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ON THE CAYLEY–DICKSON PROCESS FOR DIALGEBRAS

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ABSTRACT. We prove that the dialgebras, which are obtained by the Cayley–Dickson process from the two-dimensional commutative associative dialgebra $\mathcal{D}$, are disimple noncommutative Jordan dialgebras. Furthermore, a decomposition holds for them into the direct sum of a composition algebra and the equating ideal of the dialgebra.

Keywords: dialgebra, Cayley–Dickson process, flexible algebra, involution, noncommutative Jordan algebra, disimple dialgebra, composition algebra.

1. INTRODUCTION

A significant interest in dialgebras appeared recently (and in the structures that are closed to them, so-called Loday-type algebras). J.-L. Loday constructed the universal enveloping algebra for a Leibniz algebra (see [1]). The dialgebras, algebraical systems with two associative operations that agree among themselves, serve as these enveloping algebras. Lately, P.S. Kolesnikov [2] introduced a notion of a variety of dialgebras, which corresponds to a variety of usual algebras. A lot of functor constructions for the algebras in a variety $\mathcal{V}$ may be generalized for the dialgebras in the variety $\mathcal{V}$.

The Cayley–Dickson process of an algebra doubling is a well-known construction in algebra, which allows to construct a new algebra (with good properties) starting from an algebra with a unity and an involution (for example, see [3] for details). On this way, starting from a field, one may construct the quaternions and octonions. Moreover, it is known that all the algebras, which are obtained in this manner, are simple noncommutative Jordan algebras. The class of noncommutative Jordan algebras is extremely extensive: it includes alternative, Jordan, quasi-associative,

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quadratic flexible, and anticommutative algebras. For more information on the noncommutative Jordan algebras see, for example, [4].

The Jordan-type structures, which are related to dialgebras, were introduced and studied at the end of the past decade (for example, see [5–7]). In [8], the Cayley–Dickson process of an algebra doubling was transferred to the case of dialgebras. Let $\mathcal{D}$ be the two-dimensional commutative associative dialgebra with an involution (see the definition below). In [8], a dialgebraic analogs were constructed of quaternions and octonions, and it was proved as well, that all dialgebras, which are obtained by the Cayley–Dickson process from $\mathcal{D}$ (the *standard* CD-doubles), are symmetric flexible dialgebras with involutions; moreover, the CD-double $\mathcal{E}$ for $\mathcal{D}$ is associative, and the CD-double $\mathcal{F}$ for $\mathcal{E}$ is alternative. The main aim of the present article is a proof of the fact that all standard CD-doubles are disimple noncommutative Jordan dialgebras (see the definitions below), which are decomposed into the direct sum of a composition algebra and the equating ideal of the dialgebra.

As it was noted by P.S. Kolesnikov, the fact that the standard CD-doubles are noncommutative Jordan dialgebras probably may be obtained from [9]. However, the proof given below is short and simple. Moreover, our proof is based on a simple and interesting assertion on the flexible algebras, whose symmetric elements belong to the center.

In what follows, the symbol $:=$ denotes an equality by definition, and $\chi(F)$ stands for the characteristic of a field F (we assume that $\chi(F) \neq 2$, though this restriction is not essential in general); $\langle \Upsilon \rangle := \langle \Upsilon \rangle_F$ is the linear span of a set Υ over a field F , where we omit the symbol F if the field is clear from the context; $(x, y, z) := (xy)z - x(yz)$ and $x \circ y := xy + yx$ are the associator and the Jordan product, respectively.

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2. PRELIMINARIES

Recall some definitions. A *dialgebra* over a field F is a vector space over F with two binary bilinear operations $\dashv$ and $\vdash$.

A *0-dialgebra* over F is a dialgebra over F with the identities

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad z \dashv (x \vdash y) = z \dashv (x \dashv y).$$

A 0-dialgebra with two associative binary operations $\dashv$ and $\vdash$, which are also connected by the axiom

$$(1) \quad (x, y, z)_{\times} := (x \vdash y) \dashv z - x \vdash (y \dashv z) = 0,$$

is an *associative dialgebra*. Put

$$(x, y, z)_{\vdash} := (x \vdash y) \vdash z - x \vdash (y \vdash z), \quad (x, y, z)_{\dashv} := (x \dashv y) \dashv z - x \dashv (y \dashv z).$$

Thus, the axioms

$$(x, y, z)_{\times} = 0, \quad (x, y, z)_{\vdash} = 0, \quad (x, y, z)_{\dashv} = 0$$

together with the axioms of a 0-dialgebra hold in associative dialgebras.

Simplest examples of associative dialgebras are associative algebras ($a \dashv b = ab = a \vdash b$) and differential associative algebras (A, d) , $d^2 = 0$, $a \dashv b = a(db)$, $a \vdash b = (da)b$.

If M is an associative A -bimodule and $f : M \mapsto A$ is a bimodule homomorphism then M may be equipped with the structure of an associative dialgebra by putting $m_1 \dashv m_2 = m_1 f(m_2)$ and $m_1 \vdash m_2 = f(m_1) m_2$.

The tensor square of an associative algebra A may be equipped with the structure of an associative dialgebra:

$$(a \otimes b) \dashv (a' \otimes b') = a \otimes ba'b', \quad (a \otimes b) \vdash (a' \otimes b') = aba' \otimes b'.$$

If V is a vector space over a field F and ϕ is a linear functional in V^* then V is equipped with the structure of an associative dialgebra by

$$x \dashv y = \phi(y)x, \quad x \vdash y = \phi(x)y.$$

In dialgebras, the words of type $x_{-m} \vdash \dots \vdash x_0 \dashv \dots \dashv x_n$, under some arrangement of parentheses, are denoted by $x_{-m} \dots \dot{x}_0 \dots x_n$ (with the same arrangement of parentheses). The element x_0 is called *middle*.

Let $\mathcal{V}$ be a homogeneous variety that is defined by a set $\Sigma := \{t(x_1, \dots, x_n)\}$ of multilinear identities in the variables $X = \{x_1, x_2, \dots\}$. P.S. Kolesnikov [2], using the operad language, defined a notion of dialgebra in the variety $\mathcal{V}$ ($\mathcal{V}$ -dialgebra). In the same article, he showed that on the identity language the definition of a $\mathcal{V}$ -dialgebra may be given in the following way.

