angle φ_j becomes the angle between the segment $[v_1 v_2]$

and the tangent ray to γ_j which is directed as a continuation of γ_j . Since $u_1 \in \bar{P}_1$ and $u_2 \in \bar{P}_2$, the sum $\varphi = \varphi_1 + \varphi_2$ equals to the angle between the tangent rays to γ_1 and γ_2 at v_2 directed as a continuation of these arcs. If $\varphi \leq \pi$ the domain D^* has inner angle φ at v_2 . Then Theorem 1 yields the estimate (2.1). If $\varphi > \pi$, then the domain D has inner angle $2\pi - \varphi < \pi$ at v_2 . Then by Theorem 1 we have $\alpha(u_1, u_2; v_1, v_2) \geq \sin(\pi - \varphi/2) = \sin(\varphi/2)$ as desired. In both cases the equality situation in (2.1) follows from Theorem 1 as well. $\square$

3. RECTANGLE AS A GENERALIZED ANGLE

The main result in this note is

Theorem 2. Let $K > 0$ and $\Psi = (A_1, A_2; B_1, B_2)$ be a generalized angle with sides $A_1 = \{Re = -K, |Im| \leq 1\}$, $A_2 = \{Re = K, |Im| \leq 1\}$ and vertices $B_1 = \{|Re| \leq K, Im = -1\}$, $B_2 = \{|Re| \leq K, Im = +1\}$. Then

$$\alpha(\Psi) = \begin{cases} 1 & \text{as } K \geq 1 \\ 2K/(1 + K^2) & \text{as } 0 < K < 1 \end{cases} \quad (3.1)$$

We shall use in the proof of the Theorem two following elementary facts.

Proposition 2. Let the line segment $V \subset \mathbf{C}$ with ends v_1, v_2 be parallel to the line L . Then

$$\max_{z \in L} \angle v_1 z v_2 = \angle v_1 z_0 v_2$$

where z_0 is the orthogonal projection of $(v_1 + v_2)/2$ to L .

Proposition 3. Let the lines L_1 and L_2 be orthogonal to the line L in $\mathbf{C}$. Let points $u_1, u'_1 \in L_1$ and $u_2, u'_2 \in L_2$ be such that $(u_1 + u_2)/2 = u_0 = (u'_1 + u'_2)/2$. Let $z_0 \in L$ be such a point that the ray coming from u_1 passes through u_2 after being reflected in L at z_0 . Let the point z'_0 be defined in a similar way for points u'_1, u'_2 . Then

$$\angle u_1 z_0 u_2 = \angle u'_1 z'_0 u'_2.$$

Proof. Consider the points $s_1 = L_1 \cap L$, $s_2 = L_2 \cap L$, $\tilde{u}_2 = s_2 - (u_2 - s_2)$, $\tilde{u}'_2 = s_2 - (u'_2 - s_2)$. The condition $(u_1 + u_2)/2 = (u'_1 + u'_2)/2$ means that $(u'_1 - u_1) = -(u'_2 - u_2)$. It follows that $u_1 - \tilde{u}_2 = u'_1 - \tilde{u}'_2$. So the line H through u_1 and $\tilde{u}_2$ is parallel to the line H' through u'_1 and $\tilde{u}'_2$. Since $\{z_0\} = L \cap H$ and $\{z'_0\} = L \cap H'$, we have the equality of the angles $\angle u_2 z_0 s_2 = \angle u_1 z_0 s_1 = \angle u'_1 z'_0 s_1 = \angle u'_2 z'_0 s_2$. Thus $\angle u_1 z_0 u_2 = \pi - 2\angle u_1 z_0 s_1 = \pi - 2\angle u'_1 z'_0 s_1 = \angle u'_1 z'_0 u'_2$ as desired. $\square$

Now we shall give the proof of Theorem 2.

Proof. Given $a_1 \in A_1$, $a_2 \in A_2$ let us find a lower bound for

$$\lambda(a_1, a_2) := \sup_{b_1 \in B_1, b_2 \in B_2} \alpha(a_1, a_2; b_1, b_2).$$

According to the symmetry of the task with respect to axes $\{Re = 0\}$ and $\{Im = 0\}$ we may assume that the point $a_0 := (a_1 + a_2)/2 = it_0$ lies in $\{Im \geq 0\}$ (that is $t_0 \geq 0$), and the points $a_1 = -K + (t_0 - x)i$, $a_2 = K + (t_0 + x)i$ are such that $Im(a_2) \geq Im(a_1)$ (that is $x \geq 0$). Let $a_1^* = -K + t_0 i$, $a_2^* = K + t_0 i$.

Case 1. Let $r^2 := K^2 + t_0^2 < 1$. It means that $B := \{|z| \leq r\}$ has no intersection with $B_1 \cup B_2$, and $a_1^*, a_2^* \in \partial B$.

