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ON ALGEBRAIC ISOMORPHISMS OF RATIONAL
COHOMOLOGY OF A KÜNNEMAN COMPACTIFICATION OF
THE NÉRON MINIMAL MODEL

S.G. TANKEEV

ABSTRACT. It is proved that the Grothendieck standard conjecture of Lefschetz type holds for rational cohomology of degree 2 or 3 of a Künnemann compactification of the Néron minimal model of an absolutely simple principally polarized Abelian variety over the field of rational functions of a smooth projective curve under certain restrictions on the ring of endomorphisms of the Abelian variety.

Keywords: Abelian variety, Néron minimal model, Künnemann compactification, Grothendieck standard conjecture of Lefschetz type.

INTRODUCTION

Let H be an ample divisor on a smooth complex projective d -dimensional variety X . Then, for any natural number $i \leq d$, the map

$$L^{d-i} : H^i(X, \mathbb{Q}) \xrightarrow{\sim \text{cl}_X(H)^{\sim d-i}} H^{2d-i}(X, \mathbb{Q})$$

is an isomorphism by the strong Lefschetz theorem. The Grothendieck standard conjecture $B(X)$ of Lefschetz type [1] asserts that there exists an algebraic $\mathbb{Q}$ -cycle Z on the Cartesian product $X \times X$ which determines the inverse *algebraic* isomorphism

$$H^{2d-i}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \sim \text{cl}_{X \times X}(Z))} H^i(X, \mathbb{Q}).$$

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Denote by ${}^c\Lambda$ the dual operator for L of the classical Hodge theory. It is well known that the conjecture $B(X)$ is equivalent to the algebraicity of the operator ${}^c\Lambda$ [2, Proposition 2.3].

There is an abstract form of the standard conjecture for étale cohomology of smooth projective varieties over arbitrary fields [1]. From now on we consider only varieties over fields of characteristic zero. Provided this condition the standard conjecture $B(X)$ is equivalent to the coincidence of the numerical and homological equivalences of algebraic cycles on the Cartesian product $X \times X$ [3, Formula (1.11)]; besides, in accordance with [4, Proposition 1.7] the conjecture $B(X)$ is equivalent to the semi-simplicity of the $\mathbb{Q}$ -algebra

$$\mathcal{A}(X) = \text{cl}_{X \times X}(\text{CH}_*(X \times X)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

of algebraic self-correspondences on the variety X with the bilinear composition law [2, Formula 1.3.1]

$$g \circ f = \text{pr}_{13*}(\text{pr}_{12}^*(f) \smile \text{pr}_{23}^*(g)).$$

In abstract case, $B(X) \Rightarrow C(X)$, where the standard conjecture $C(X)$ of Künneth type asserts the algebraicity of Künneth components of the class of the diagonal $\Delta_X \hookrightarrow X \times X$ [2, Lemma 2.4]. The conjecture $B(X)$ is compatible with Cartesian product [2, Corollary 2.5], hyperplane section [2, Theorem 2.13] and specialization (with possible change of characteristic) [2, Introduction]. In characteristic zero it is compatible with monoidal transformations along smooth centres [5, Theorem 4.3].

By definition, a d -dimensional elliptic variety is birationally equivalent to a variety containing a smooth family of elliptic curves parameterized by some affine variety of dimension $d - 1$.

It is known that the standard conjecture $B(X)$ is true for all smooth complex projective curves, surfaces, Abelian varieties [6] and threefolds of Kodaira dimension $\kappa(X) < 3$ [7] (in particular, it holds for all complex elliptic threefolds and for compactifications of Néron minimal models of Abelian surfaces over fields of algebraic functions of one variable with the field of constants $\mathbb{C}$). Besides, $B(X)$ holds for Hilbert schemes of points on surfaces [8, Corollary 7.5], for hyperkähler varieties deformation equivalent to Hilbert schemes of points of $K3$ surfaces [9], for the fibre product $X_1 \times_C X_2$ of two projective non-isotrivial smooth families $\pi_k : X_k \rightarrow C$ ($k = 1, 2$) of regular surfaces with geometric genus 1 over a smooth projective curve C under the assumption that ranks of lattices of transcendental cycles on generic geometric fibres X_{k_s} ($k = 1, 2$) are different prime odd numbers [10], [11].

If S is a $K3$ or Abelian surface, H an ample linear bundle on S and X the Gieseker - Maruyama - Simpson moduli space of H -stable torsion-free sheaves of rank r on S with fixed Chern classes c_1, c_2 , then the standard conjecture of Lefschetz type holds for X under the assumption that X is projective [8, Theorem 7.8, Corollary 7.9].

Besides the standard conjecture holds for the Altman - Kleiman compactification X of the relative Jacobian of a family $\mathcal{C} \rightarrow \mathbb{P}^2$ of hyperelliptic curves of genus 2 with weak degeneracies under the condition that the canonical projection $X \rightarrow \mathbb{P}^2$ is a Lagrangian fibration [12].

The conjecture $B(X)$ holds for every smooth projective compactification X of the Néron minimal model of an Abelian scheme of relative dimension 3 over an affine

curve provided that the generic scheme fibre of the Abelian scheme has reductions of multiplicative type at all infinite places [12]. Besides, it holds for a 4-dimensional smooth projective complex variety, fibred over a smooth projective curve, if every degenerated fibre is a union of smooth irreducible components of multiplicity 1 with normal crossings, for generic geometric fibre $X_{\bar{\eta}}$ the standard conjecture $B(X_{\bar{\eta}})$ holds, there exists at least one degenerated fibre X_{δ} , for irreducible components V_i of every degenerated fibre $X_{\delta} = V_1 + \cdots + V_m$ the rings of rational cohomology $H^*(V_i, \mathbb{Q})$ and $H^*(V_i \cap V_j, \mathbb{Q})$ are generated by classes of algebraic cycles [13].

As it was shown by Charles [14, Theorem 1], an algebraic isomorphism

$$H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$$

(inverse to the Lefschetz isomorphism) exists iff, for some smooth quasi-projective variety S and for an appropriate algebraic cycle $Z \in \text{CH}^2(X \times S)$ of codimension 2, there exists a point $s \in S$ such that the canonical map $\phi_{Z,s} : \wedge^2 \Theta_{S,s} \rightarrow H^2(X, \mathcal{O}_X)$ is surjective (where $\Theta_{S,s}$ is the tangent space to the variety S at the point s).

Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of the Abelian variety $\mathcal{M}_{\eta}$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C . Suppose that at any place $s \in C$ the reduction of the Abelian variety $\mathcal{M}_{\eta}$ is stable in Grothendieck's sense. In this case the connected component $\mathcal{M}_s^0$ of the neutral element of the algebraic group $\mathcal{M}_s$ is an extension of an Abelian variety by a linear torus whose dimension r_s is called *the toric (reductive) rank* at the place s [15, Section 2.1.12].

Let R be a Dedekind domain with the fraction field K and let A_{η} be an Abelian variety over $\eta = \text{Spec } K$ such that all reductions are stable in Grothendieck's sense. As it was shown by Künnemann [16, Section 5.8], in this case there exists a finite extension K' of the field K such that the Abelian variety $A_{\eta} \otimes_K K'$ has (not necessarily unique) a flat projective regular model P' over the integral closure R' of the ring R in the field K' ; this model P' has strict semi-stable reductions over each localization of the ring R' (in particular, every special fibre P'_s is a union of smooth divisors of multiplicity 1 with normal crossings [17, Section 1.9]), and the scheme P' contains the Néron minimal model $\mathcal{A}'$ of the variety $A_{\eta} \otimes_K K'$ in the case when all residue fields of the scheme $\text{Spec } R'$ are perfect [17, Section 4.4, Theorem 4.6].

Since the coordinate ring $\mathbb{C}[C']$ of any smooth affine curve C' over the field $\mathbb{C}$ is a Dedekind domain, then after the base change determined by an appropriate ramified covering $\tilde{C} \rightarrow C$, we may assume by the results of Künnemann cited above that, for the Néron minimal model $\mathcal{M} \rightarrow C$, there exists a smooth compactification X of the variety $\mathcal{M}$ which is flat and projective over the curve C such that the following conditions hold:

- (i) the model X/C has strictly semi-stable reductions (in particular, all fibres of the structure morphism $\pi : X \rightarrow C$ are unions of smooth irreducible components of multiplicity 1 with normal crossings);
- (ii) the variety X contains the variety $\mathcal{M}$ as an open dense subscheme;
- (iii) the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \rightarrow C$ coincides with the structure morphism of the Néron model;
- (iv) the connected component $\mathcal{M}_s^0$ of the neutral element of any fibre $\mathcal{M}_s$ ($s \in C$) is an extension of an Abelian variety by a linear torus of dimension r_s ;
- (v) C -group law $\mathcal{M}^0 \times_C \mathcal{M}^0 \rightarrow \mathcal{M}^0$ can be expanded to a group C -action $\mathcal{M}^0 \times_C X \rightarrow X$.

We call such compactifications of the Néron model by *Künnemann compactifications*.

By definition, the Abelian variety $\mathcal{M}_\eta$ has a *trivial* trace if, for any finite ramified covering $\tilde{C} \rightarrow C$, the group scheme $\mathcal{M} \times_C \tilde{C} \rightarrow \tilde{C}$ has no non-trivial *constant* Abelian subscheme.

Let $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ be the set of all non-zero natural numbers. In this article we prove the following main result:

Theorem. *Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of an absolutely simple $(d-1)$ -dimensional principally polarized Abelian variety $\mathcal{M}_\eta$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C .*

Assume that the trace of the Abelian variety $\mathcal{M}_\eta$ is trivial and the centre $Z_{\mathbb{Q}}$ of the division $\mathbb{Q}$ -algebra $E_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{End}_{\kappa(\eta)}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a totally real field of degree e over $\mathbb{Q}$.

If one of the following conditions holds:

(i) *$\text{End}_{\kappa(\eta)}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) = \mathbb{Z}$ and, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the complexification $\text{Lie Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C}$ of the Lie algebra of the Hodge group of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}$ is a simple Lie algebra of type C_{d-1} (this condition automatically holds if*

$$d-1 \notin \text{Ex}(1) \stackrel{\text{def}}{=} \left\{ 4^l, \frac{1}{2} \binom{4l+2}{2l+1}^{2m-1}, 2^{8lm+4l-4m-3}, 4^l(m+1)^{2l+1} \mid l, m \in \mathbb{N}^+ \right\} \\ = \{4, 10, 16, 32, 64, 108, 126, 256, 500, 512, 864, 1024, 1372, 1716, 2048, \dots\};$$

(ii) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type I of Albert's classification and $\frac{d-1}{e}$ is an odd integer;*

(iii) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type II of Albert's classification and $\frac{d-1}{e}$ is not divisible by 4;*

(iv) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type III of Albert's classification, $\frac{d-1}{e}$ is not divisible by 4 and $(\forall r \in \mathbb{N}^+) \quad \frac{d-1}{e} \neq \binom{4r}{2r}$,*

then there exists a finite ramified covering $\tilde{C} \rightarrow C$ such that, for any Künnemann compactification $\tilde{X}$ of the Néron minimal model of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$, there exist algebraic isomorphisms

$$H^{2d-2}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^2(\tilde{X}, \mathbb{Q}), \quad H^{2d-3}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Q}).$$

Remark. In the case under consideration, there exist algebraic isomorphisms

$$H^{2d-2}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^2(\tilde{X}, \mathbb{Q}), \quad H^{2d-3}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Q})$$

which are inverse to the Lefschetz isomorphisms in degrees 2 or 3 of rational cohomology of the variety $\tilde{X}$ [14, Lemma 6].

The author is grateful to a referee who suggested essential improvements of the initial text.

§ 1. SOME REMARKS ON POINCARÉ CLASSES, GLOBAL MONODROMY AND COHOMOLOGY OF LOCAL SYSTEMS

1.1. It is well known that the Hodge decomposition

$$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} V_{\mathbb{C}}^{p,q}$$

of the Hodge $\mathbb{Q}$ -substructure $V_{\mathbb{Q}} \hookrightarrow H^n(X, \mathbb{Q})$ yields the action $h : U^1 \rightarrow \mathrm{GL}(V_{\mathbb{R}})$ of the group $U^1 \stackrel{\mathrm{def}}{=} \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ on the real space $V_{\mathbb{R}} \stackrel{\mathrm{def}}{=} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, such that $h(e^{i\theta})(v^{p,q}) = e^{i\theta(p-q)} \cdot v^{p,q}$ for any element $v^{p,q} \in V_{\mathbb{C}}^{p,q}$ ([18], Section 2.1.5). By definition, the Hodge group $\mathrm{Hg}(V_{\mathbb{Q}})$ is the smallest algebraic $\mathbb{Q}$ -subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ whose group of $\mathbb{R}$ -points contains the group $h(U^1)$ ([19], Definition B51). It is known that the group $\mathrm{Hg}(V_{\mathbb{Q}})$ is a connected reductive group, and in the case $(r-l)n = 2p$ the space of invariants $[V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}]^{\mathrm{Hg}(V_{\mathbb{Q}})}$ coincides with the space of Hodge cycles $[V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}] \cap [V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}]_{\mathbb{C}}^{p,p}$ ([19], Corollary B55).

A polarization of the Hodge $\mathbb{Q}$ -substructure $V_{\mathbb{Q}} \hookrightarrow H^n(X, \mathbb{Q})$ is a morphism of rational Hodge structures $\psi_{V_{\mathbb{Q}}} : V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \rightarrow \mathbb{Q}(-n)$ such that the real bilinear form $(u, v) \mapsto (2\pi i)^n \psi_{V_{\mathbb{Q}}}(u, h(i)v)$ is symmetric and positive-definite on $V_{\mathbb{R}}$ ([18], Section 2.1.14).