Define a linear mapping Ψ_n^i from the set of multilinear identities of the algebra into the set of dialgebra identities (*di-identities*) in the following way:

if $v = (x_{i_1} \dots x_i \dots x_{i_n})$ is a monomial with some arrangement of parentheses then $\Psi_n^i(v) = (x_{i_1} \vdash \dots x_i \dots \dashv x_{i_n})$ is a dimonomial with the same arrangement of parentheses.

In [2], the following theorem was proved.

Theorem. *A dialgebra A belongs to a variety $\mathcal{V}$ if and only if A is a 0-dialgebra and it satisfies the di-identities $\Psi_n^i(t)$, $t \in \Sigma$, $n = \deg t$, $i = 1, \dots, n$.*

Consider some examples of dialgebras in different varieties [2,10], which further will appear in this article.

Associative dialgebras: $\Sigma = \{(x_1, x_2, x_3)\}$; $\Psi_3^1(x_1, x_2, x_3) = (x_1, x_2, x_3)_{\dashv}$,

$$\Psi_3^2(x_1, x_2, x_3) = (x_1, x_2, x_3)_{\times}, \quad \Psi_3^3(x_1, x_2, x_3) = (x_1, x_2, x_3)_{\vdash}.$$

Commutative dialgebras: $\Sigma = \{x_1 x_2 - x_2 x_1\}$. In the dialgebra case we get

$$x_1 \vdash x_2 - x_2 \dashv x_1.$$

Hence, a commutative dialgebra A may be considered as a usual algebra with respect to the operation $ab = a \vdash b$, $a, b \in A$. In this case we obtain an algebra with the identity

$$(x_1 x_2 - x_2 x_1) x_3 = 0.$$

Alternative dialgebras:

$$\Sigma = \{(x_1, x_2, x_3) + (x_2, x_1, x_3), (x_1, x_2, x_3) + (x_1, x_3, x_2)\}.$$

The defining identities lead to the following dialgebra identities:

$$(x_1, x_2, x_3)_{\dashv} + (x_2, x_1, x_3)_{\times}, \quad (x_1, x_2, x_3)_{\vdash} + (x_2, x_1, x_3)_{\vdash},$$

$$(x_1, x_2, x_3)_{\dashv} + (x_1, x_3, x_2)_{\dashv}, \quad (x_1, x_2, x_3)_{\times} + (x_1, x_3, x_2)_{\vdash},$$

which are equivalent to the di-identities from [11].

Lie dialgebras: $\Sigma = \{(x_1, x_2, x_3) - x_2(x_1x_3), x_1x_2 + x_2x_1\}$. The corresponding dialgebra di-identities include $x_1 \dashv x_2 + x_2 \vdash x_1$. A Lie dialgebra, which is considered as a usual algebra with respect to the operation $a \dashv b$, is a right Leibniz algebra.

Recall the definition of a bimodule in the sense of Eilenberg and another approach to the definition of dialgebra varieties, which allows to understand better the structure of obtained dialgebras. Let $\mathcal{V}$ be a variety of algebras, and let $A \in \mathcal{V}$. A $\mathcal{V}$ -bimodule in the sense of Eilenberg is a vector space M equipped with some actions $am, ma \in M$ of elements in A such that the split null-extension $E = A \oplus M$ is an algebra in $\mathcal{V}$ with respect to the standard product

$$(a, m)(b, n) = (ab, an + mb).$$

Let D be a 0-dialgebra over a field F , let $\tilde{D}$ be an isomorphic copy of D as a vector space, and let $I^\circ := I^\circ(D) := \text{ideal}\langle x \dashv y - x \vdash y : x, y \in D \rangle = \langle x \dashv y - x \vdash y : x, y \in D \rangle_F$ be the *equating* ideal of D (in detail, see [10]). Then the rule

$$(2) \quad \bar{d}_1 \cdot \tilde{d}_2 = \widetilde{d_1 \vdash d_2}, \quad \tilde{d}_2 \cdot \bar{d}_1 = \widetilde{d_2 \dashv d_1}, \quad d_1, d_2 \in D,$$

correctly defines a bimodule action $\bar{D} := D/I^\circ$ on $\tilde{D}$.

Conversely, assuming that the action (2) is correctly defined, we arrive at the axioms of a 0-dialgebra.

A 0-dialgebra D is a $\mathcal{V}$ -dialgebra in the sense of Eilenberg provided that $\bar{D} := D/I^\circ \in \mathcal{V}$ and $\tilde{D}$ with respect to the operations (2) is a $\bar{D}$ -bimodule of the variety $\mathcal{V}$ in the sense of Eilenberg.

Theorem. [10] *A 0-dialgebra D is a $\mathcal{V}$ -dialgebra in the sense of Kolesnikov if and only if D is a $\mathcal{V}$ -dialgebra in the sense of Eilenberg.*

3. THE CAYLEY–DICKSON PROCESS FOR DIALGEBRAS

The *associative center* $Z_{Ass}(A)$ of an algebra A is defined as follows:

$$Z_{Ass}(A) := \{a \in A : (a, x, y) = (x, a, y) = (x, y, a) = 0 \ \forall x, y \in A\}.$$

Recall that an algebra A is *flexible* provided that the identity

$$(3) \quad (x, y, z) + (z, y, x) = 0$$

holds in A .

A flexible algebra is a *noncommutative Jordan algebra* if it possesses the identity $(x^2y)x = x^2(yx)$, which may be written as

$$(4) \quad (x \circ y, t, z) + (z \circ x, t, y) + (y \circ z, t, x) = 0,$$

after linearization.

A linear mapping $*$ of an algebra A is an *involution* provided that $(a^*)^* = a$, $(ab)^* = b^*a^*$ for all $a, b \in A$, and $A = H \oplus S$, where $H := H(A) = \{a \in A : a^* = a\}$ is the subspace of symmetric elements, and $S := S(A) = \{s \in A : s^* = -s\}$ is the subspace of skew-symmetric elements.