Let $b_j \in B_j$ be such a point that the ray coming from a_1 and reflecting in B_j at b_j passes through a_2 ($j = 1, 2$). By Corollary 1 we have the inequality $\alpha(a_1, a_2; b_1, b_2) \geq$

$\sin(\varphi/2)$ where $\varphi = \angle a_1 b_1 a_2 + \angle a_1 b_2 a_2$. Then the Proposition 3 being applied to the points a_1, a_1^*, a_2, a_2^* yields the equality $\varphi = \angle a_1^*(i) a_2^* + \angle a_1^*(-i) a_2^*$. For the angle $\psi := \angle(i) a_1^*(-i)$ we have $\varphi = 2\pi - 2\psi$. By Proposition 2 $\psi \leq \psi_0 := \angle(i)(-K)(-i)$, so that $\varphi \geq 2\pi - 2\psi_0 = 2\angle(-K)(i)(+K)$. In the case under consideration, $\varphi = \angle a_1^*(-i) a_2^* + \angle a_1^*(i) a_2^* < \pi$, so the inequality $\varphi/2 \geq \angle(-K)(i)(+K)$ implies the inequality $\sin(\varphi/2) \geq \sin(\angle(-K)(i)(+K)) = 2K/(K^2 + 1)$. Hence we obtain the estimate

$$\lambda(a_1, a_2) \geq \alpha(a_1, a_2; b_1, b_2) \geq 2K/(K^2 + 1) = \alpha(-K, +K; i, -i). \quad (3.3)$$

Case 2. Let $r^2 := K^2 + t_0^2 \geq 1$. It means that $B := \{|z| \leq r\}$ has non-empty intersections both with B_1 and B_2 (in particular, $i \in B$ and $-i \in B$), and $a_1^*, a_2^* \in \partial B$. Consider the angle $\beta := \angle a_2 a_0 a_2^* = \angle a_1 a_0 a_1^*$ and the mapping $f(z) := it_0 + t(z - it_0)e^{i\beta}$ where $t = |a_1 - a_2|/K = \sqrt{1 + (x/K)^2} = \sqrt{1 + \tan^2 \beta} = 1/\cos \beta$. The calculation gives

$$f(i) = it_0 + \frac{e^{i\beta}}{\cos \beta} i(1 - t_0) = -(1 - t_0) \tan \beta = -\frac{(1 - t_0)x}{K} + i;$$

$$f(-i) = it_0 - \frac{e^{i\beta}}{\cos \beta} i(1 + t_0) = -i + (1 + t_0) \tan \beta = \frac{(1 + t_0)x}{K} - i.$$

Since $(1 - t_0)x \leq (1 - t_0)(1 + t_0) = 1 - t_0^2 \leq K^2$ and $(1 + t_0)x \leq (1 + t_0)(1 - t_0) \leq K^2$, we have $f(i) \in B_1$, $f(-i) \in B_2$. The closed disk $f(B)$ has its center at point $f(0) = t_0 x/K$ and the radius $r_* = \sqrt{K^2 + t_0^2}/\cos \beta = \sqrt{(1 + x^2/K^2)(K^2 + t_0^2)}$. Since

$$\begin{aligned} \left| (-K - i) - \frac{t_0 x}{K} \right|^2 - r_*^2 &= 1 + \left(\frac{t_0 x}{K} + K \right)^2 - \left(1 + \frac{x^2}{K^2} \right) (K^2 + t_0^2) = \\ &= 2t_0 x - t_0^2 - x^2 + 1 = 1 - (t_0 - x)^2 \geq 0, \end{aligned}$$

the point $(-K - i) \in B_1$ is out of the open disk B . Hence the line segment with end points $(-K - i)$ and $f(-i) \in B_1$ has a point $b'_1 \in B_1$ on the circle $C := \partial f(B)$.

Since

$$\begin{aligned} \left| (K + i) - \frac{t_0 x}{K} \right|^2 - r_*^2 &= 1 + \left(K - \frac{t_0 x}{K} \right)^2 - \left(1 + \frac{x^2}{K^2} \right) (K^2 + t_0^2) = \\ &= 1 - 2t_0 x - t_0^2 - x^2 = 1 - (t_0 + x)^2 \geq 0, \end{aligned}$$

the point $(K + i) \in B_2$ is out of the open disk B . Hence the line segment with end points $(K + i)$ and $f(i) \in B_2$ has a point $b'_2 \in C$. Now we obtain four points $a_1, a_2, b'_1, b'_2 \in C$ such that the pair a_1, a_2 divides the pair b'_1, b'_2 on the circle C . By Ptolemy's Theorem it means that $\alpha(a_1, a_2; b'_1, b'_2) = 1$. Thus in this case the equality $\lambda(a_1, a_2)$ holds.

If $K < 1$ the inequality $\lambda(a_1, a_2) \geq 2K/(1 + K^2)$ holds in each of cases 1 and 2. Thus

$$\alpha(\Psi) = \inf_{a_1 \in A_1, a_2 \in A_2} \lambda(a_1, a_2) \geq 2K/(1 + K^2) = \alpha(\{-K\}, \{K\}; i, (-i)).$$

Consequently, $\alpha(\Psi) = 2K/(1 + K^2)$ as $K < 1$.

If $K \geq 1$ only the case 2 is possible. So $\lambda(a_1, a_2) = 1$ for all $a_1 \in A_1$ and $a_2 \in A_2$. It means that $\alpha(\Psi) = \inf_{a_1, a_2} \lambda(a_1, a_2) = 1$. $\square$

In order to obtain usefull estimates for the mappings of bounded angular distortion we need to have some more examples of explicit expression for values of generalized angles.

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