Denote by $\psi_{V_{\mathbb{Q}}}^0$ the composite $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \xrightarrow{\psi_{V_{\mathbb{Q}}}} \mathbb{Q}(-n) \xrightarrow{\sim} \mathbb{Q} \xrightarrow{(2\pi i)^n} \mathbb{Q}$. Clearly, the bilinear form $\psi_{V_{\mathbb{Q}}}^0$ is non-degenerate; besides, this property holds for the restriction $\psi_{V_{\mathbb{Q}}}^0|_{W_{\mathbb{Q}}}$ of the form $\psi_{V_{\mathbb{Q}}}^0$ to any non-trivial Hodge $\mathbb{Q}$ -substructure $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$, because the form $(u, v) \mapsto (2\pi i)^n \psi_{V_{\mathbb{Q}}}(u, h(i)v)$ is positive-definite on $V_{\mathbb{R}}$ and, consequently, it's restriction is positive-definite on $W_{\mathbb{R}}$. It is well-known that the classical rational Hodge structure $H^n(X, \mathbb{Q})$ is polarizable. Therefore every Hodge $\mathbb{Q}$ -substructure $V_{\mathbb{Q}} \subset H^n(X, \mathbb{Q})$ is polarizable too.

It is known that the bilinear form $\psi_{V_{\mathbb{Q}}}^0$ is invariant with respect to the action of the group U^1 ([18], Section 2.1.16). On the other hand, the group U^1 trivially acts on the structure $\mathbb{Q}(-n)_{\mathbb{R}}$, therefore the bilinear form $\psi_{V_{\mathbb{Q}}}^0$ also is U^1 -invariant. Thus the form $\psi_{V_{\mathbb{Q}}}^0$ is invariant with respect to the canonical action of the Hodge group $\mathrm{Hg}(V_{\mathbb{Q}})$ in the space $V_{\mathbb{Q}}$. In particular, there exists an embedding of *connected* algebraic $\mathbb{Q}$ -groups $\mathrm{Hg}(V_{\mathbb{Q}}) \hookrightarrow [\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0$, where $[\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0$ is the connected component of unity of the $\mathbb{Q}$ -group $\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)$.

Consider the diagonal action of the group $[\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0$ in the space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$, determined by the formula $\mu(x \otimes y) = \mu(x) \otimes \mu(y)$. It is clear that elements of the subspace $[V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{[\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0} \subset [V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{\mathrm{Hg}(V_{\mathbb{Q}})}$ are Hodge cycles. Assume that the number n is odd or n is even and $\dim_{\mathbb{Q}} V_{\mathbb{Q}} \neq 2$. Then the standard representation of the group $[\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0$ in the space $V_{\mathbb{Q}}$ is an absolutely irreducible *orthogonal* or *symplectic* representation ([18], Section 2.1.16) of a *semi-simple* $\mathbb{Q}$ -group ([20], Ch. I, § 6, n° 7, Proposition 9) and, according to the Schur lemma, the 1-dimensional $\mathbb{Q}$ -space $[V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{[\mathrm{Aut}(\psi_{V_{\mathbb{Q}}}^0)]^0}$ is generated by some element $\wp_0(V_{\mathbb{Q}})$ (which is determined uniquely up to a non-zero scalar multiple). We call the element $\wp_0(V_{\mathbb{Q}})$ the *Poincaré class* of the polarizable rational Hodge structure $(V_{\mathbb{Q}}, \psi_{V_{\mathbb{Q}}})$.

In particular, if the number n is odd or n is even and $\dim_{\mathbb{Q}} H^n(X, \mathbb{Q}) \neq 2$, then there exists the Poincaré class $\wp_0(H^n(X, \mathbb{Q}))$ associated with some polarization form $\psi_{H^n(X, \mathbb{Q})}$ on the classical rational Hodge structure $H^n(X, \mathbb{Q})$, so that the Hodge cycle $\wp_0(H^n(X, \mathbb{Q}))$, considered as (non necessarily algebraic) correspondence, yields an isomorphism of $[\mathrm{Aut}(\psi_{H^n(X, \mathbb{Q})}^0)]^0$ -modules $H^n(X, \mathbb{Q})^{\vee} \xrightarrow{\sim} H^n(X, \mathbb{Q})$.

1.2. By the strong Lefschetz theorem and by the Poincaré duality the bilinear form

$$\Phi : H^n(X, \mathbb{Q}) \times H^n(X, \mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \mathrm{cl}_X(H) \smile^{d-n}} H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow{\sim} \mathbb{Q}$$

is non-degenerate [2, Section 1.2A]. Let $\langle \rangle: H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$ be the orientation isomorphism. Since $\text{cl}_X(H) \in H^2(X, \mathbb{Q})^{\text{Hg}(H^2(X, \mathbb{Q}))} = \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \stackrel{\text{def}}{=} \text{NS}_{\mathbb{Q}}(X)$ by the Lefschetz theorem on divisors, the group U^1 acts trivially on the subset $\text{NS}_{\mathbb{Q}}(X) \hookrightarrow H^{1,1}(X, \mathbb{C})$, so that we have (with a trivial action of U^1 on $\mathbb{R}$):

$$\begin{aligned} \forall \sigma \in U^1 \quad \Phi_{\mathbb{R}}(x, y) &= [\Phi_{\mathbb{R}}(x, y)]^{\sigma} = \langle x \smile \text{cl}_X(H)^{d-n} y \rangle^{\sigma} = \\ &= \langle x^{\sigma} \smile \text{cl}_X(H)^{d-n} \smile y^{\sigma} \rangle = \Phi_{\mathbb{R}}(x^{\sigma}, y^{\sigma}). \end{aligned}$$

Therefore the form $\Phi_{\mathbb{R}}$ is U^1 -invariant, so there exists a canonical embedding

$$\text{Hg}(H^n(X, \mathbb{Q})) \hookrightarrow \text{Aut}(\Phi)^0 = \begin{cases} \text{Sp}(\Phi), & \text{for odd } n, \\ \text{SO}(\Phi), & \text{for even } n, \end{cases}$$

and Φ is a $\text{Hg}(H^n(X, \mathbb{Q}))$ -invariant form.

If the number n is odd or n is even and $\dim_{\mathbb{Q}} H^n(X, \mathbb{Q}) \neq 2$, then the 1-dimensional $\mathbb{Q}$ -space $[H^n(X, \mathbb{Q}) \otimes H^n(X, \mathbb{Q})]^{\text{Aut}(\Phi)^0}$ of invariants of the diagonal action of the group $\text{Aut}(\Phi)^0$ is generated by a Hodge cycle $\wp(H^n(X, \mathbb{Q}))$, which is again called *the Poincaré class*. It determines an isomorphism of $[\text{Aut}(\Phi)]^0$ -modules $H^n(X, \mathbb{Q})^{\vee} \xrightarrow{\sim} H^n(X, \mathbb{Q})$. The Poincaré duality theorem yields an identification of the Weil cohomology $H^n(X, \mathbb{Q})^{\vee} = H^{2d-n}(X, \mathbb{Q})$ ([2], Section 1.2A), therefore the isomorphism under consideration is the composite ([2], Section 1.3)

$$\begin{aligned} H^{2d-n}(X, \mathbb{Q}) &\xrightarrow{\text{pr}_1^*} H^{2d-n}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^0(X, \mathbb{Q}) \\ &\xrightarrow{\smile \wp(H^n(X, \mathbb{Q}))} H^{2d}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X, \mathbb{Q}) \xrightarrow{\text{pr}_{2*}} H^n(X, \mathbb{Q}). \end{aligned}$$

In contrast to the situation of Section 1.1, a restriction of the form Φ to a non-trivial rational Hodge substructure $V_{\mathbb{Q}} \subset H^n(X, \mathbb{Q})$ may be degenerated. A simple example is suggested by a referee: if X_s is a smooth fibre of a morphism $\pi: X \rightarrow C$ of the variety X onto a smooth projective curve C , then in virtue of the equality $\text{cl}_X(X_s) \smile \text{cl}_X(X_s) = 0$ the restriction of the form $\Phi: H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \rightarrow \mathbb{Q}$ to the non-trivial rational Hodge substructure $\mathbb{Q} \cdot \text{cl}_X(X_s) \subset H^2(X, \mathbb{Q})$ is trivial.

Nevertheless, if a restriction of the form Φ to a non-trivial rational Hodge substructure $V_{\mathbb{Q}} \subset H^n(X, \mathbb{Q})$ is non-degenerate, then there is a decomposition of Hodge $\mathbb{Q}$ -structures [21, Ch. IX, § 4, n° 1, Corollary of Proposition 1]

$$H^n(X, \mathbb{Q}) = V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^{\perp},$$

where $V_{\mathbb{Q}}^{\perp}$ is an orthogonal complement of the $\mathbb{Q}$ -space $V_{\mathbb{Q}}$ with respect to the form Φ and the Poincaré classes

$$\wp(V_{\mathbb{Q}}) \in [V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{\text{Aut}(\Phi|_{V_{\mathbb{Q}}})^0}, \quad \wp(V_{\mathbb{Q}}^{\perp}) \in [V_{\mathbb{Q}}^{\perp} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}^{\perp}]^{\text{Aut}(\Phi|_{V_{\mathbb{Q}}^{\perp}})^0}$$

are well defined under the assumption that the number n is odd or n is even and $\dim_{\mathbb{Q}} V_{\mathbb{Q}} \neq 2$, $\dim_{\mathbb{Q}} V_{\mathbb{Q}}^{\perp} \neq 2$. Moreover, in this situation the Poincaré classes $\wp(V_{\mathbb{Q}})$, $\wp(V_{\mathbb{Q}}^{\perp})$ are *Hodge cycles*.

1.3. We may assume that

$$\{s \in C \mid \text{the fibre } \mathcal{M}_s \text{ is non-compact}\} = \Delta \stackrel{\text{def}}{=} \{\delta \in C \mid \text{Sing}(X_{\delta}) \neq \emptyset\}.$$

Set $C' = C \setminus \Delta$. It is evident that the structure morphism $\pi: X \rightarrow C$ is smooth over C' . Let $C' \xrightarrow{j} C$ be the canonical embedding, $X' = X \setminus \pi^{-1}(\Delta)$, $\pi' = \pi|_{X'}: X' \rightarrow C'$.

Considering, if necessary, a ramified covering $\tilde{C} \rightarrow C$ and a projective Knnemann model $\tilde{X} \rightarrow \tilde{C}$ of the corresponding Nron model $\tilde{\mathcal{M}} \rightarrow \tilde{C}$ of the generic scheme fibre of the canonical projection $X \times_C \tilde{C} \rightarrow \tilde{C}$, we may assume in virtue of [16, Section 5.8]; [17, Section 4.4, Theorem 4.6] that any singular fibre X_δ is a union of smooth irreducible components of multiplicity 1 with normal crossings. One may also assume that there is a *countable* subset $\Delta_{\text{countable}} \subset C'$ such that, for any point $s \in C' \setminus \Delta_{\text{countable}}$, the closure G of the image of the monodromy representation $\pi_1(C', s) \rightarrow \text{GL}(H^1(X_s, \mathbb{Q}))$ in the Zariski topology of the algebraic group $\text{GL}(H^1(X_s, \mathbb{Q}))$ is a connected semi-simple [18, Corollary 4.2.9] *normal* [22, Theorem 7.3] subgroup of the Hodge group

$$\text{Hg}(X_s) \stackrel{\text{def}}{=} \text{Hg}(H^1(X_s, \mathbb{Q}))$$

of the Abelian variety X_s . In accordance with Mumford's result, the reductive Hodge group $\text{Hg}(X_s)$ is commutative (and, consequently, it is a linear $\mathbb{Q}$ -torus) if and only if X_s is an Abelian variety of CM-type [23]. Therefore it follows from the triviality of the trace of the Abelian variety X_η that the generic scheme fibre X_η can not be an Abelian variety of CM-type (because the group G is a *non-trivial* [18, Sections (4.1.3.3), 4.4.3] connected *semi-simple* group). By the same reasons the variety X_s can not be an Abelian variety of CM-type for any point $s \in C' \setminus \Delta_{\text{countable}}$. We may also assume that local monodromies (Picard - Lefschetz transformations) are *unipotent* and $\text{End}_{\kappa(\eta)}(X_\eta) = \text{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)})$.

Consider the normalization $f : Z \rightarrow \pi^{-1}(\Delta)$ of the scheme $\pi^{-1}(\Delta)$. Then Z is a disjoint union of smooth irreducible components of the divisor $\pi^{-1}(\Delta)$. Since $f : Z \rightarrow \pi^{-1}(\Delta)$ is a resolution of singularities of the subscheme $i_\Delta : \pi^{-1}(\Delta) \hookrightarrow X$, there is a canonical exact sequence of mixed Hodge $\mathbb{Q}$ -structures [24, Corollary (8.2.8)]:

$$H^{n-2}(Z, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q}),$$

where $(i_\Delta f)_*$ is a morphism of bidegree $(1, 1)$ of pure Hodge structures and φ_n is the restriction map. In particular,

$$(1.1) \quad (i_\Delta f)_* H^{n-2}(Z, \mathbb{Q}) = \text{Ker}[H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q})].$$

By the conditions (i) - (iv) of the theorem, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the Hodge group $\text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ is a $\mathbb{Q}$ -simple algebraic group [25, § 4, Deduction of Theorem 4.1 from Lemmas 1 - 3]. Therefore

$$(1.2) \quad G = \text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}) = \text{Hg}(X_s) \quad \forall s \in C' \setminus \Delta_{\text{countable}}$$

by Proposition 4.1 in [25]. It is well known that the canonical representation of the Lie algebra $\text{Lie Hg}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ in the space $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is determined by *minuscule weights* [26], [22, Theorem 0.5.1] in Bourbaki's sense [20].

1.4. Note that if $\text{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) = \mathbb{Z}$ and $d - 1 \notin \text{Ex}(1)$ then, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the complexification $\text{Lie Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C}$ of the Lie algebra of the Hodge group of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}$ is a simple Lie algebra of type C_{d-1} [27, Theorem 1.1].