Theorem 1. *Let A be a flexible algebra with an involution such that $H(A) \subseteq Z_{Ass}(A)$. Then A is a noncommutative Jordan algebra.*

Proof. Let $H := H(A)$ be the subspace of symmetric elements in A (with respect to the involution), and let $S := S(A)$ be the subspace of skew-symmetric elements. Then $A = H \oplus S$. Since the defining identities of a noncommutative Jordan algebra are multilinear; therefore, it suffices to verify them on the elements in H and in S .

If $t \in H$ in (4) then (4) holds, since $t \in Z_{Ass}(A)$. An analogous argument holds in the case when two of x, y, z belong to H . Without loss of generality we may assume that $x \in H$, and $y, z \in S$. Show that $(xy, t, z) + (xz, t, y) = 0$. Using the inclusion $x \in Z_{Ass}(A)$, we get

$$(xy, t, z) = ((xy)t)z - xy \cdot tz = (x \cdot yt)z - x(y \cdot tz) = x(yt \cdot z) - x(y \cdot tz) = x \cdot (y, t, z).$$

Then $(xz, t, y) = x \cdot (z, t, y)$, and we obtain the required assertion by the flexibility. The equality $(yx, t, z) + (zx, t, y) = 0$ is proved analogously, since it is equivalent to $(z, t, yx) + (y, t, zx) = 0$ by the flexibility, and x stands on the right. Finally, (4) holds in A . $\square$

A 0-dialgebra D is *flexible* provided that the flexibility di-identities

$$(\dot{x}, y, z)_{\dashv} + (z, y, \dot{x})_{\vdash} = 0, \quad (x, \dot{y}, z)_{\times} + (z, \dot{y}, x)_{\times} = 0$$

hold in D .

A 0-dialgebra is a *noncommutative Jordan dialgebra* if it satisfies two flexibility di-identities and two Jordan di-identities

$$(5) \quad (\dot{x} \circ y, t, z)_{\dashv} + (z \circ \dot{x}, t, y)_{\dashv} + (y \circ z, t, \dot{x})_{\vdash} = 0,$$

$$(6) \quad (x \circ y, \dot{t}, z)_{\times} + (z \circ x, \dot{t}, y)_{\times} + (y \circ z, \dot{t}, x)_{\times} = 0.$$

Since the identities (5) and (6) are obtained by Kolesnikov's theorem, which was cited above; therefore, every Jordan dialgebra is a noncommutative Jordan dialgebra. There is a notion of a quasi-Jordan algebra, which is close to a Jordan dialgebra. The difference between quasi-Jordan algebras and noncommutative Jordan algebra was stated in [5].

Let D be a dialgebra with an *involution of the second type*, i. e., a linear mapping $*$ is defined on D such that $(a^*)^* = a$ for every $a \in D$, and $D = H \oplus S$, where $H := H(D) = \{a \in D : a^* = a\}$, $S := S(D) = \{s \in D : s^* = -s\}$, and $(a \dashv b)^* = b^* \vdash a^*$, $(a \vdash b)^* = b^* \dashv a^*$ for all $a, b \in D$. In what follows, a pair $(D; *)$ stands for a dialgebra D with an involution $*$ of the second type. Moreover, talking about involutions on a dialgebra, we always have in mind the involutions of the second type.

The *associative center* $Z_{Ass}(D)$ of a dialgebra D is defined as follows:

$$Z_{Ass}(D) := \{a \in D : (a, x, y)_{\diamond} = (x, a, y)_{\diamond} = (x, y, a)_{\diamond} = 0 \ \forall x, y \in D, \ \diamond \in \{\dashv, \times, \vdash\}\}.$$

Theorem 2. *Let $(A; *)$ be a flexible dialgebra such that $H(A) \subseteq Z_{Ass}(A)$. Then A is a noncommutative Jordan dialgebra.*

Proof. We follow the proof and notations of Theorem 1. A difference in the proof mainly consists in the fact that we have to consider now in (5) two possibilities: 1) $x \in H$; 2) $y \in H$. This is due to the fact that the elements x and y are unequal now: x is a middle element in contrast to y .

Case 1). Show that $(x \dashv y, t, z)_{\dashv} + (x \dashv z, t, y)_{\dashv} = 0$. Following the proof of Theorem 1, we get

$$(x \dashv y, t, z)_{\dashv} + (x \dashv z, t, y)_{\dashv} = x \dashv (y, t, z)_{\dashv} + x \dashv (z, t, y)_{\dashv}.$$

By the 0-identities we have $x \dashv (z, t, y)_{\dashv} = x \dashv (z, t, y)_{\vdash}$, and now the first flexibility di-identity gives the required equality.

Now, show that $(y \vdash x, t, z)_{\vdash} + (z \vdash x, t, y)_{\vdash} = 0$. By the first flexibility identity we have $(y \vdash x, t, z)_{\vdash} = -(z, t, y \vdash x)_{\vdash}$. Continuing analogously to the previous argument, finally we have

$$\begin{aligned} (y \vdash x, t, z)_{\vdash} + (z \vdash x, t, y)_{\vdash} &= -(z, t, y \vdash x)_{\vdash} - (y, t, z \vdash x)_{\vdash} = \\ &= -(z, t, y)_{\vdash} \vdash x - (y, t, z)_{\vdash} \vdash x = -(z, t, y)_{\vdash} \vdash x - (y, t, z)_{\vdash} \vdash x = 0. \end{aligned}$$

Case 2). Let $y \in H$. Show that $(y \vdash x, t, z)_{\vdash} + (y \vdash z, t, \dot{x})_{\vdash} = 0$. We have

$$(y \vdash x, t, z)_{\vdash} = y \vdash (x, t, z)_{\vdash}, \quad (y \vdash z, t, \dot{x})_{\vdash} = y \vdash (z, t, \dot{x})_{\vdash},$$

whence the required equality follows. Prove that

$$(x \dashv y, t, z)_{\vdash} + (z \vdash y, t, x)_{\vdash} = -(z, t, x \dashv y)_{\vdash} - (x, t, z \dashv y)_{\vdash} = 0.$$

Indeed,

$$(z, t, x \dashv y)_{\vdash} = (z, t, x)_{\vdash} \dashv y, \quad (x, t, z \dashv y)_{\vdash} = (x, t, z)_{\vdash} \dashv y.$$

It remains to consider (6).