1.5. It follows from the conditions (i) - (iv) of the theorem and from the triviality of the trace that there is a canonical isomorphism [25, § 4, Corollary of Theorem (Hom - End), Theorem 4.1]

$$(1.3) \quad \text{End}_{C'}(X') \xrightarrow{\sim} \text{End}_{C'}(R^1 \pi'_* \mathbb{Z}).$$

A choice of a point $s \in C' \setminus \Delta_{\text{countable}}$ determines the canonical embeddings

$$\text{Im}[\pi_1(C', s) \rightarrow \text{GL}(H^1(X_s, \mathbb{Q}))] \subset G \subset \text{Hg}(X_s).$$

Therefore it follows from (1.3) and from the well known equality [19, Lemma B.60]

$$\text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that there are canonical maps

$$(1.4) \quad \begin{aligned} & \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\pi_1(C', s)} H^1(X_s, \mathbb{Q}) \\ & \hookrightarrow \text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

The restriction map $\text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, hence it follows from (1.4) that there exists a canonical isomorphism

$$(1.5) \quad \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the Abelian variety X_s is simple.

§ 2. A CONSTRUCTION OF AN ALGEBRAIC ISOMORPHISM OF RATIONAL COHOMOLOGY OF DEGREES $d-2$ AND 2

2.1. From now on we denote by

$$K_{nX} \stackrel{\text{def}}{=} \text{Ker}[H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n \pi_* \mathbb{Q})]$$

the kernel of the edge map of the Leray spectral sequence $E_2^{p,q}(\pi)$ of the structure morphism $\pi : X \rightarrow C$. Besides, for any irreducible smooth projective variety W , we denote by $\langle \rangle : H^{2 \dim W}(W, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$ the orientation isomorphism of Weil cohomology [2, Section 1.2.A] determined by a choice of an element $\sqrt{-1} \in \mathbb{C}$.

By the assumption of the theorem, the generic scheme fibre $\mathcal{M}_\eta$ is a principally polarized Abelian variety; hence, for every point $s \in C'$, the Abelian variety X_s has a principal polarization determined by some ample divisor H_s on the variety X_s . Taking into account arguments of [3, § 4], we may assume that there exists a rigid Poincaré bundle $\mathcal{P}'$ on the Abelian scheme $X' \times_{C'} X'$, inducing the Poincaré bundle $\mathcal{P}'_s$ on the Cartesian product $X_s \times X_s$ for every point $s \in C'$.

One may assume that

$$c_1(\mathcal{P}'_s) = \wp(H^1(X_s, \mathbb{Q})) \in H^2(X_s \times X_s, \mathbb{Q}) \cap H^{1,1}(X_s \times X_s, \mathbb{C}),$$

where $\wp(H^1(X_s, \mathbb{Q}))$ is the Poincaré class in the sense of Section 1.2; it is *algebraic* by the Lefschetz theorem on divisors and it is determined uniquely (up to a non-zero scalar multiple) by the corresponding bilinear form

$$\Phi_s : H^1(X_s, \mathbb{Q}) \times H^1(X_s, \mathbb{Q}) \xrightarrow{x \times y \mapsto \langle x \smile \text{cl}_{X_s}(H_s) \smile^{d-2} \smile y \rangle} \mathbb{Q}.$$

Besides, for the group G (defined in Section 1.3) and for arbitrary point $s \in C' \setminus \Delta_{\text{countable}}$, there are embeddings

$$G \hookrightarrow \text{Hg}(H^1(X_s, \mathbb{Q})) \hookrightarrow \text{Sp}(H^1(X_s, \mathbb{Q}), \Phi_s).$$

Since the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ is a generator of the 1-dimensional subspace

$$\begin{aligned} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\text{Sp}(H^1(X_s, \mathbb{Q}), \Phi_s)} & \hookrightarrow [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^G \\ & = [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C', s)} \end{aligned}$$

under the diagonal action $\pi_1(C', s)$ on $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})$, then the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ determines the section

$$\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \xrightarrow{\sim} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C', s)}$$

of type $(1, 1)$ of the local system of Hodge structures $R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$ inducing the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ for every point $s \in C'$.

Consider the canonical diagram of the fibre product

$$\begin{array}{ccccc} X' \times_{C'} X' & & \xrightarrow{p'_1} & & X' \\ & \searrow p'_2 & & \searrow \tau' & \downarrow \pi' \\ & X' & & \xrightarrow{\pi'} & C'. \end{array}$$

As it was explained in Section 1.2, the *algebraic* Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ determines an algebraic isomorphism $H^{2d-3}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^1(X_s, \mathbb{Q})$. Therefore the section $\Lambda'_{1,1}$ yields the isomorphism of local systems $R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$(2.1) \quad \begin{array}{c} R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{(p'_1)^*} R^{2d-3}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} \xrightarrow{\sim \Lambda'_{1,1}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q} \\ \xrightarrow{(p'_2)^*} R^1\pi'_*\mathbb{Q}. \end{array}$$

Let

$$\begin{array}{ccccc} X \times_C X & & \xrightarrow{p_1} & & X \\ & \searrow p_2 & & \searrow \tau & \downarrow \pi \\ & X & & \xrightarrow{\pi} & C \end{array}$$

be the canonical diagram of the fibre product, $\iota : X \times_C X \hookrightarrow X \times X$ the canonical embedding, $\sigma : Y \rightarrow X \times_C X$ a resolution of singularities of the variety $X \times_C X$. One may assume that σ induces an isomorphism over C' . In particular, Y can be considered as a smooth projective compactification of the fibre product $X' \times_{C'} X'$. Besides, we may assume in virtue of Hironaka's results that $(\tau\sigma)^{-1}(\Delta)$ is a union of smooth divisors (of some positive multiplicities) with normal crossings.

By Deligne's theorem, the canonical morphism $H^2(Y, \mathbb{Q}) \rightarrow H^0(C', R^2\tau'_*\mathbb{Q})$ is a surjective morphism of Hodge $\mathbb{Q}$ -structures [18, Theorem 4.1.1, Proof of Corollary 4.1.2]. Since $\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \subset H^0(C', R^2\tau'_*\mathbb{Q})$ is an element of Hodge type $(1, 1)$, then from Lefschetz' theorem on divisors it follows that there exists an algebraic $\mathbb{Q}$ -cycle $D^{(1)}$ on the variety Y such that the image of the class $\text{cl}_Y(D^{(1)}) \in H^2(Y, \mathbb{Q}) \cap H^{1,1}(Y, \mathbb{C})$ with respect to the canonical surjective morphism $H^2(Y, \mathbb{Q}) \rightarrow H^0(C', R^2\tau'_*\mathbb{Q})$ coincides with the section $\Lambda'_{1,1}$.

For any open subset $U \hookrightarrow C^{\text{an}}$, there is a canonical proper morphism $p_k\sigma : (\tau\sigma)^{-1}(U) \rightarrow \pi^{-1}(U)$ inducing canonical maps [28, Ch. II, Section 4.16]

$$\begin{aligned} (p_k\sigma)^* &: H^q(\pi^{-1}(U), \mathbb{Q}) \rightarrow H^q((\tau\sigma)^{-1}(U), \mathbb{Q}), \\ (p_k\sigma)_c^* &: H_c^q(\pi^{-1}(U), \mathbb{Q}) \rightarrow H_c^q((\tau\sigma)^{-1}(U), \mathbb{Q}), \end{aligned}$$

and, by the Poincaré duality theorem [29, Ch. VI, § 11, Corollary 11.2], there is a canonical map

$$(p_k\sigma)_* = [(p_k\sigma)_c^*]^\vee : H^{2d-2+i}((\tau\sigma)^{-1}(U), \mathbb{Q}) \rightarrow H^i(\pi^{-1}(U), \mathbb{Q}).$$

Consequently, there are canonical morphisms of sheaves

$$(p_k\sigma)^* : R^q\pi_*\mathbb{Q} \rightarrow R^q(\tau\sigma)_*\mathbb{Q},$$

$$(p_k\sigma)_* : R^{2d-2+i}(\tau\sigma)_*\mathbb{Q} \rightarrow R^i\pi_*\mathbb{Q},$$

and

$$(p_k\sigma)^*|_{C'} = (p'_k)^*, \quad (p_k\sigma)_*|_{C'} = (p'_k)_*.$$

Therefore the compatibility of the $\smile$ -products with the Leray spectral sequence

$$E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$$

([30], Vol. II, Ch. 4, Lemma 4.13) allows us to expand (2.1) to a sequence of morphisms of sheaves

$$(2.2) \quad R^{2d-3}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-3}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile_{\text{cl}_Y(D^{(1)})}} R^{2d-1}(\tau\sigma)_*\mathbb{Q} \xrightarrow{(p_2\sigma)_*} R^1\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ .

2.2. It follows from the existence of the natural embedding

$$R^2\pi'_*\mathbb{Q} = \wedge^2 R^1\pi'_*\mathbb{Q} \hookrightarrow R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$$

and from (1.3) that $H^0(C', R^2\pi'_*\mathbb{Q})$ is a rational Hodge structure of type $(1, 1)$ because a polarization on X determines an isomorphism of families of Hodge structures [18, Section 4.2.3]

$$[R^1\pi'_*\mathbb{Q}]^\vee \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}(1),$$

the ring $\text{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the component of type $(0, 0)$ of the Hodge $\mathbb{Q}$ -structure $\text{End}_{C'}(R^1\pi'_*\mathbb{Q})$ [18, Section 4.4.6] and there are morphisms of rational Hodge structures

$$\begin{aligned} H^0(C', R^2\pi'_*\mathbb{Q}) &\hookrightarrow H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \\ &\xrightarrow{\sim} H^0(C', [R^1\pi'_*\mathbb{Q}]^\vee \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})(-1) = \text{End}_{C'}(R^1\pi'_*\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-1) \\ &\xrightarrow{\sim} \text{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}(-1). \end{aligned}$$

Consequently, if $\Delta = \emptyset$, then the $\mathbb{Q}$ -space

$$[H^0(C, R^2\pi_*\mathbb{Q}) \otimes_{\mathbb{Q}} H^0(C, R^2\pi_*\mathbb{Q})] \cap H^{2,2}(X \times X, \mathbb{C})$$

is generated by classes of algebraic cycles, therefore by [3, Theorem 10.1] there exists an *algebraic* isomorphism $H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$. On the other hand, $H^0(C, R^3\pi_*\mathbb{Q}) = 0$ in virtue of Lemma 3.2 below, therefore by [3, Theorem 10.1] there exists an *algebraic* isomorphism $H^{2d-3}(X, \mathbb{Q}) \xrightarrow{\sim} H^3(X, \mathbb{Q})$.

2.3. From now on we assume that $\Delta \neq \emptyset$. Then there exists a *singular* fibre, whose components are smooth varieties of multiplicity 1 with normal crossings, therefore $\text{rank NS}(X) \geq 3$ [13, Formula (2.24)].

The sequence (2.2) yields a sequence of canonical maps of cohomology

$$(2.3) \quad \begin{aligned} H^1(C, R^{2d-3}\pi_*\mathbb{Q}) &\xrightarrow{[(p_1\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) \\ &\xrightarrow{[\smile_{\text{cl}_Y(D^{(1)})}]_1} H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \xrightarrow{[(p_2\sigma)_*]_1} H^1(C, R^1\pi_*\mathbb{Q}). \end{aligned}$$

By the functoriality properties of cohomology [31, Ch. II, Theorem 3.11] the composite of these maps coincides with the canonical map

$$(2.4) \quad H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile_{\text{cl}_Y(D^{(1)})})]_1} H^1(C, R^1\pi_*\mathbb{Q}),$$

corresponding to the morphism of sheaves

$$(2.5) \quad R^{2d-3}\pi_*\mathbb{Q} \xrightarrow{x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)}))} R^1\pi_*\mathbb{Q}.$$

Since the kernel and the cokernel of the map (2.5) are concentrated on Δ , their higher cohomology vanish, therefore the map (2.4) is surjective.

The composite of morphisms of sheaves (2.1) yields an isomorphism of local systems $R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}$, therefore there is an isomorphism of sheaves

$$(2.6) \quad j_*R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{\sim} j_*R^1\pi'_*\mathbb{Q},$$

where $j : C' \hookrightarrow C$ is the canonical embedding. Finitely, by the theorem on local invariant cycles ([32, Section (3.7)]; [33, Proposition (15.12)]) the canonical map $R^p\pi_*\mathbb{Q} \rightarrow j_*R^p\pi'_*\mathbb{Q}$ is surjective with the kernel concentrated on a finite set Δ . Consequently, there is a canonical isomorphism

$$H^1(C, R^p\pi_*\mathbb{Q}) \xrightarrow{\sim} H^1(C, j_*R^p\pi'_*\mathbb{Q}).$$

Therefore by the surjectivity of the map (2.4) it follows from (2.6) that there exists an isomorphism of bidegree $(2-d, 2-d)$ of mixed Hodge structures

$$(2.7) \quad H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)}))]_1} H^1(C, R^1\pi_*\mathbb{Q}).$$

2.4. The Leray spectral sequences $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ and $E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$ are degenerated: $E_2^{p,q} = E_\infty^{p,q}$ [33, Corollary (15.15)]. Therefore in notations of Section 2.1, for any natural number n , there are exact sequences of Hodge $\mathbb{Q}$ -structures ([13], Formula (2.4))

$$(2.8) \quad 0 \rightarrow H^2(C, R^{n-2}\pi_*\mathbb{Q}) \rightarrow K_{nX} \xrightarrow{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}) \rightarrow 0,$$

$$(2.9) \quad 0 \rightarrow H^2(C, R^{n-2}(\tau\sigma)_*\mathbb{Q}) \rightarrow K_{nY} \xrightarrow{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \rightarrow 0.$$

Let $\text{NS}_{\mathbb{Q}}(X) = \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $T_{\mathbb{Q}}^2(X)$ be the sum of all irreducible Hodge $\mathbb{Q}$ -substructures of dimension greater than 1 in $H^2(X, \mathbb{Q})$. By the Lefschetz theorem on divisors and by the strong Lefschetz theorem we have *canonical* decompositions

$$(2.10) \quad H^2(X, \mathbb{Q}) = T_{\mathbb{Q}}^2(X) \oplus \text{NS}_{\mathbb{Q}}(X);$$

$$H^{2d-2}(X, \mathbb{Q}) = [T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}] \oplus [\text{NS}_{\mathbb{Q}}(X) \smile \text{cl}_X(H)^{\smile d-2}],$$

where $\text{NS}_{\mathbb{Q}}(X) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{C})$ and $\text{NS}_{\mathbb{Q}}(X) \smile \text{cl}_X(H)^{\smile d-2}$ are identified with spaces of algebraic cohomology classes, $T_{\mathbb{Q}}^2(X)$ and $T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}$ are identified with spaces of transcendental cohomology classes.

2.5. Lemma. *Rational Hodge structure $H^0(C, R^2\pi_*\mathbb{Q})$ has type $(1, 1)$.*

Proof. One may assume that $\Delta \neq \emptyset$ and, consequently, $H^2(C', \mathbb{Q}) = 0$, because the affine curve C' has cohomological dimension 1 ([29], Ch. VI, § 7, Theorem 7.2). Let $D^*(\delta)$ be a small open punctured disc on the curve C with a centre at the point $\delta \in \Delta$. The Leray spectral sequence for the embedding $j : C' \subset C$ yields an exact sequence of mixed Hodge structures ([33], P. 457, Corollary (13.10), Remark (14.5))

$$0 \rightarrow H^1(C, j_*R^1\pi'_*\mathbb{Q}) \rightarrow H^1(C', R^1\pi'_*\mathbb{Q}) \rightarrow H^0(C, R^1j_*R^1\pi'_*\mathbb{Q})$$

$$\rightarrow H^2(C, j_*R^1\pi'_*\mathbb{Q}),$$

where

$$H^0(C, R^1 j_* R^1 \pi'_* \mathbb{Q}) \\ = \oplus_{\delta \in \Delta} H^1(D^*(\delta), R^1 \pi'_* \mathbb{Q}) \xrightarrow{\sim} \oplus_{\delta \in \Delta} H^1(X_s, \mathbb{Q}) / N_\delta H^1(X_s, \mathbb{Q}),$$

the space $H^1(X_s, \mathbb{Q})$ ($s \in C'$) has the limit mixed Hodge structure associated with the local monodromy γ around the point $\delta \in C$ (the Picard - Lefschetz transformation) and $N_\delta = \log \gamma$. By the theorem on local invariant cycles this sequence takes the form

$$0 \rightarrow H^1(C, R^1 \pi_* \mathbb{Q}) \rightarrow H^1(C', R^1 \pi'_* \mathbb{Q}) \rightarrow \oplus_{\delta \in \Delta} H^1(X_s, \mathbb{Q}) / N_\delta H^1(X_s, \mathbb{Q}).$$

On the other hand, for the case $n = 2$, the exact sequence (2.8) yields canonical isomorphisms of rational Hodge structures

$$H^1(C, R^1 \pi_* \mathbb{Q}) \xrightarrow{\sim} \text{Ker}[H^2(X, \mathbb{Q}) \rightarrow H^0(C, R^2 \pi_* \mathbb{Q})] / H^2(C, \mathbb{Q}) \\ \xrightarrow{\sim} \text{Ker}[H^2(X, \mathbb{Q}) / H^2(C, \mathbb{Q}) \rightarrow H^0(C, R^2 \pi_* \mathbb{Q})],$$

therefore there is a canonical exact sequence of Hodge $\mathbb{Q}$ -structures

$$0 \rightarrow H^1(C, R^1 \pi_* \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q}) / H^2(C, \mathbb{Q}) \rightarrow H^0(C, R^2 \pi_* \mathbb{Q}) \rightarrow 0.$$

By similar reasons, the degenerated Leray spectral sequence

$$E_2^{p,q}(\pi') = H^p(C', R^q \pi'_* \mathbb{Q})$$

yields the canonical exact sequence of mixed Hodge $\mathbb{Q}$ -structures

$$0 \rightarrow H^1(C', R^1 \pi'_* \mathbb{Q}) \rightarrow H^2(X', \mathbb{Q}) \rightarrow H^0(C', R^2 \pi'_* \mathbb{Q}) \rightarrow 0,$$

such that the canonical morphism of rational Hodge structures

$$H^2(X, \mathbb{Q}) \rightarrow H^0(C', R^2 \pi'_* \mathbb{Q})$$

is surjective ([18], Theorem 4.1.1). As a result, denoting by

$$\overline{\varphi}_2 : H^2(X, \mathbb{Q}) / H^2(C, \mathbb{Q}) \rightarrow H^2(X', \mathbb{Q})$$

the canonical map induced by the restriction map $\varphi_2 : H^2(X, \mathbb{Q}) \rightarrow H^2(X', \mathbb{Q})$, and taking into account the evident equality $\text{Im}(\overline{\varphi}_2) = \text{Im}(\varphi_2)$ and the commutativity of the diagram of morphisms

$$\begin{array}{ccc} X' & \subset & X \\ \downarrow \pi' & & \downarrow \pi \\ C' & \subset & C, \end{array}$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(C, R^1 \pi_* \mathbb{Q}) & \rightarrow & H^2(X, \mathbb{Q}) / H^2(C, \mathbb{Q}) & \rightarrow & H^0(C, R^2 \pi_* \mathbb{Q}) \rightarrow 0 \\ & & \cap & & \downarrow \overline{\varphi}_2 & & \downarrow \psi_2 \\ 0 & \rightarrow & H^1(C', R^1 \pi'_* \mathbb{Q}) & \rightarrow & H^2(X', \mathbb{Q}) & \rightarrow & H^0(C', R^2 \pi'_* \mathbb{Q}) \rightarrow 0. \end{array}$$

Evidently this diagram yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(C, R^1 \pi_* \mathbb{Q}) & \rightarrow & H^2(X, \mathbb{Q}) / H^2(C, \mathbb{Q}) & \rightarrow & H^0(C, R^2 \pi_* \mathbb{Q}) \rightarrow 0 \\ & & \cap & & \downarrow \overline{\varphi}_2 & & \downarrow \psi_2 \\ 0 & \rightarrow & H^1(C', R^1 \pi'_* \mathbb{Q}) \cap \text{Im}(\varphi_2) & \rightarrow & \text{Im}(\varphi_2) & \rightarrow & H^0(C', R^2 \pi'_* \mathbb{Q}) \rightarrow 0 \end{array}$$

and the corresponding exact sequence of Hodge $\mathbb{Q}$ -structures ([13], Section 2.6) of the snake-like diagram ([34], § 1, Proposition 2)

$$0 \rightarrow \text{Ker}(\overline{\varphi}_2) \rightarrow \text{Ker}(\psi_2) \rightarrow \frac{H^1(C', R^1 \pi'_* \mathbb{Q}) \cap \text{Im}(\varphi_2)}{H^1(C, R^1 \pi_* \mathbb{Q})} \rightarrow 0.$$

Note that $\text{Ker}(\varphi_2)$ is generated by classes of divisors in virtue of Corollary (8.2.8) in [24]. Therefore the Hodge $\mathbb{Q}$ -structure $\text{Ker}(\overline{\varphi_2})$ has type $(1, 1)$. It is well known that the weight filtration of the limit mixed Hodge structure $H^1(X_s, \mathbb{Q})$ has the form $0 \subset W_0 \subset W_1 \subset W_2 = H^1(X_s, \mathbb{Q})$, and the map $N_\delta : W_2/W_1 \xrightarrow{\sim} W_0$ is an isomorphism of bidegree $(-1, -1)$ ([35], Lemma (6.4), Theorem (6.16)). Clearly the Hodge structure W_2/W_1 is of type $(1, 1)$, because W_0 is a pure rational Hodge structure of type $(0, 0)$. On the other hand, W_1/W_0 is a pure rational Hodge structure of type $(1, 0) + (0, 1)$, therefore the exact sequence of morphisms of mixed Hodge structures

$$0 \rightarrow W_1/W_0 \rightarrow W_2/W_0 \rightarrow W_2/W_1 \rightarrow 0$$

shows that $H^1(X_s, \mathbb{Q})/N_\delta H^1(X_s, \mathbb{Q}) = W_2/N_\delta(W_2) = W_2/W_0$ is a mixed Hodge structure of type $(1, 1) + (1, 0) + (0, 1)$ ([18], Sections 2.3.6 - 2.3.7). It is easy to see that there exists a canonical embedding of mixed Hodge structures

$$\frac{H^1(C', R^1\pi'_*\mathbb{Q}) \cap \text{Im}(\varphi_2)}{H^1(C, R^1\pi_*\mathbb{Q})} \hookrightarrow \bigoplus_{\delta \in \Delta} \frac{H^1(X_s, \mathbb{Q})}{N_\delta H^1(X_s, \mathbb{Q})}.$$

Thus the rational Hodge structure $\frac{H^1(C', R^1\pi'_*\mathbb{Q}) \cap \text{Im}(\varphi_2)}{H^1(C, R^1\pi_*\mathbb{Q})}$ of type $(2, 0) + (1, 1) + (0, 2)$ in fact has type $(1, 1)$. Therefore the Hodge $\mathbb{Q}$ -structure $\text{Ker}(\psi_2)$ is of type $(1, 1)$. Consequently, the Hodge $\mathbb{Q}$ -structure $H^0(C, R^2\pi_*\mathbb{Q})$ is of type $(1, 1)$, because the map ψ_2 is surjective and, by the result of Section 2.2, the rational Hodge structure $H^0(C', R^2\pi'_*\mathbb{Q})$ has type $(1, 1)$. Lemma is proved.

2.6. The space $H^2(C, \mathbb{Q})$ is a 1-dimensional Hodge substructure of type $(1, 1)$ generated by the class $\text{cl}_X(X_s)$ of a fibre, therefore it easily follows from Lemma 2.5, from the exactness of the sequence (2.8) of rational Hodge structures for $n = 2$, from Lefschetz' theorem on divisors and from the surjectivity of the canonical edge morphism $H^2(X, \mathbb{Q}) \rightarrow H^0(C, R^2\pi_*\mathbb{Q})$ of rational Hodge structures [33, Corollary (15.14)] that the sum of all irreducible Hodge $\mathbb{Q}$ -substructures of dimension greater than 1 in the space $H^1(C, R^1\pi_*\mathbb{Q})$ is canonically identified with the space $T_{\mathbb{Q}}^2(X)$. In particular, there is a canonical embedding of rational Hodge structures

$$(2.11) \quad T_{\mathbb{Q}}^2(X) \hookrightarrow H^1(C, R^1\pi_*\mathbb{Q}).$$

According to the strong Lefschetz theorem on fibres of a smooth morphism π' and to the theorem on local invariant cycles we have the equalities

$$\begin{aligned} H^1(C, R^{2d-3}\pi_*\mathbb{Q}) &= H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \\ &= H^1(C, j_*R^1\pi'_*\mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-2} = H^1(C, R^1\pi_*\mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-2}, \end{aligned}$$

therefore by (2.7) and (2.11) the space $T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}$ is canonically identified with the space of transcendental classes of the rational Hodge structure $H^1(C, R^{2d-3}\pi_*\mathbb{Q})$ and, in particular, there exists a canonical embedding

$$(2.12) \quad T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2} \hookrightarrow H^1(C, R^{2d-3}\pi_*\mathbb{Q}).$$

Finally, the surjectivity of canonical edge morphism of rational Hodge structures $H^{2d-2}(X, \mathbb{Q}) \rightarrow H^0(C, R^{2d-2}\pi_*\mathbb{Q})$ [33, Corollary (15.14)], formulae (2.8) for the case $n = 2d - 2$, (2.10) and (2.12) show that the rational Hodge structure $H^0(C, R^{2d-2}\pi_*\mathbb{Q})$ is generated by images of *algebraic* cohomology classes, so the formula (2.8) yields the existence of the canonical embedding

$$(2.13) \quad T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2} \hookrightarrow K_{(2d-2)X}.$$

According to the theorem on local invariant cycles the canonical map

$$R^{2d-3}(\tau\sigma)_*\mathbb{Q} \rightarrow j_*R^{2d-3}(\tau\sigma)'_*\mathbb{Q} = j_*R^{2d-3}\tau'_*\mathbb{Q}$$

is surjective and it determines a canonical isomorphism

$$\begin{aligned} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) &\xrightarrow{\sim} H^1(C, j_*R^{2d-3}\tau'_*\mathbb{Q}) \\ &= \oplus_{a+b=2d-3} H^1(C, j_*R^a\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^b\pi'_*\mathbb{Q}). \end{aligned}$$

Consequently, the map

$$H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[(p_1\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q})$$

corresponds to the canonical embedding

$$\begin{aligned} &H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \\ &\xrightarrow{x \mapsto x \oplus [0]} H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \oplus [H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^1\pi'_*\mathbb{Q}) \\ (2.14) \quad &\oplus H^1(C, j_*R^{2d-5}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^2\pi'_*\mathbb{Q}) \oplus \dots \oplus H^1(C, j_*\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^{2d-3}\pi'_*\mathbb{Q})]. \end{aligned}$$

2.7. Admitting some freedom in notation, we denote by

$$(p_k\sigma)^* : H^n(X, \mathbb{Q}) \rightarrow H^n(Y, \mathbb{Q})$$

a canonical *injective* map of Weil cohomology [2, Section 1.2.A, Proposition 1.2.4] (determined by a *surjective* morphism $p_k\sigma : Y \rightarrow X$), which is different from the morphism of sheaves $(p_k\sigma)^* : R^n\pi_*\mathbb{Q} \xrightarrow{(p_k\sigma)^*} R^n(\tau\sigma)_*\mathbb{Q}$ defined above.

For any point $s \in C$, denote by $\iota_{X_s/X} : X_s \hookrightarrow X$ and by $\iota_{Y_s/Y} : Y_s \hookrightarrow Y$ the canonical embeddings. Set $p_{1s} = p_1|_{X_s \times X_s}$, $\sigma_s = \sigma|_{Y_s}$.

The morphism π is proper, therefore the fibre of the sheaf $R^n\pi_*\mathbb{Q}$ over a point $s \in C$ coincides with the space $H^n(X_s, \mathbb{Q})$ [28, Ch. II, § 4, Remark 4.17.1].