As before, we may assume that $x \in H, y, z, t \in S$. Then we have to prove that

$$(x \circ y, t, z)_{\times} + (x \circ z, t, y)_{\times} = 0.$$

We have

$$\begin{aligned} (x \vdash y, t, z)_{\times} &= x \vdash (y, t, z)_{\times}; \quad (x \vdash z, t, y)_{\times} = x \vdash (z, t, y)_{\times}; \\ (y \vdash x, t, z)_{\times} &= -(z, t, y \vdash x)_{\times} = -(z, t, y)_{\times} \dashv x; \\ (z \vdash x, t, y)_{\times} &= -(y, t, z \vdash x)_{\times} = -(y, t, z)_{\times} \dashv x, \end{aligned}$$

which proves the theorem. $\square$

In [8], the authors transferred the Cayley–Dickson construction of algebra doubling to the case of dialgebras. Give this construction.

Let $(A; *)$ be a dialgebra over a field F . Consider the direct sum $(A, A) := A \oplus A$ of vector spaces and define on $A \oplus A$ the left and right products

$$\begin{aligned} (a, b) \dashv (c, d) &= (a \dashv c - d \vdash b^*, a^* \dashv d + c \vdash b), \\ (a, b) \vdash (c, d) &= (a \vdash c - d \dashv b^*, a^* \vdash d + c \dashv b). \end{aligned}$$

Extending the involution on $A \oplus A$ by the rule $(a, b)^* = (a^*, -b)$, we obtain an involution of second type on $A \oplus A$ [8]. The obtained dialgebra $CD(A) := A \oplus A$ is the *Cayley–Dickson double* of the dialgebra A . It is easy to see that $CD(A)$ is a $\mathbb{Z}_2$ -graded dialgebra: $CD(A) = CD(A)_0 + CD(A)_1$, where $CD(A)_0 := \{(a, 0) : a \in A\}$ and $CD(A)_1 := \{(0, a) : a \in A\}$.

We notice that this Cayley–Dickson double is obtained by the same scheme as P.S. Kolesnikov used for the definition of the varieties of dialgebras. However, as correctly notice the authors in [8], application of this scheme in the case under consideration is not standard, since it is used for the multiplication rule, not for an identity. Therefore, a direct transfer of algebraical identities to the dialgebra case is not justified in this situation.

It was shown in [8] that if A is a commutative associative dialgebra with involution then its Cayley–Dickson double is an associative dialgebra.

A dialgebra $(D; *)$ is *partially symmetric* provided that

$$(\dot{x} + \dot{\bar{x}})y = y(\dot{x} + \dot{\bar{x}}), \quad (x + \bar{x})\dot{y} = \dot{y}(x + \bar{x})$$

for all $x, y \in D$. If A is a partially symmetric dialgebra then $CD(A)$ is partially symmetric as well [8]. If A is a partially symmetric associative dialgebra then its Cayley–Dickson double is a partially symmetric alternative dialgebra [8]. A partially symmetric dialgebra is *symmetric* if its symmetric elements belong to the associative center. If A is a symmetric flexible dialgebra then $CD(A)$ is a symmetric flexible dialgebra as well [8]. In [8], starting from the unique two-dimensional associative commutative dialgebra $(\mathcal{D}; *)$, the authors constructed some dialgebra analogs of quaternions and octonions (note that $\mathcal{D}$ is spanned by two bar-units e and e^*). The dialgebras that are obtained by the consecutive application of the Cayley–Dickson process to $\mathcal{D}$ are the *standard* Cayley–Dickson doubles.

Recall that a *bar-unity* of a dialgebra $(A; \dashv, \vdash)$ is an element $e \in A$ such that $e \vdash x = x \dashv e = x$ for all $x \in A$. (It is easy to notice that if e is a bar-unity in $(A; *)$ then e^* is a bar-unity in A as well.) The *commutative center* $Z_{\text{Comm}}(A)$ of a dialgebra A is defined as follows:

$$Z_{\text{Comm}}(A) := \{a \in A : a \dashv y = y \vdash a, a \vdash y = y \dashv a \ \forall y \in D\};$$

$$Z(A) := Z_{\text{Comm}}(A) \cap Z_{\text{Ass}}(A)$$

is the *center* of a dialgebra A .

Lemma. [8] *Up to a multiplication by a scalar, there exists unique two-dimensional associative commutative dialgebra $(\mathcal{D}; \dashv, \vdash, *)$ with an involution $*$ and nonzero equating ideal.* $\square$

For the convenience of reader, write the multiplication tables of dialgebras $\mathcal{D}, E$. The multiplication table for $(\mathcal{D}; \dashv, \vdash, *)$ may be chosen as follows:

$\dashv$	x	y
x	x	x
y	y	y

$\vdash$	x	y
x	x	y
y	y	y

The involution is the following one: $x^* = y$, $y^* = x$. It is easy to see that x and y are bar-unities in $\mathcal{D}$, and $\frac{1}{2}(x + y)$ is a symmetric bar-unity in $Z(\mathcal{D})$.

Applying the Cayley–Dickson process to $\mathcal{D}$, we obtain a four-dimensional associative dialgebra $\mathcal{E}$ with an involution $*$: for some basis $\{p, q, r, s\}$, its multiplication table is the following one:

$\dashv$	p	q	r	s
p	p	p	s	s
q	q	q	r	r
r	r	r	$\neg q$	$\neg q$
s	s	s	$\neg p$	$\neg p$

$\vdash$	p	q	r	s
p	p	q	r	s
q	p	q	r	s
r	r	s	$\neg p$	$\neg q$
s	r	s	$\neg p$	$\neg q$

The involution is the following one: $p^* = q$, $q^* = p$, $r^* = -r$, $s^* = -s$.

It is easy to see that p and q are bar-unities in $\mathcal{E}$, and $\frac{1}{2}(p + q)$ is a symmetric bar-unity in $Z(\mathcal{E})$.

Theorem 2 implies

Theorem 3. *Let $(A; *)$ be a symmetric flexible dialgebra. Then its Cayley–Dickson double is a noncommutative Jordan dialgebra.*

Thus, starting from $\mathcal{D}$, all obtained Cayley–Dickson doubles are noncommutative Jordan dialgebras, which is an analog to the well-known result for algebras.

It is well-known [4] that A is a noncommutative Jordan algebra if and only if A is flexible and $A^{(+)}$ is a Jordan algebra, where $A^{(+)}$ is the adjoint algebra to A with the product $x \bullet y = xy + yx$. By [10, Theorem 4], A is a noncommutative Jordan dialgebra if and only if A is a flexible dialgebra and $A^{(+)}$ is a Jordan dialgebra with the product $x \bullet y = x \dashv y + y \vdash x$.

In what follows, we are interested in the structure of the Cayley–Dickson doubles.

Proposition 1. *Let $(A; \dashv, \vdash, *)$ be a dialgebra with an involution $*$ and a symmetric bar-unity e over a field F of characteristic not 2. If $A = \langle e \rangle \oplus_F S(A)$, $S(A) \neq 0$, and $D := CD(A)$ then $H(D) = \langle f \rangle_F$, $S(D) = (S(A), A)$, where $f = (e, 0)$ is a symmetric bar-unity in D . Furthermore, $Z_{\text{Comm}}(D) = \langle f \rangle$, if $e \in Z_{\text{Comm}}(A)$, and $Z_{\text{Comm}}(D) = 0$ otherwise.*

Proof. The assertions about $H(D)$ and $S(D)$ are obvious. Let $(a, b) \in Z_{\text{Comm}}(D)$. Then

$$\begin{aligned} (a, b) \dashv (c, d) &= (a \dashv c - d \vdash b^*, a^* \dashv d + c \vdash b) = \\ (c, d) \vdash (a, b) &= (c \vdash a - b \dashv d^*, c^* \vdash b + a \dashv d), \end{aligned}$$

whence with $d = 0$ we get $c \vdash a = a \dashv c$, $c^* \vdash b = c \vdash b$ for every $c \in A$; and putting $c = 0$ we have $b \dashv d^* = d \vdash b^*$, $a \dashv d = a^* \dashv d$ for every $d \in A$. In particular, putting $d = e$ we have $a, b \in H(A)$. On the other hand,

$$\begin{aligned} (a, b) \vdash (c, d) &= (a \vdash c - d \dashv b^*, a^* \vdash d + c \dashv b) = \\ (c, d) \dashv (a, b) &= (c \dashv a - b \vdash d^*, c^* \dashv b + a \vdash d), \end{aligned}$$

whence with $d = 0$ we get $a \vdash c = c \dashv a$, and $c \dashv b = c^* \dashv b$ for every $c \in A$. Thus, $a \in Z_{\text{Comm}}(A) \cap H(A)$. Since $b \in H(A)$; therefore, $b = \alpha e$ for some $\alpha \in F$, whence $c \dashv b = \alpha c = c^* \dashv b = \alpha c^*$ for every $c \in A$, i. e., $\alpha = 0$, and $b = 0$. $\square$

Corollary 1. *Let D be the standard Cayley–Dickson double with a symmetric bar-unity e . Then*

$$Z(D) = Z_{\text{Comm}}(D) = \langle e \rangle = H(D).$$

Proof. It follows from Proposition 1 and the results of [8], which were cited above. $\square$

Lemma 1. *Let $D := CD(A)$ be the Cayley–Dickson double of a dialgebra A , and let $I^\circ(A)$ be the equating ideal of A . Then $I^\circ(D) = I^\circ(A) \oplus I^\circ(A)$.*

Proof. Introduce new operation $x \star y := x \dashv y - x \vdash y$. By the definitions of the operations in D , we have

$$\begin{aligned} I^\circ(D) &= \langle (a, b) \star (c, d) : a, b, c, d \in A \rangle \\ &= \langle (a \star c + d \star b^*, a^* \star d - c \star b) : a, b, c, d \in A \rangle \subseteq I^\circ(A) \oplus I^\circ(A). \end{aligned}$$

The inverse inclusion is obvious. $\square$

Apparently, the notion of simple dialgebra firstly appeared for Leibniz algebras, since every Leibniz algebra is a Lie dialgebra. This notion for the associative dialgebras appeared in [6].

A dialgebra A is *disimple* provided that A/I° is simple.

Theorem 4. *Let $(A; \dashv, \vdash, *)$ be a disimple dialgebra with an involution $*$ and a symmetric bar-unity e over a field F of characteristic not 2. Let $A = \langle e \rangle \oplus_F S(A)$ and $S(A) \neq 0$. Assume that F does not contain the roots of the equation $x^2 = -1$. Then the Cayley–Dickson double $CD(A)$ is a disimple dialgebra.*

Proof. Let $J \trianglelefteq D := CD(A)$ and $I^\circ(D) \subsetneq J$. By Lemma 1

$$(x \dashv y, 0) - (x \vdash y, 0) \in I^\circ(D) \subseteq J, \quad (0, x \dashv y) - (0, x \vdash y) \in I^\circ(D) \subseteq J$$

for all $x, y \in A$. Therefore, the operations $\dashv$ and $\vdash$ may be identified (and may not be written). Further, $\equiv$ denotes an equivalence modulo J . Let $(x, y) \in J$ and $x \notin I^\circ(A)$. Then

$$(7) \quad (x, y)(c, 0) \equiv (xc, cy) \in J, \quad (c, 0)(x, y) \equiv (cx, c^*y) \in J,$$

$$(8) \quad (x, y)(0, d) \equiv (-dy^*, x^*d) \in J, \quad (0, d)(x, y) \equiv (-yd^*, xd) \in J$$

for all $c, d \in A$. Put $J_A := \{a \in A : \exists b \in A(a, b) \in J\}$. The inclusions (7)–(8) say that $J_A \trianglelefteq A$ and $I^\circ(A) \subsetneq J_A$, whence the disimplicity of A implies $J_A = A$. Thus, we may assume that $(e, y) \in J$. Then (7) implies that $(c, cy), (c, c^*y) \in J$, whence $(2c, (c + c^*)y) \in J$ for every $c \in A$. Let $c = \alpha e + s$ for some $\alpha \in F, s \in S(A)$. Then $(2(\alpha e + s), 2\alpha ey) \in J$, whence $(s, 0) \in J$ for every $s \in S(A)$. Since $(s, 0)(0, e) = (0, s^*e)$, we have $(0, s) \in J$ for every $s \in S(A)$. If $y \in S(A)$ then $(e, 0) \in J$, and $J = D$.