On the other hand, the Leray spectral sequence is functorial, therefore the restriction homomorphism $\iota_{X_s/X}^* : H^n(X, \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q})$ transforms the Leray spectral sequence $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ into the Leray spectral sequence corresponding to the morphism $X_s \rightarrow s$; consequently, the homomorphism $\iota_{X_s/X}^*$ coincides with the composite ([36], Ch. 9, the beginning of § 5)

$$H^n(X, \mathbb{Q}) = F^0 H^n(X, \mathbb{Q}) \rightarrow E_{\infty}^{0,n}(\pi) \rightarrow E_2^{0,n}(\pi) = H^0(C, R^n\pi_*\mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q}).$$

Thus the map $\iota_{X_s/X}^*$ is the composite of canonical maps

$$H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n\pi_*\mathbb{Q}) \hookrightarrow \prod_{s \in C} H^n(X_s, \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q})$$

and, consequently, for all elements $\omega \in \text{Ker}[H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n\pi_*\mathbb{Q})]$, we have the equality $\iota_{X_s/X}^*(\omega) = 0$. On the other hand, the commutative diagram of morphisms

$$\begin{array}{ccc} X & \xleftarrow{p_{1\sigma}} & Y \\ \uparrow \iota_{X_s/X} & & \uparrow \iota_{Y_s/Y} \\ X_s & \xleftarrow{p_{1s}\sigma_s} & Y_s \end{array}$$

yields the commutative diagram of canonical maps

$$\begin{array}{ccc} H^n(X, \mathbb{Q}) & \xrightarrow{(p_1\sigma)^*} & H^n(Y, \mathbb{Q}) \\ \downarrow \iota_{X_s/X}^* & & \downarrow \iota_{Y_s/Y}^* \\ H^n(X_s, \mathbb{Q}) & \xrightarrow{(p_{1s}\sigma_s)^*} & H^n(Y_s, \mathbb{Q}), \end{array}$$

so that, for any point $s \in C$, it follows from the equality $\iota_{X_s}^*(\omega) = 0$ that

$$\iota_{Y_s/Y}^*(p_1\sigma)^*(\omega) = 0.$$

The morphism $\tau\sigma$ is proper, therefore the canonical restriction map

$$\iota_{Y_s/Y}^* : H^n(Y, \mathbb{Q}) \rightarrow H^n(Y_s, \mathbb{Q})$$

is the composite of canonical maps

$$H^n(Y, \mathbb{Q}) \rightarrow H^0(C, R^n(\tau\sigma)_*\mathbb{Q}) \hookrightarrow \prod_{s \in C} H^n(Y_s, \mathbb{Q}) \rightarrow H^n(Y_s, \mathbb{Q}).$$

Thus, it follows from the equalities $\iota_{Y_s/Y}^*(p_1\sigma)^*(\omega) = 0$ ($s \in C$) that

$$(p_1\sigma)^*(\omega) \in \text{Ker}[H^n(Y, \mathbb{Q}) \rightarrow H^0(C, R^n(\tau\sigma)_*\mathbb{Q})].$$

As a result, we obtain the following inclusion

$$(2.15) \quad (p_1\sigma)^*K_{nX} \hookrightarrow K_{nY}.$$

2.8. The functoriality of constructions under consideration and (2.8) - (2.15) yield a commutative diagram of canonical morphisms of rational Hodge structures

$$(2.16) \quad \begin{array}{ccc} K_{nX} & \xrightarrow{(p_k\sigma)^*|_{K_{nX}}} & K_{nY} \\ \downarrow \alpha_{nX} & & \downarrow \alpha_{nY} \\ H^1(C, R^{n-1}\pi_*\mathbb{Q}) & \xrightarrow{[(p_k\sigma)^*]_1} & H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(C, j_*R^{n-1}\pi'_*\mathbb{Q}) & & H^1(C, j_*R^{n-1}\tau'_*\mathbb{Q}). \end{array}$$

It is well known that a linear map of finite-dimensional linear $\mathbb{Q}$ -spaces $u : E \rightarrow F$ is surjective iff the conjugate map $u^\vee \stackrel{\text{def}}{=} {}^t u : F^\vee \rightarrow E^\vee$ of dual spaces is injective [37, Ch. II, § 4, Section 9, Corollary of Theorem 3]; on the other hand, it follows from [37, Ch. II, § 4, Section 9, Theorem 4] that if the map u is injective, then the conjugate map ${}^t u : F^\vee \rightarrow E^\vee$ is surjective.

It is evident that the canonical map of cohomology

$$H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \xrightarrow{[(p_k\sigma)^*]_1} H^1(C, R^1\pi_*\mathbb{Q})$$

coincides, by the theorem on local invariant cycles, with the map

$$H^1(C, j_*R^{2d-1}(\tau'\sigma')_*\mathbb{Q}) \rightarrow H^1(C, j_*R^1\pi'_*\mathbb{Q}),$$

induced by the map of sheaves $R^{2d-1}(\tau'\sigma')_*\mathbb{Q} = R^{2d-1}\tau'_*\mathbb{Q} \xrightarrow{(p'_k)^*} R^1\pi'_*\mathbb{Q}$.

It follows from (2.8) and from the equalities $R^{2d}\pi_*\mathbb{Q} = R^{2d-1}\pi_*\mathbb{Q} = 0$ that

$$H^2(C, R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d).$$

On the other hand, the choice of an element $\sqrt{-1} \in \mathbb{C}$ determines the orientation isomorphism $H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$ of Weil cohomology [2, Section 1.2.A] and the non-degenerate pairing

$$H^{2d-2}(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \xrightarrow{x \times x' \mapsto x \smile x'} H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$$

of the Poincaré duality. Besides, in accordance with the theorem on local invariant cycles, there is a non-degenerate canonical pairing [33, Proposition (10.5)]

$$\begin{aligned} H^1(C, j_* R^{2d-3} \pi'_* \mathbb{Q}) \times H^1(C, j_* R^1 \pi'_* \mathbb{Q}) &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_* R^{2d-2} \pi'_* \mathbb{Q}) \\ &= H^2(C, R^{2d-2} \pi_* \mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}. \end{aligned}$$

By the similar reasons there exist non-degenerate pairings

$$\begin{aligned} H^{2d}(Y, \mathbb{Q}) \times H^{2d-2}(Y, \mathbb{Q}) &\xrightarrow{y \times y' \mapsto y \smile y'} H^{4d-2}(Y, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}; \\ H^1(C, j_* R^{2d-1}(\tau\sigma)'_* \mathbb{Q}) \times H^1(C, j_* R^{2d-3}(\tau\sigma)'_* \mathbb{Q}) &\xrightarrow{y \times y' \mapsto y \smile y'} H^2(C, j_* R^{4d-4}(\tau\sigma)'_* \mathbb{Q}) = H^2(C, R^{4d-4}(\tau\sigma)_* \mathbb{Q}) \\ &= H^{4d-2}(Y, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}. \end{aligned}$$

Therefore the map $H^1(C, R^{2d-1}(\tau\sigma)_* \mathbb{Q}) \xrightarrow{[(p_k\sigma)_*]_1} H^1(C, R^1 \pi_* \mathbb{Q})$ coincides with the surjective map which is dual to the canonical injection

$$H^1(C, R^{2d-3} \pi_* \mathbb{Q}) \xrightarrow{[(p_k\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_* \mathbb{Q})$$

determined in the diagram (2.16) for $n = 2d - 2$. In virtue of the surjectivity of maps $\alpha_{(2d-2)X}$ and $\alpha_{(2d-2)Y}$ in (2.8), (2.9), the commutative diagram dual to the diagram (2.16) in the case $n = 2d - 2$, takes the form

$$(2.17) \quad \begin{array}{ccc} K_{(2d-2)X}^\vee & \xleftarrow{((p_k\sigma)^*|_{K_{(2d-2)X}})^\vee} & K_{(2d-2)Y}^\vee \\ \cup \uparrow [\alpha_{(2d-2)X}]^\vee & & \cup \uparrow [\alpha_{(2d-2)Y}]^\vee \\ H^1(C, R^1 \pi_* \mathbb{Q}) & \xleftarrow{[(p_k\sigma)_*]_1} & H^1(C, R^{2d-1}(\tau\sigma)_* \mathbb{Q}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(C, j_* R^1 \pi'_* \mathbb{Q}) & & H^1(C, j_* R^{2d-1} \tau'_* \mathbb{Q}). \end{array}$$

2.9. Since the choice of an element $\sqrt{-1} \in \mathbb{C}$ identifies cohomology with coefficients in the Hodge - Tate structure $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ and Weil cohomology, then the canonical embedding of Hodge $\mathbb{Q}$ -structures $K_{nX} \hookrightarrow H^n(X, \mathbb{Q})$ and the Poincaré duality [2, Section 1.2.A] yield the canonical exact sequence of rational Hodge structures $0 \rightarrow G_{X/C}^{2d-n} \rightarrow H^{2d-n}(X, \mathbb{Q}) \rightarrow K_{nX}^\vee \rightarrow 0$ and the identification $K_{nX}^\vee = H^{2d-n}(X, \mathbb{Q})/G_{X/C}^{2d-n}$. Finally, the diagram dual to the commutative diagram

$$\begin{array}{ccc} K_{(2d-2)X} & \hookrightarrow & H^{2d-2}(X, \mathbb{Q}) \\ \cap \downarrow (p_2\sigma)^*|_{K_{(2d-2)X}} & & \cap \downarrow (p_2\sigma)^* \\ K_{(2d-2)Y} & \hookrightarrow & H^{2d-2}(Y, \mathbb{Q}), \end{array}$$

yields (in obvious notations) the commutative diagram

$$\begin{array}{ccccc} H^2(X, \mathbb{Q})/G_{X/C}^2 & = & K_{(2d-2)X}^\vee & \leftarrow & H^2(X, \mathbb{Q}) \\ \uparrow ((p_2\sigma)^*|_{K_{(2d-2)X}})^\vee & & \uparrow ((p_2\sigma)^*|_{K_{(2d-2)X}})^\vee & & \uparrow (p_2\sigma)^* \\ H^{2d}(Y, \mathbb{Q})/G_{Y/C}^{2d} & = & K_{(2d-2)Y}^\vee & \leftarrow & H^{2d}(Y, \mathbb{Q}). \end{array}$$

Therefore, for all $y \in H^{2d}(Y, \mathbb{Q})$, we have:

$$((p_2\sigma)^*|_{K_{(2d-2)X}})^\vee (y + G_{Y/C}^{2d}) = (p_2\sigma)_*(y) + G_{X/C}^2.$$

Consequently, the diagram (2.17) yields the commutative diagram of morphisms of Hodge structures

$$(2.18) \quad \begin{array}{ccc} H^2(X, \mathbb{Q})/G_{X/C}^2 & \xleftarrow{y+G_{Y/C}^{2d} \mapsto (p_2\sigma)_*(y)+G_{X/C}^2} & H^{2d}(Y, \mathbb{Q})/G_{Y/C}^{2d} \\ \cup \uparrow \beta_{2X} & & \cup \uparrow \beta_{(2d)Y} \\ H^1(C, R^1\pi_*\mathbb{Q}) & \xleftarrow{[(p_2\sigma)_*]_1} & H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

In accordance with (2.13) and (2.15) there is a commutative diagram of morphisms of Hodge structures

$$(2.19) \quad \begin{array}{ccc} (p_1\sigma)^*(T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) & \xrightarrow{\smile \text{cl}_Y(D^{(1)})} & (p_1\sigma)^*(T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) \smile \text{cl}_Y(D^{(1)}) \\ \cap & & \cap \\ K_{(2d-2)Y} & \xrightarrow{\smile \text{cl}_Y(D^{(1)})} & K_{(2d-2)Y} \smile \text{cl}_Y(D^{(1)}) \\ \downarrow \alpha_{(2d-2)Y} & & \downarrow \alpha_{(2d-2)Y} \smile \text{cl}_Y(D^{(1)}) \\ H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile \text{cl}_Y(D^{(1)})]_1} & H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(C, j_*R^{2d-3}\tau'_*\mathbb{Q}) & & H^1(C, j_*R^{2d-1}\tau'_*\mathbb{Q}). \end{array}$$

Since restrictions of elements $w \in K_{(2d-2)Y}$ and $w \smile \text{cl}_Y(D^{(1)})$ to the fibre Y_s are trivial for all $s \in C$, then

$$K_{(2d-2)Y} \smile \text{cl}_Y(D^{(1)}) \subset K_{(2d)Y},$$

therefore the diagram (2.19) yields the commutative diagram of morphisms of Hodge structures

$$(2.20) \quad \begin{array}{ccc} (p_1\sigma)^*(T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) & \xrightarrow{\smile \text{cl}_Y(D^{(1)})} & (p_1\sigma)^*(T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) \smile \text{cl}_Y(D^{(1)}) \\ \cap & & \cap \\ K_{(2d-2)Y} & \xrightarrow{\smile \text{cl}_Y(D^{(1)})} & K_{(2d)Y} \\ \downarrow \alpha_{(2d-2)Y} & & \downarrow \alpha_{(2d)Y} \\ H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile \text{cl}_Y(D^{(1)})]_1} & H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(C, j_*R^{2d-3}\tau'_*\mathbb{Q}) & & H^1(C, j_*R^{2d-1}\tau'_*\mathbb{Q}). \end{array}$$

On the other hand, it follows from (2.3) - (2.4) and (2.11) - (2.12) that the restriction of the isomorphism (2.7) to the subspace $T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2} \hookrightarrow H^1(C, R^{2d-3}\pi_*\mathbb{Q})$ (coinciding with the sum of all irreducible Hodge $\mathbb{Q}$ -substructures greater than 1 in the Hodge $\mathbb{Q}$ -structure $H^1(C, R^{2d-3}\pi_*\mathbb{Q})$) yields the equality

$$[(p_2\sigma)_*]_1 \circ [\smile \text{cl}_Y(D^{(1)})]_1 \circ [(p_1\sigma)^*]_1 (T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) = T_{\mathbb{Q}}^2(X).$$