If $y = \beta e + s$ for some $0 \neq \beta \in F, s \in S(A)$ then $(e, y) = (e, \beta e) + (0, s)$ and $(e, \beta e) \in J$. Then for all $c, d \in A$ we have

$$(e, \beta e)(c, d) = (c - \beta d, d + \beta c) = (c, \beta c) + (-\beta d, d) \in J,$$

whence $(c, \beta c), (-\beta c, c) \in J$ for every $c \in A$, i. e., $(0, (1 + \beta^2)c) \in J$, and if $1 + \beta^2 \neq 0$ in F then $(0, e) \in J$ and $J = D$. $\square$

Corollary 2. *Let $(D; *)$ be the standard Cayley–Dickson double over a field without the roots from -1 . Then $(D; *)$ is a disimple noncommutative Jordan dialgebra.*

Let $(A; \dashv, \vdash, *)$ be a disimple dialgebra with an involution $*$ and a symmetric bar-unity $e \in Z_{\text{Comm}}(A)$ over a field F of characteristic not 2, which contains an element $\mathbf{i}$ such that $\mathbf{i}^2 = -1$. Let $A = \langle e \rangle \oplus_F S(A)$ and $S(A) \neq 0$. Consider $J := \langle (\mathbf{i}e, e) \rangle + (S(A), S(A))$. Show that $J \triangleleft D(A)$.

It is easy to see that $(\mathbf{i}a, a) \equiv (\mathbf{i}a, a^*) \equiv (\mathbf{i}e, e) \equiv 0 \pmod{J}$ for every $a \in A$. Since $e \in Z_{\text{Comm}}(A)$ and $(a \vdash e)^* = a^* \vdash e$ for every $a \in A$; therefore, the equalities

$$(a, b) \dashv (\mathbf{i}e, e) = (\mathbf{i}a - b^*, a^* + \mathbf{i}b) = (\mathbf{i}a, a^*) + (-b^*, \mathbf{i}b),$$

$$(a, b) \vdash (\mathbf{i}e, e) = (\mathbf{i}a \vdash e, a^* \vdash e) - (e \dashv b^*, \mathbf{i}e \dashv b),$$

$$(\mathbf{i}e, e) \vdash (a, b) = (\mathbf{i}a - b, a + \mathbf{i}b) = (\mathbf{i}a, a) + (-b, \mathbf{i}b),$$

$$(\mathbf{i}e, e) \dashv (a, b) = (\mathbf{i}e \dashv a, a \vdash e) - (b \vdash e, -\mathbf{i}e \dashv b)$$

prove our assertion. As it is easy to see we have $\text{codim } J = 1$. Note that J is not $*$ -invariant, since otherwise J coincides with D . Thus, in the given case $D(A)$ contains a unique maximal ideal J of codimension 1. Furthermore, the quotient-algebra D/J is isomorphic to the main field. Thus, in the given case the standard Cayley–Dickson doubles are not disimple.

Let A be a dialgebra, let $I := I^\circ(A)$ be its equating ideal, and let A/I be isomorphic to an algebra, which is obtained by the consecutive application of the Cayley–Dickson process to the field $\mathbb{R}$ of real numbers. Let $D := CD(A)$, and let

$J := I^\circ(D)$ be the equating ideal of D . Then $J = (I, I)$. Note that $(A/J, 0)$ is a subalgebra of D/J , which is isomorphic to A/I . Considering $(A/J, A/J)$, we see that the multiplication in this algebra is obtained from the multiplication in A/I by the Cayley–Dickson process. Hence, $(A/J, A/J)$ is a Cayley–Dickson double of A/I . Thus, we have

Theorem 5. *Let $(A; *)$ be a dialgebra over $\mathbb{R}$ and $B := A/I^\circ(A)$. Then the algebra $CD(A)/I^\circ(CD(A))$ is isomorphic to an algebra, which is obtained by the consecutive application of the Cayley–Dickson process to B .*

In particular, applying Theorem 5 to the dialgebras $\mathcal{D}, E, F$, which were constructed in [8], we obtain the fields $\mathbb{R}$ and $\mathbb{C}$ as some quotient algebras by the equating ideals, and the quaternion algebra $\mathbb{H}$ as well (considering $\mathcal{G} := CD(\mathcal{F})$, we also get the algebra of Cayley numbers).

To prove the following theorem we need to give the multiplication tables for the dialgebras $\mathcal{F}$ and $\mathcal{G}$. The first table we take from [8], and we directly compute the second one.

The multiplication table for the eight-dimensional alternative dialgebra $\mathcal{F}$ with an involution $*$ for some basis $\{p, q, r, s, t, u, v, w\}$:

$\dashv$	p	q	r	s	t	u	v	w
p	p	p	s	s	u	u	v	v
q	q	q	r	r	t	t	w	w
r	r	r	$-q$	$-q$	$-v$	$-v$	u	u
s	s	s	$-p$	$-p$	$-w$	$-w$	t	t
t	t	t	v	v	$-q$	$-q$	$-s$	$-s$
u	u	u	w	w	$-p$	$-p$	$-r$	$-r$
v	v	v	$-t$	$-t$	r	r	$-p$	$-p$
w	w	w	$-u$	$-u$	s	s	$-q$	$-q$

$\vdash$	p	q	r	s	t	u	v	w
p	p	q	r	s	t	u	v	w
q	p	q	r	s	t	u	v	w
r	r	s	$-p$	$-q$	$-v$	$-w$	t	u
s	r	s	$-p$	$-q$	$-v$	$-w$	t	u
t	t	u	v	w	$-p$	$-q$	$-r$	$-s$
u	t	u	v	w	$-p$	$-q$	$-r$	$-s$
v	w	v	$-u$	$-t$	s	r	$-q$	$-p$
w	w	v	$-u$	$-t$	s	r	$-q$	$-p$

The involution is the following one:

$$p^* = q, \quad q^* = p, \quad r^* = -r, \quad s^* = -s, \quad t^* = -t, \quad u^* = -u, \quad v^* = -v, \quad w^* = -w.$$

The multiplication table for the 16-dimensional alternative dialgebra $\mathcal{G}$ with an involution $*$ for some basis $\{p, q, r, s, t, u, v, w, a, b, c, d, e, f, g, h\}$:

$\dashv$	p	q	r	s	t	u	v	w	a	b	c	d	e	f	g	h
p	p	p	s	s	u	u	v	v	b	b	c	c	e	e	h	h
q	q	q	r	r	t	t	w	w	a	a	d	d	f	f	g	g
r	r	r	$-q$	$-q$	$-v$	$-v$	u	u	$-c$	$-c$	b	b	g	g	$-f$	$-f$
s	s	s	$-p$	$-p$	$-w$	$-w$	t	t	$-d$	$-d$	a	a	h	h	$-e$	$-e$
t	t	t	v	v	$-q$	$-q$	$-s$	$-s$	$-e$	$-e$	$-g$	$-g$	b	b	d	d
u	u	u	w	w	$-p$	$-p$	$-r$	$-r$	$-f$	$-f$	$-h$	$-h$	a	a	c	c
v	v	v	$-t$	$-t$	r	r	$-p$	$-p$	$-g$	$-g$	e	e	$-c$	$-c$	a	a
w	w	w	$-u$	$-u$	s	s	$-q$	$-q$	$-h$	$-h$	f	f	$-d$	$-d$	b	b
a	a	a	c	c	e	e	h	h	$-q$	$-q$	$-s$	$-s$	$-u$	$-u$	$-v$	$-v$
b	b	b	d	d	f	f	g	g	$-p$	$-p$	$-r$	$-r$	$-t$	$-t$	$-w$	$-w$
c	c	c	$-a$	$-a$	g	g	$-f$	$-f$	r	r	$-p$	$-p$	v	v	$-u$	$-u$
d	d	d	$-b$	$-b$	h	h	$-e$	$-e$	s	s	$-q$	$-q$	w	w	$-t$	$-t$
e	e	e	$-g$	$-g$	$-a$	$-a$	d	d	t	t	$-v$	$-v$	$-p$	$-p$	s	s
f	f	f	$-h$	$-h$	$-b$	$-b$	c	c	u	u	$-w$	$-w$	$-q$	$-q$	r	r
g	g	g	e	e	$-c$	$-c$	$-b$	$-b$	v	v	t	t	$-r$	$-r$	$-q$	$-q$
h	h	h	f	f	$-d$	$-d$	$-a$	$-a$	w	w	u	u	$-s$	$-s$	$-p$	$-p$

$\vdash$	p	q	r	s	t	u	v	w	a	b	c	d	e	f	g	h
p	p	q	r	s	t	u	v	w	a	b	c	d	e	f	g	h
q	p	q	r	s	t	u	v	w	a	b	c	d	e	f	g	h
r	r	s	$-p$	$-q$	$-v$	$-w$	t	u	$-c$	$-d$	a	b	g	h	$-e$	$-f$
s	r	s	$-p$	$-q$	$-v$	$-w$	t	u	$-c$	$-d$	a	b	g	h	$-e$	$-f$
t	t	u	v	w	$-p$	$-q$	$-r$	$-s$	$-e$	$-f$	$-g$	$-h$	a	b	c	d
u	t	u	v	w	$-p$	$-q$	$-r$	$-s$	$-e$	$-f$	$-g$	$-h$	a	b	c	d
v	w	v	$-u$	$-t$	s	r	$-q$	$-p$	$-h$	$-g$	f	e	$-d$	$-c$	b	a
w	w	v	$-u$	$-t$	s	r	$-q$	$-p$	$-h$	$-g$	f	e	$-d$	$-c$	b	a
a	a	b	c	d	e	f	g	h	$-p$	$-q$	$-r$	$-s$	$-t$	$-u$	$-v$	$-w$
b	a	b	c	d	e	f	g	h	$-p$	$-q$	$-r$	$-s$	$-t$	$-u$	$-v$	$-w$
c	d	c	$-b$	$-a$	g	h	$-e$	$-f$	s	r	$-q$	$-p$	v	w	$-t$	$-u$
d	d	c	$-b$	$-a$	g	h	$-e$	$-f$	s	r	$-q$	$-p$	v	w	$-t$	$-u$
e	f	e	$-g$	$-h$	$-b$	$-a$	c	d	u	t	$-v$	$-w$	$-q$	$-p$	r	s
f	f	e	$-g$	$-h$	$-b$	$-a$	c	d	u	t	$-v$	$-w$	$-q$	$-p$	r	s
g	g	h	f	e	$-d$	$-c$	$-a$	$-b$	v	w	u	t	$-s$	$-r$	$-p$	$-q$
h	g	h	f	e	$-d$	$-c$	$-a$	$-b$	v	w	u	t	$-s$	$-r$	$-p$	$-q$

The involution is the following one: $p^* = q$, $q^* = p$, where the remaining basis elements are skew-symmetric.

Theorem 6. Let $(A; *)$ be one of the dialgebras $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ over a field F of characteristic not 2, and let $B := A/I^\circ(A)$. Then $A = S \oplus I$, where S is a composition subalgebra of A , which is isomorphic to B , and $I := I^\circ(A)$. Furthermore, $I \diamond I = 0$, $\diamond \in \{-, \vdash\}$, $S \dashv I = I \vdash S = 0$, and I is an irreducible module over S , if F is formally real.