Therefore (2.16) in the case $n = 2d - 2$, the diagrams (2.20), (2.17) and the injectivity of maps β_{2X} , $\beta_{(2d)Y}$ in the diagram (2.18) show that

$$(2.21) \quad (p_2\sigma)_* \left((p_1\sigma)^*(T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}) \smile \text{cl}_Y(D^{(1)}) \right) = T_{\mathbb{Q}}^2(X).$$

Let $\text{pr}_k : X \times X \rightarrow X$ be the canonical projection. Then $\text{pr}_k \iota\sigma = p_k\sigma$. Since pr_k and $\iota\sigma$ are morphisms of smooth projective varieties, then by the projection

formula [2, Section 1.2.A] for all $x \in T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2}$ we have:

$$\begin{aligned} (p_2\sigma)_* \left((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)}) \right) &= (\text{pr}_2 \iota\sigma)_* \left((\text{pr}_1 \iota\sigma)^*x \smile \text{cl}_Y(D^{(1)}) \right) = \\ \text{pr}_{2*}(\iota\sigma)_* \left((\iota\sigma)^* \text{pr}_1^*x \smile \text{cl}_Y(D^{(1)}) \right) &= \text{pr}_{2*} \left(\text{pr}_1^*x \smile (\iota\sigma)_* \text{cl}_Y(D^{(1)}) \right). \end{aligned}$$

Therefore it follows from (2.21) that *an algebraic* class

$$u \stackrel{\text{def}}{=} (\iota\sigma)_* \text{cl}_Y(D^{(1)}) \in H^4(X \times X, \mathbb{Q})$$

determines an *algebraic* isomorphism of type $(2-d, 2-d)$ of rational Hodge structures

$$(2.22) \quad T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2} \xrightarrow{\sim} T_{\mathbb{Q}}^2(X).$$

2.10. By the Lefschetz theorem on divisors we have the relations

$$\text{NS}_{\mathbb{Q}}(X) = H^2(X, \mathbb{Q})^{\text{Hg}(H^2(X, \mathbb{Q}))} = H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{C});$$

$$T_{\mathbb{Q}}^2(X)^{\text{Hg}(H^2(X, \mathbb{Q}))} = 0.$$

It is evident that $u = u_{0,4} + u_{1,3} + u_{2,2} + u_{3,1} + u_{4,0}$, where $u_{i,4-i} \in [H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^{4-i}(X, \mathbb{Q})] \cap H^{2,2}(X \times X, \mathbb{C})$, hence by [19, Corollary B.55] we have:

$$\begin{aligned} u_{2,2} \in [T_{\mathbb{Q}}^2(X) \oplus \text{NS}_{\mathbb{Q}}(X)]^{\otimes 2} \cap H^{2,2}(X \times X, \mathbb{C}) &= \\ [T_{\mathbb{Q}}^2(X) \otimes_{\mathbb{Q}} T_{\mathbb{Q}}^2(X)]^{\text{Hg}(H^2(X, \mathbb{Q}))} \oplus [\text{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X)]^{\text{Hg}(H^2(X, \mathbb{Q}))} \oplus \\ [T_{\mathbb{Q}}^2(X) \otimes_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X)]^{\text{Hg}(H^2(X, \mathbb{Q}))} \oplus [\text{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T_{\mathbb{Q}}^2(X)]^{\text{Hg}(H^2(X, \mathbb{Q}))} &= \\ [T_{\mathbb{Q}}^2(X) \otimes_{\mathbb{Q}} T_{\mathbb{Q}}^2(X)]^{\text{Hg}(H^2(X, \mathbb{Q}))} \oplus [\text{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X)]. \end{aligned}$$

In particular, $u_{2,2} = t_{2,2} + n_{2,2}$, where $t_{2,2} \in T_{\mathbb{Q}}^2(X) \otimes_{\mathbb{Q}} T_{\mathbb{Q}}^2(X)$ and $n_{2,2} \in \text{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X)$.

It is evident that

$$(2.23) \quad T_{\mathbb{Q}}^2(X) \smile \text{NS}_{\mathbb{Q}}(X)^{\smile d-1} = 0,$$

because otherwise this space coincides with a sum of certain irreducible Hodge $\mathbb{Q}$ -structures of dimension > 1 , contrary to the equality $\dim_{\mathbb{Q}} H^{2d}(X, \mathbb{Q}) = 1$.

By the results of Section 1.2 and (2.23), the restriction of the non-degenerate bilinear form

$$\Phi : H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \text{cl}_X(H)^{\smile d-2}} H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow{(2\pi i)^d} \mathbb{Q}$$

to the subspace $\text{NS}_{\mathbb{Q}}(X) \subset H^2(X, \mathbb{Q})$ is non-degenerate. Hence it follows from the inequality $\dim_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X) \geq 3$ that there exists the Poincaré class $\wp(\text{NS}_{\mathbb{Q}}(X))$ as a generator of the 1-dimensional space

$$[\text{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \text{NS}_{\mathbb{Q}}(X)]^{\text{SO}(\text{NS}_{\mathbb{Q}}(X), \Phi|_{\text{NS}_{\mathbb{Q}}(X)})}.$$

Correspondences $u_{0,4}, u_{1,3}, u_{3,1}, u_{4,0}$ annihilate the space $H^{2d-2}(X, \mathbb{Q})$ in virtue of [38, Formula (1.2)]. Therefore by (2.23) the correspondence $-n_{2,2} + \wp(\text{NS}_{\mathbb{Q}}(X))$ annihilates the subspace

$$T_{\mathbb{Q}}^2(X) \smile \text{cl}_X(H)^{\smile d-2} \subset H^{2d-2}(X, \mathbb{Q}),$$

and the algebraic class $u - n_{2,2} + \wp(\text{NS}_{\mathbb{Q}}(X))$ determines the isomorphism (2.22). Finally, the correspondence $t_{2,2}$ annihilates the subspace

$$\text{NS}_{\mathbb{Q}}(X) \smile \text{cl}_X(H)^{\smile d-2} \subset H^{2d-2}(X, \mathbb{Q})$$

in virtue of (2.23).

On the other hand, the correspondence $\wp(\mathrm{NS}_{\mathbb{Q}}(X))$ determines an algebraic isomorphism

$$\mathrm{NS}_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile d-2} \xrightarrow{\sim} \mathrm{NS}_{\mathbb{Q}}(X);$$

this result is an easy consequence of (2.23), because in virtue of the results of Section 1.2 and arguments of Section 3.5 below applied to the decomposition $H^2(X, \mathbb{Q}) = T_{\mathbb{Q}}^2(X) \oplus \mathrm{NS}_{\mathbb{Q}}(X)$ of rational Hodge structures, to the symmetric forms

$$\Psi^{\perp} = \Phi|_{T_{\mathbb{Q}}^2(X)}, \quad \Psi = \Phi|_{\mathrm{NS}_{\mathbb{Q}}(X)}$$

and to the canonical embedding of algebraic $\mathbb{Q}$ -groups

$$\mathrm{SO}(\mathrm{NS}_{\mathbb{Q}}(X), \Psi) \times \mathrm{SO}(T_{\mathbb{Q}}^2(X), \Psi^{\perp}) \hookrightarrow \mathrm{SO}(H^2(X, \mathbb{Q}), \Phi),$$

the Poincaré class $\wp(H^2(X, \mathbb{Q}))$ belongs to the space

$$T_{\mathbb{Q}}^2(X) \otimes_{\mathbb{Q}} T_{\mathbb{Q}}^2(X) \oplus [\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)]^{\mathrm{SO}(\mathrm{NS}_{\mathbb{Q}}(X), \Phi|_{\mathrm{NS}_{\mathbb{Q}}(X)})}$$

and it determines an isomorphism $H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$, which is the composite of maps

$$\begin{aligned} H^{2d-2}(X, \mathbb{Q}) &\xrightarrow{\mathrm{pr}_1^*} H^{2d-2}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^0(X, \mathbb{Q}) \\ &\xrightarrow{\smile \wp(H^2(X, \mathbb{Q}))} H^{2d}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^2(X, \mathbb{Q}) \xrightarrow{\mathrm{pr}_2^*} H^2(X, \mathbb{Q}). \end{aligned}$$

Therefore it follows from (2.10) and (2.22) that the algebraic class

$$u - n_{2,2} + \wp(\mathrm{NS}_{\mathbb{Q}}(X)) = u_{0,4} + u_{1,3} + t_{2,2} + u_{3,1} + u_{4,0} + \wp(\mathrm{NS}_{\mathbb{Q}}(X))$$

determines an algebraic isomorphism $H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$.

§ 3. SOME ISOMORPHISMS AND CANONICAL DECOMPOSITIONS OF RATIONAL HODGE STRUCTURES

3.1. First of all we are going to construct an algebraic isomorphism

$$H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow{\sim} H^1(C, R^2\pi_*\mathbb{Q}).$$

By [2, Lemma 2A12, Remark 2A13] the algebraic correspondence $c_1(\mathcal{P}'_s)^{\smile 2}$ yields an algebraic isomorphism $H^{2d-4}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_s, \mathbb{Q})$. Using arguments of Sections 2.1 and 2.3, we may assume that the element $\Lambda'_{1,1} \smile^2$ yields the isomorphism of local systems $R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^2\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$(3.1) \quad \begin{aligned} &R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{(p'_1)^*} R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} \\ &\xrightarrow{\smile \Lambda'_{1,1} \smile^2} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^2\pi'_*\mathbb{Q} \xrightarrow{(p'_2)^*} R^2\pi'_*\mathbb{Q}. \end{aligned}$$

We may also expand (3.1) to a sequence of morphisms of sheaves

$$R^{2d-4}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-4}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \mathrm{cl}_Y(D^{(1)})^{\smile 2}} R^{2d-2}(\tau\sigma)_*\mathbb{Q} \xrightarrow{(p_2\sigma)^*} R^2\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ . Therefore there exists an isomorphism of bidegree $(3-d, 3-d)$ of mixed Hodge structures

$$(3.2) \quad H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow{\smile [x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \mathrm{cl}_Y(D^{(1)})^{\smile 2})]_1} H^1(C, R^2\pi_*\mathbb{Q}).$$

3.2. Lemma. *For any odd natural number n , there are the equalities*

$$H^0(C', R^n \tau'_* \mathbb{Q}) = H^0(C', R^n \pi'_* \mathbb{Q}) = H^2(C, R^n(\tau\sigma)_* \mathbb{Q}) = H^2(C, R^n \pi_* \mathbb{Q}) = 0.$$

Proof. For a point $s \in C' \setminus \Delta_{\text{countable}}$ one has by (1.2)

$$H^0(C', R^n \tau'_* \mathbb{Q}) \xrightarrow{\sim} H^n(X_s \times X_s, \mathbb{Q})^G = H^n(X_s \times X_s, \mathbb{Q})^{\text{Hg}(X_s)} = 0$$

because in notations of Section 1.1, for any $\theta \in \mathbb{R}$, an element $e^{i\theta} \in U^1$ acts on the Hodge component $H^{p,n-p}(X_s \times X_s, \mathbb{C})$ as a multiplication by an element $e^{ip\theta} e^{-i(n-p)\theta} = e^{i(2p-n)\theta}$, therefore the group $\text{Hg}(X_s) \otimes_{\mathbb{Q}} \mathbb{C} = G \otimes_{\mathbb{Q}} \mathbb{C}$ acts on $H^n(X_s \times X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ without non-zero fixed points. Similarly one has the equality $H^0(C', R^n \pi'_* \mathbb{Q}) = 0$. It remains to note, in virtue of the theorem on local invariant cycles and [33, Proposition (10.5)], that

$$\begin{aligned} H^2(C, R^n(\tau\sigma)_* \mathbb{Q}) &= H^2(C, j_* R^n(\tau'\sigma')_* \mathbb{Q}) \\ &\xrightarrow{\sim} H^0(C', R^n(\tau'\sigma')_* \mathbb{Q})^\vee = H^0(C', R^n \tau'_* \mathbb{Q})^\vee; \\ H^2(C, R^n \pi_* \mathbb{Q}) &= H^2(C, j_* R^n \pi'_* \mathbb{Q}) \xrightarrow{\sim} H^0(C', R^n \pi'_* \mathbb{Q})^\vee. \end{aligned}$$

3.3. It follows from Lemma 3.2 and from (2.8) - (2.9) that, for any odd natural number n , one has canonical identifications of rational Hodge structures

$$(3.3) \quad K_{nX} \xrightarrow{\sim} H^1(C, R^{n-1} \pi_* \mathbb{Q});$$

$$(3.4) \quad K_{nY} \xrightarrow{\sim} H^1(C, R^{n-1}(\tau\sigma)_* \mathbb{Q}).$$

Besides, there is a non-degenerate canonical pairing [33, Proposition (10.5)]

$$\begin{aligned} H^1(C, j_* R^{2d-4} \pi'_* \mathbb{Q}) \times H^1(C, j_* R^2 \pi'_* \mathbb{Q}) &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_* R^{2d-2} \pi'_* \mathbb{Q}) \\ &= H^2(C, R^{2d-2} \pi_* \mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}, \end{aligned}$$

identifying (in accordance with the theorem on local invariant cycles) the space $H^1(C, R^{2d-4} \pi'_* \mathbb{Q})^\vee$ with $H^1(C, R^2 \pi'_* \mathbb{Q})$. By the similar reason we have the identification $H^1(C, R^{2d-4}(\tau\sigma)_* \mathbb{Q})^\vee = H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q})$. Hence from (3.3) and (3.4) we obtain the canonical isomorphisms

$$(3.5) \quad K_{(2d-3)X}^\vee \xleftarrow{\sim} H^1(C, R^{2d-4} \pi'_* \mathbb{Q})^\vee = H^1(C, R^2 \pi'_* \mathbb{Q}) \xrightarrow{[\alpha_{3X}]^{-1}} K_{3X};$$

$$\begin{aligned} K_{(2d-3)Y}^\vee &\xleftarrow{\sim} H^1(C, R^{2d-4}(\tau\sigma)_* \mathbb{Q})^\vee \\ (3.6) \quad &= H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}) \xrightarrow{[\alpha_{(2d+1)Y}]^{-1}} K_{(2d+1)Y}. \end{aligned}$$

On the other hand, acting by the algorithm of Section 2.9, we see that the Poincaré duality yields the identifications

$$K_{(2d-3)X}^\vee = H^3(X, \mathbb{Q})/G_{X/C}^3, \quad K_{(2d-3)Y}^\vee = H^{2d+1}(Y, \mathbb{Q})/G_{Y/C}^{2d+1}$$

and the equality $((p_2\sigma)^*|_{K_{(2d-3)X}})^\vee(y + G_{Y/C}^{2d+1}) = (p_2\sigma)_*(y) + G_{X/C}^3$ for any element $y \in H^{2d+1}(Y, \mathbb{Q})$. It follows from (3.5) - (3.6) that exact sequences of rational Hodge structures

$$\begin{aligned} 0 &\rightarrow G_{X/C}^3 \rightarrow H^3(X, \mathbb{Q}) \rightarrow H^3(X, \mathbb{Q})/G_{X/C}^3 \rightarrow 0, \\ 0 &\rightarrow G_{Y/C}^{2d+1} \rightarrow H^{2d+1}(Y, \mathbb{Q}) \rightarrow H^{2d+1}(Y, \mathbb{Q})/G_{Y/C}^{2d+1} \rightarrow 0 \end{aligned}$$

have *canonical splittings*, so that we obtain a commutative diagram

$$(3.7) \quad \begin{array}{ccc} H^3(X, \mathbb{Q})/G_{X/C}^3 & \xleftarrow{y+G_{Y/C}^{2d+1} \mapsto (p_2\sigma)_*(y)+G_{X/C}^3} & H^{2d+1}(Y, \mathbb{Q})/G_{Y/C}^{2d+1} \\ || & & || \\ K_{3X} & \xleftarrow{\left((p_2\sigma)^*|_{K_{(2d-3)X}}\right)^\vee = (p_2\sigma)^*|_{K_{(2d+1)Y}}} & K_{(2d+1)Y} \\ \simeq \uparrow [\alpha_{3X}]^{-1} & & \simeq \uparrow [\alpha_{(2d+1)Y}]^{-1} \\ H^1(C, R^2\pi_*\mathbb{Q}) & \xleftarrow{[(p_2\sigma)^*]_1} & H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

Finally, there is a commutative diagram

$$(3.8) \quad \begin{array}{ccc} K_{(2d-3)Y} & \xrightarrow{\smile \text{cl}_Y(D^{(1)}) \smile^2} & K_{(2d+1)Y} \\ \simeq \downarrow \alpha_{(2d-3)Y} & & \simeq \downarrow \alpha_{(2d+1)Y} \\ H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile \text{cl}_Y(D^{(1)}) \smile^2]_1} & H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

For $x \in H^{2d-3}(X, \mathbb{Q})$, the projection formula yields:

$$\begin{aligned} (p_2\sigma)_* \left((p_1\sigma)^*x \smile [\text{cl}_Y(D^{(1)})] \smile^2 \right) &= [\text{pr}_2 \iota\sigma]_* \left([\text{pr}_1 \iota\sigma]^*x \smile [\text{cl}_Y(D^{(1)})] \smile^2 \right) \\ &= \text{pr}_{2*}(\iota\sigma)_* \left((\iota\sigma)^* \text{pr}_1^*x \smile [\text{cl}_Y(D^{(1)})] \smile^2 \right) \\ &= \text{pr}_{2*} \left(\text{pr}_1^*x \smile (\iota\sigma)_* [\text{cl}_Y(D^{(1)})] \smile^2 \right). \end{aligned}$$

Therefore diagrams (2.16) for $n = 2d - 3$, (3.7) - (3.8) show that an algebraic class

$$(\iota\sigma)_* \left[[\text{cl}_Y(D^{(1)})] \smile^2 \right] \in H^6(X \times X, \mathbb{Q})$$

determines the isomorphism (3.2), which in virtue of the identification (3.3) takes the form

$$(3.9) \quad K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \text{pr}_{2*} \left(\text{pr}_1^*x \smile (\iota\sigma)_* [\text{cl}_Y(D^{(1)})] \smile^2 \right)} K_{3X}.$$

3.4. It follows from the theorem on local invariant cycles and from the strong Lefschetz theorem for fibres of the smooth morphism π' that the $\smile$ -multiplication by the class $\text{cl}_X(H)^{\smile d-3}$ yields the equalities

$$\begin{aligned} &\text{cl}_X(H)^{\smile d-3} \smile H^1(C, R^2\pi_*\mathbb{Q}) \\ &= \text{cl}_X(H)^{\smile d-3} \smile H^1(C, j_*R^2\pi'_*\mathbb{Q}) = H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) = H^1(C, R^{2d-4}\pi_*\mathbb{Q}), \end{aligned}$$

therefore from (3.3) we get the equality

$$(3.10) \quad K_{(2d-3)X} = \text{cl}_X(H)^{\smile d-3} \smile K_{3X}.$$

Taking into account (3.3), the theorem on locally invariant cycles and the non-degeneracy of the canonical pairing [33, Proposition (10.5)]

$$\begin{aligned} &H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^2\pi'_*\mathbb{Q}) \\ &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_*R^{2d-2}\pi'_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \end{aligned}$$

we see, that the canonical pairing $K_{(2d-3)X} \times K_{3X} \xrightarrow{x \times x' \mapsto x \smile x'} H^{2d}(X, \mathbb{Q})$ is non-degenerate. Therefore the restriction of the *non-degenerate* bilinear form

$$\Phi : H^3(X, \mathbb{Q}) \times H^3(X, \mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \text{cl}_X(H)^{\smile d-3}} H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow[\sim]{(2\pi i)^d} \mathbb{Q}.$$

to the subspace $K_{3X} \subset H^3(X, \mathbb{Q})$ is non-degenerate. Hence, as it was noticed in Section 1.2, there exists the decomposition of rational Hodge structures

$$H^3(X, \mathbb{Q}) = K_{3X} \oplus K_{3X}^\perp,$$

where $K_{3X}^\perp$ is the orthogonal complement of the subspace $K_{3X} \hookrightarrow H^3(X, \mathbb{Q})$ with respect to the *non-degenerate* bilinear form Φ .

It follows from (3.10) that the subspace

$$(3.11) \quad \begin{aligned} K_{3X}^\perp &= \{x \in H^3(X, \mathbb{Q}) \mid x \smile y \smile \text{cl}_X(H)^{\smile d-3} = 0 \quad \forall y \in K_{3X}\} \\ &= \{x \in H^3(X, \mathbb{Q}) \mid x \smile y = 0 \quad \forall y \in K_{(2d-3)X}\} \end{aligned}$$

does not depend on the choice of a divisor H . In particular, the decomposition of rational Hodge structures

$$(3.12) \quad H^3(X, \mathbb{Q}) = K_{3X} \oplus K_{3X}^\perp$$

is *canonical*.

Therefore an exact sequence of rational Hodge structures

$$(3.13) \quad 0 \rightarrow K_{3X} \rightarrow H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3\pi_*\mathbb{Q}) \rightarrow 0$$

allows us *canonically* identify the Hodge $\mathbb{Q}$ -structure $H^0(C, R^3\pi_*\mathbb{Q})$ with the Hodge substructure $K_{3X}^\perp \hookrightarrow H^3(X, \mathbb{Q})$.

Besides, it follows from the strong Lefschetz theorem that the subspace

$$\begin{aligned} &\text{cl}_X(H)^{\smile d-3} \smile K_{3X}^\perp \\ &= \text{cl}_X(H)^{\smile d-3} \smile \{x \in H^3(X, \mathbb{Q}) \mid x \smile y \smile \text{cl}_X(H)^{\smile d-3} = 0 \quad \forall y \in K_{3X}\} \\ &= \{x \in H^{2d-3}(X, \mathbb{Q}) \mid x \smile y = 0 \quad \forall y \in K_{3X}\} \end{aligned}$$

does not depend on the choice of a divisor H . Therefore, in accordance with (3.10), (3.12) and the strong Lefschetz theorem, an exact sequence of rational Hodge structures

$$0 \rightarrow K_{(2d-3)X} \rightarrow H^{2d-3}(X, \mathbb{Q}) \rightarrow H^0(C, R^{2d-3}\pi_*\mathbb{Q}) \rightarrow 0$$

allows us *canonically* identify the rational Hodge structure $H^0(C, R^{2d-3}\pi_*\mathbb{Q})$ with the Hodge substructure

$$(3.14) \quad \text{cl}_X(H)^{\smile d-3} \smile K_{3X}^\perp = \{x \in H^{2d-3}(X, \mathbb{Q}) \mid K_{3X} \smile x = 0\} \stackrel{\text{def}}{=} K_{(2d-3)X}^\perp.$$

In particular, there is the *canonical* decomposition of rational Hodge structures

$$(3.15) \quad H^{2d-3}(X, \mathbb{Q}) = K_{(2d-3)X} \oplus K_{(2d-3)X}^\perp.$$

In virtue of (3.10) - (3.11) we have:

$$(3.16) \quad K_{(2d-3)X} \smile K_{3X}^\perp = \text{cl}_X(H)^{\smile d-3} \smile K_{3X} \smile K_{3X}^\perp = 0.$$

3.5. The non-degeneracy of the skew-symmetric form $\Psi \stackrel{\text{def}}{=} \Phi|_{K_{3X}}$ implies the non-degeneracy of the form $\Psi^\perp \stackrel{\text{def}}{=} \Phi|_{K_{3X}^\perp}$ [21, Ch. IX, § 4, n° 1, Corollary of Proposition 1], therefore the decomposition (3.12) determines a canonical embedding of algebraic groups

$$\text{Sp}(K_{3X}, \Psi) \times \text{Sp}(K_{3X}^\perp, \Psi^\perp) \hookrightarrow \text{Sp}(H^3(X, \mathbb{Q}), \Phi),$$

which in turn yields the inclusion

$$\mathbb{Q} \cdot \wp(H^3(X, \mathbb{Q})) \subset \mathbb{Q} \cdot \wp(K_{3X}) + \mathbb{Q} \cdot \wp(K_{3X}^\perp),$$

because for elements

$$x_i \in K_{3X} \ (i = 1, 2), \quad x'_i \in K_{3X}^\perp \ (i = 1, 2), \quad \sigma \in \mathrm{Sp}(K_{3X}, \Psi), \quad \tau \in \mathrm{Sp}(K_{3X}^\perp, \Psi^\perp)$$

the action of an element $\sigma \times \tau \in \mathrm{Sp}(H^3(X, \mathbb{Q}), \Phi)$ is given by the formulae

$$\begin{aligned} (\sigma \times \tau)(x_1 \otimes x_2) &= \sigma x_1 \otimes \sigma x_2 = \sigma(x_1 \otimes x_2), \\ (\sigma \times \tau)(x'_1 \otimes x'_2) &= \tau x'_1 \otimes \tau x'_2 = \tau(x'_1 \otimes x'_2), \\ (\sigma \times \tau)(x_1 \otimes x'_1) &= \sigma x_1 \otimes \tau x'_1, \\ (\sigma \times \tau)(x'_1 \otimes x_1) &= \tau x'_1 \otimes \sigma x_1 \end{aligned}$$

and, in particular, invariants of tensor products

$$K_{3X} \otimes K_{3X}^\perp, \quad K_{3X}^\perp \otimes K_{3X}$$

with respect to actions of the algebraic group $\mathrm{Sp}(K_{3X}, \Psi) \times \mathrm{Sp}(K_{3X}^\perp, \Psi^\perp)$ are trivial in virtue of the irreducibility of the standard representations of symplectic groups. On the other hand, it was noticed in Section 1.2 that the correspondence $\wp(H^3(X, \mathbb{Q}))$ determines the isomorphism

$$(3.17) \quad H^{2d-3}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x) \smile \wp(H^3(X, \mathbb{Q})))} H^3(X, \mathbb{Q}).$$

Therefore there is the inclusion

$$\mathbb{Q}^\times \cdot \wp(H^3(X, \mathbb{Q})) \subset \mathbb{Q}^\times \cdot \wp(K_{3X}) + \mathbb{Q}^\times \cdot \wp(K_{3X}^\perp)$$

and, consequently, one may assume that

$$(3.18) \quad \wp(H^3(X, \mathbb{Q})) = \wp(K_{3X}) + \wp(K_{3X}^\perp).$$

It follows from (3.14) that the correspondence $\wp(K_{3X})$ annihilates the subspace $K_{(2d-3)X}^\perp \subset H^{2d-3}(X, \mathbb{Q})$ in the decomposition (3.15). Therefore the restriction of the isomorphism (3.17) to the subspace $K_{(2d-3)X}^\perp \subset H^{2d-3}(X, \mathbb{Q})$ determines the isomorphism

$$(3.19) \quad K_{(2d-3)X}^\perp \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x) \smile \wp(K_{3X}^\perp))} K_{3X}^\perp.$$

3.6. By Lemma 3.2 we have the equalities $H^0(C', R^3\pi'_*\mathbb{Q}) = H^2(C, R^1\pi_*\mathbb{Q}) = 0$. On the other hand, the theorem on local invariant cycles and the Leray spectral sequence for the embedding $j : C' \subset C$ yields an embedding of mixed Hodge structures ([33], P. 457, Corollary (13.10), Remark (14.5))

$$H^1(C, R^2\pi_*\mathbb{Q}) \hookrightarrow H^1(C', R^2\pi'_*\mathbb{Q}).$$

Consequently, taking into account the equality $H^2(C', R^1\pi'_*\mathbb{Q}) = 0$ in virtue of ([29, Ch. VI, § 7, Theorem 7.2]), (1.1), (3.3) and arguments of Section 2.5, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & H^1(C, R^2\pi_*\mathbb{Q}) & \rightarrow & H^3(X, \mathbb{Q}) & \rightarrow & H^0(C, R^3\pi_*\mathbb{Q}) & \rightarrow 0 \\ & \cap & & \downarrow \varphi_3 & & \downarrow \psi_3 & \\ 0 \rightarrow & H^1(C', R^2\pi'_*\mathbb{Q}) \cap \mathrm{Im}(\varphi_3) & \rightarrow & \mathrm{Im}(\varphi_3) & \rightarrow & 0 & \rightarrow 0 \end{array}$$

and the corresponding exact sequence of Hodge $\mathbb{Q}$ -structures ([13], Section 2.6) of the snake-like diagram ([34], § 1, Proposition 2)

$$(3.20) \quad 0 \rightarrow (i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \rightarrow H^0(C, R^3\pi_*\mathbb{Q}) \rightarrow \frac{H^1(C', R^2\pi'_*\mathbb{Q}) \cap \text{Im}(\varphi_3)}{H^1(C, R^2\pi_*\mathbb{Q})} \rightarrow 0.$$