Proof. Considering the multiplication tables of the dialgebras $\mathcal{D}, \mathcal{E}$, which were given above, we infer that $I^\circ(\mathcal{D}) = \langle x - y \rangle_F$ and $S = \langle x \rangle_F \cong F$ in the case $\mathcal{D}$; and $I^\circ(\mathcal{E}) = \langle p - q, r - s \rangle_F$, $S = \langle p + q, r + s \rangle_F \cong \mathbb{C}$ in the case $\mathcal{E}$. In the case $\mathcal{F}$, we

have $I^\circ(\mathcal{F}) = \langle p - q, r - s, u - t, v - w \rangle_F$, and $S = \langle p + q, r + s, u + t, v + w \rangle_F \cong \mathbb{H}$. In the last case, considering the multiplication table of $\mathcal{G}$, we get

$$\begin{aligned} I^\circ(\mathcal{G}) &= \langle p - q, r - s, u - t, v - w, a - b, c - d, e - f, g - h \rangle_F, \\ S &= \langle p + q, r + s, u + t, v + w, a + b, c + d, e + f, g + h \rangle_F \cong \mathbb{O}. \end{aligned}$$

To see the last isomorphism one may put

$$\begin{aligned} e_0 &= \frac{1}{2}(p + q), \quad e_1 = \frac{1}{2}(r + s), \quad e_2 = \frac{1}{2}(v + w), \quad e_3 = \frac{1}{2}(t + u), \\ e_4 &= \frac{1}{2}(c + d), \quad e_5 = \frac{1}{2}(a + b), \quad e_6 = \frac{1}{2}(e + f), \quad e_7 = -\frac{1}{2}(g + h), \end{aligned}$$

obtaining the multiplication table of the Cayley–Dickson algebra $C(-1, -1, -1)$ (for example, see [3]).

The proof of the equalities $I \diamond I = 0$, $\diamond \in \{\dashv, \vdash\}$, $S \dashv I = I \vdash S = 0$ follows easily from the multiplication tables of the dialgebras, and the fact that I is an irreducible module over S if F is formally real may be also proved directly (it is convenient to use in the proof induction on the dimension of the Cayley–Dickson double, because of the specific module structure). $\square$

Finally, note that in every algebra of generalized quaternions Q over F the identity $xy^*z - zy^*x = yz^*x - xz^*y$ holds, since $xy^* + yx^* = x^*y + y^*x \in F$ for all $x, y \in Q$. An analog to this result holds in $\mathcal{E}$ as well.

Lemma 2. *For all $x, y, z \in \mathcal{E}$, we have*

$$\begin{aligned} x \dashv y^* + y \vdash x^* &= x^* \dashv y + y^* \vdash x \in Z(\mathcal{E}); \\ x \dashv y^* \dashv z - z \vdash y^* \vdash x &= y \vdash z^* \vdash x - x \dashv z^* \dashv y, \\ x \vdash y^* \dashv z - z \vdash y^* \dashv x &= y \dashv z^* \dashv x - x \vdash z^* \vdash y, \\ x \vdash y^* \vdash z - z \dashv y^* \dashv x &= y \vdash z^* \dashv x - x \vdash z^* \dashv y. \end{aligned}$$

Proof. Since $x + x^* \in Z(\mathcal{E}) := \langle e \rangle$ and $x \dashv y^* + y \vdash x^* \in H(\mathcal{E})$ for all $x, y \in \mathcal{E}$; therefore, the first assertion is obvious (in particular, one may use the decomposition $y = \alpha e + s$, where e is the symmetric bar-unity in the center, and $s \in S(\mathcal{E})$). The remaining assertions follow from the first one, regrouping the summands and using the 0-di-identities and the diassociativity. For example,

$$\begin{aligned} x \dashv y^* \dashv z - z \vdash y^* \vdash x - y \vdash z^* \vdash x + x \dashv z^* \dashv y &= \\ x \dashv (y^* \dashv z + z^* \vdash y) - (z \vdash y^* + y \vdash z^*) \vdash x &= 0, \end{aligned}$$

as it was required. $\square$

Now, using the results of [12] and equipping $\mathcal{E}$ with a ternary operation $[x, y, z] = (x \dashv y^* \dashv z) - (z \vdash y^* \vdash x)$, we obtain a nontrivial example of a Filippov algebra $\mathcal{FE}$ with split product (so called *sp*-algebra) (*sp*-algebras are a generalization of dialgebras to the case of Ω -algebras; the definition of *sp*-algebras is slightly cumbersome, and it may be found in [12]).

Recall that a ternary Filippov *sp-algebra* is a ternary algebra, which satisfies the identities

$$\begin{aligned} [a, b, b] &= 0, \\ [d, [a, a, b], e] &= 0, \\ [a, b, [c, d, e]] &= [[a, b, c], d, e] + [[a, b, d], e, c] + [[a, b, e], c, d], \\ [[c, d, e], b, a] &= [[c, b, a], d, e] + [c, [d, b, a], e] + [c, d, [e, b, a]]. \end{aligned}$$

Following [12], if $(D; *)$ is an associative dialgebra with an involution of second type then we can equip D with three ternary operations $\{x, y, z\}_i$ ($i = 1, 2, 3$):

$$\{x, y, z\}_1 = x \dashv y^* \dashv z, \quad \{x, y, z\}_2 = x \vdash y^* \dashv z, \quad \{x, y, z\}_3 = x \vdash y^* \vdash z,$$

and denote the obtained Ω -algebra by $\mathcal{A}(D, *)$. Then by [12] $\mathcal{A}(D, *)$ is an associative *sp*-system of second type, which satisfies the identities

$$\begin{aligned} \{x, y, z\}_1 + \{y, x, z\}_2 &= \{z, x, y\}_2 + \{z, y, x\}_3, \\ \{x, y, z\}_3 + \{y, x, z\}_3 &= \{z, x, y\}_1 + \{z, y, x\}_1, \end{aligned}$$

if $D = \mathcal{E}$. By [12], $\mathcal{FE}$ is a ternary Filippov *sp*-algebra. The multiplication table for $\mathcal{FE}$ may be easily written, using the multiplication table for $\mathcal{E}$ from [8]. The obtained multiplication is nontrivial, i. e., it is nonzero and it is not a multiplication of a Filippov algebra, since it is not anticommutative in all variables.

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