3.7. Lemma. *There is the equality $K_{3X}^\perp = (i_{\Delta}f)_* H^1(Z, \mathbb{Q})$.*

Proof. First of all we are going to check that

$$(3.21) \quad (i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \smile K_{(2d-3)X} = 0.$$

Irreducible components of a smooth variety Z are naturally identified with irreducible components $X_{\delta i}$ of the divisor $\pi^{-1}(\Delta) = \sum_{\delta \in \Delta} X_{\delta}$. Denote by $\iota_{X_{\delta i}/X} : X_{\delta i} \hookrightarrow X$ the canonical embedding. From the commutativity of the diagram

$$(3.22) \quad \begin{array}{ccc} X_{\delta i} & \xrightarrow{\iota_{X_{\delta i}/X}} & X \\ \parallel & & \uparrow i_{\Delta}f \\ X_{\delta i} & \hookrightarrow & Z \end{array}$$

of canonical morphisms it follows the equality

$$(3.23) \quad (i_{\Delta}f)_* |_{H^1(X_{\delta i}, \mathbb{Q})} = \iota_{X_{\delta i}/X*},$$

therefore in order to prove the formula (3.21) it suffices to verify that

$$\iota_{X_{\delta i}/X*} H^1(X_{\delta i}, \mathbb{Q}) \smile K_{(2d-3)X} = 0.$$

By definition, the operator $\iota_{X_{\delta i}/X*}$ is conjugate to the operator $\iota_{X_{\delta i}/X}^*$ ([2], Section 1.2.A), therefore

$$< \iota_{X_{\delta i}/X*} H^1(X_{\delta i}, \mathbb{Q}) \smile K_{(2d-3)X} > = < H^1(X_{\delta i}, \mathbb{Q}) \smile \iota_{X_{\delta i}/X}^* K_{(2d-3)X} >$$

and, consequently, it suffices to verify the equality

$$(3.24) \quad \iota_{X_{\delta i}/X}^* K_{(2d-3)X} = 0.$$

Since $\iota_{X_{\delta i}/X}^* = \iota_{X_{\delta i}/X_{\delta}}^* \iota_{X_{\delta}/X}^*$ and $\iota_{X_{\delta}/X}^* K_{(2d-3)X} = 0$ by arguments of Section 2.7, we see that the formulae (3.24) and (3.21) are true.

There is a canonical exact sequence of rational Hodge structures [33, P. 473]

$$0 \rightarrow \mathcal{A}_3 \rightarrow H^0(C, R^3\pi_*\mathbb{Q}) \rightarrow H^0(C, j_* R^3\pi'_*\mathbb{Q}) \rightarrow 0,$$

so $H^0(C, R^3\pi_*\mathbb{Q}) = \mathcal{A}_3$ in virtue of Lemma 3.2 and the equality

$$H^0(C, j_* R^3\pi'_*\mathbb{Q}) = H^0(C', R^3\pi'_*\mathbb{Q}).$$

Denote by $D(\delta) \subset C$ a small open disc with the centre at the point $\delta \in \Delta$. Set $X_{D(\delta)} = X \times_C D(\delta)$. Degenerated fibres of the morphism π are unions of *smooth* $(d-1)$ -dimensional varieties of multiplicity 1 with normal crossings, therefore in accordance with [33, the 3d row from bottom on P. 473] we have:

$$\mathcal{A}_3 = \oplus_{\delta \in \Delta} \text{Ker}[H^3(X_{\delta}, \mathbb{Q}) \rightarrow R^3\pi'_*\mathbb{Q}],$$

where, in virtue of the theorem on local invariant cycles [33, Proposition (15.12)], for any point s of the punctured disc $D^*(\delta)$ with a centre at δ , the composite

$$H^3(X_{\delta}, \mathbb{Q}) \xrightarrow{\sim} H^3(X_{D(\delta)}, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})$$

of the isomorphism of deformation retraction (determined by the Clemens map of the variety $X_{D(\delta)}$ onto the degenerated fibre X_{δ} compatible with the radial retraction $D(\delta) \rightarrow \{\delta\}$ ([32]; [39, Ch. 5, Section 1.2, Section 3.3]) and of the restriction has as the image the space of cohomology classes invariant under the local

monodromy [33, proof of Proposition (15.12)]. In other words, there is a surjective composite $H^3(X_\delta, \mathbb{Q}) \xrightarrow{\sim} H^3(X_{D(\delta)}, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})^{\pi_1(D^*(\delta), s)}$. Thus

$$(3.25) \quad H^0(C, R^3\pi_*\mathbb{Q}) = \oplus_{\delta \in \Delta} \text{Ker}[H^3(X_\delta, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})^{\pi_1(D^*(\delta), s)}].$$

It follows from (3.21), (3.11) - (3.13) and (3.25) that

$$(3.26) \quad \begin{aligned} (i_\Delta f)_* H^1(Z, \mathbb{Q}) &\subset K_{3X}^\perp \xrightarrow{\sim} H^0(C, R^3\pi_*\mathbb{Q}) \\ &= \oplus_{\delta \in \Delta} \text{Ker}[H^3(X_\delta, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})^{\pi_1(D^*(\delta), s)}]. \end{aligned}$$

Let Z_δ be the normalization of the divisor $\pi^{-1}(\delta) = X_\delta$. It follows from (3.26) that for the proof of the lemma it suffices to prove the equality

$$(i_\Delta f)_* H^1(Z_\delta, \mathbb{Q}) = \text{Ker}[H^3(X_\delta, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})^{\pi_1(D^*(\delta), s)}]$$

for any point $\delta \in \Delta$.

It is well known that there is a long exact sequence of cohomology with compact supports [29, Ch. III, § 1, Remark 1.30]

$$\dots \rightarrow H_c^{2d-3}(X_{D(\delta)} \setminus \pi^{-1}(\delta), \mathbb{Q}) \rightarrow H_c^{2d-3}(X_{D(\delta)}, \mathbb{Q}) \rightarrow H_c^{2d-3}(\pi^{-1}(\delta), \mathbb{Q}) \rightarrow \dots,$$

therefore the Poincaré duality [29, Ch. VI, § 11, Corollary 11.2] yields the exact sequence [24, Corollary (8.2.8)]

$$(3.27) \quad H^1(Z_\delta, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*|_{H^1(Z_\delta, \mathbb{Q})}} H^3(X_{D(\delta)}, \mathbb{Q}) \rightarrow H^3(X_{D^*(\delta)}, \mathbb{Q}).$$

Thus the Clemens theory ([32]; [39, Ch. 5, Section 1.2, Section 3.3]), the theorem on local invariant cycles [33, Proof of Proposition (15.12)] and (3.27) yield the equalities

$$\begin{aligned} (i_\Delta f)_* H^1(Z_\delta, \mathbb{Q}) &= \text{Ker}[H^3(X_{D(\delta)}, \mathbb{Q}) \rightarrow H^3(X_{D^*(\delta)}, \mathbb{Q})] \\ &= \text{Ker}[H^3(X_\delta, \mathbb{Q}) \rightarrow H^3(X_s, \mathbb{Q})^{\pi_1(D^*(\delta), s)}]. \end{aligned}$$

Lemma is proved.

3.8. Lemma. *The Poincaré class $\wp(K_{3X}^\perp)$ is algebraic.*

Proof. According to Lemma 3.7 we have:

$$\wp(K_{3X}^\perp) = \wp((i_\Delta f)_* H^1(Z, \mathbb{Q})).$$

It follows directly from the definition of the Poincaré class in Section 1.2 that

$$\wp((i_\Delta f)_* H^1(Z, \mathbb{Q})) \in H^6(X \times X, \mathbb{Q}) \cap H^6(X \times X, \mathbb{C})^{3,3} \stackrel{\text{def}}{=} H^6(X \times X, \mathbb{Q})^{3,3}.$$

On the other hand, the morphism of pure Hodge $\mathbb{Q}$ -structures $(i_\Delta f)_*$ has bidegree $(1, 1)$ [30, Vol. I, P. 179] and it induces a surjection of pure Hodge structures $H^1(Z, \mathbb{Q}) \rightarrow (i_\Delta f)_* H^1(Z, \mathbb{Q})$. According to the definition of the Hodge group of rational Hodge structure [19, Definition B.51], this surjection yields a canonical surjection of Hodge groups $\text{Hg}(H^1(Z, \mathbb{Q})) \rightarrow \text{Hg}((i_\Delta f)_* H^1(Z, \mathbb{Q}))$, so that the $\mathbb{Q}$ -space $(i_\Delta f)_* H^1(Z, \mathbb{Q})$ has a natural structure of a $\text{Hg}(H^1(Z, \mathbb{Q}))$ -module.

In the situation of Section 1.1, there are polarizations $\psi_{H^1(X_{\delta i}, \mathbb{Q})}$ and bilinear skew-symmetric $\text{Hg}(H^1(X_{\delta i}, \mathbb{Q}))$ -invariant non-degenerate forms $\psi_{H^1(X_{\delta i}, \mathbb{Q})}^0$ such that the form

$$\psi_{H^1(Z, \mathbb{Q})}^0 \stackrel{\text{def}}{=} \sum_{\delta \in \Delta, i} \psi_{H^1(X_{\delta i}, \mathbb{Q})}^0$$

is a $\text{Hg}(H^1(Z, \mathbb{Q}))$ -invariant bilinear skew-symmetric non-degenerate form.

Let $K^\perp \subset H^1(Z, \mathbb{Q})$ be the orthogonal complement of the $\mathbb{Q}$ -space

$$K \stackrel{\text{def}}{=} \text{Ker}[H^1(Z, \mathbb{Q}) \rightarrow (i_{\Delta}f)_* H^1(Z, \mathbb{Q})]$$

with respect to the form $\psi_{H^1(Z, \mathbb{Q})}^0$. We have the decomposition of $\text{Hg}(H^1(Z, \mathbb{Q}))$ -modules $H^1(Z, \mathbb{Q}) = K \oplus K^\perp$ and the isomorphism of $\text{Hg}(H^1(Z, \mathbb{Q}))$ -modules $K^\perp \xrightarrow{\sim} [(i_{\Delta}f)_* H^1(Z, \mathbb{Q})](1)$.

By the properties of Hodge groups discussed in Section 1.1, we have:

$$\begin{aligned} & [(i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} (i_{\Delta}f)_* H^1(Z, \mathbb{Q})]^{3,3} \\ &= [(i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} (i_{\Delta}f)_* H^1(Z, \mathbb{Q})]^{\text{Hg}((i_{\Delta}f)_* H^1(Z, \mathbb{Q}))} \\ &= [(i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} (i_{\Delta}f)_* H^1(Z, \mathbb{Q})]^{\text{Hg}(H^1(Z, \mathbb{Q}))} \\ &= [[(i_{\Delta}f)_* \otimes_{\mathbb{Q}} (i_{\Delta}f)_*] (K^\perp \otimes_{\mathbb{Q}} K^\perp)]^{\text{Hg}(H^1(Z, \mathbb{Q}))} \\ &= [(i_{\Delta}f)_* \otimes_{\mathbb{Q}} (i_{\Delta}f)_*] ([K^\perp \otimes_{\mathbb{Q}} K^\perp]^{\text{Hg}(H^1(Z, \mathbb{Q}))}) \\ &\hookrightarrow [(i_{\Delta}f)_* \otimes_{\mathbb{Q}} (i_{\Delta}f)_*] ([H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z, \mathbb{Q})]^{\text{Hg}(H^1(Z, \mathbb{Q}))}) \\ &= [(i_{\Delta}f)_* \otimes_{\mathbb{Q}} (i_{\Delta}f)_*] ([H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z, \mathbb{Q})] \cap H^{1,1}(Z \times Z, \mathbb{C})). \end{aligned}$$

Taking into account that, by Lefschetz' theorem on divisors, there is an inclusion

$$[H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z, \mathbb{Q})] \cap H^{1,1}(Z \times Z, \mathbb{C}) \hookrightarrow \text{NS}_{\mathbb{Q}}(Z \times Z),$$

we see that the Poincaré class

$$\varphi((i_{\Delta}f)_* H^1(Z, \mathbb{Q})) \in [(i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} (i_{\Delta}f)_* H^1(Z, \mathbb{Q})]^{3,3}$$

is algebraic. Lemma is proved.

3.9. Denote by $\Delta^{\text{multiplicative}}$ the set of all places $\delta \in \Delta$ such that the Abelian variety X_η has a totally degenerated reduction of multiplicative type at the place δ (in other words, the toric rank r_δ equals $d-1$). Since, for such places δ , the fibre X_δ is a union of smooth $(d-1)$ -dimensional varieties $X_{\delta i}$ of multiplicity 1 with normal crossings, and the variety $X_{\delta i}$ is the closure of the torus Gm^{d-1} in the Zariski topology of the fibre X_δ , then $H^1(X_{\delta i}, \mathbb{Q}) = 0$ and, consequently, $H^1(Z_\delta, \mathbb{Q}) = 0$. In particular, Lemma 3.7 yields the equality

$$K_{3X}^\perp = \sum_{\delta \in \Delta \setminus \Delta^{\text{multiplicative}}} (i_{\Delta}f)_* H^1(Z_\delta, \mathbb{Q}).$$

Therefore it follows from (3.3) and (3.12) that there exists the canonical decomposition of rational Hodge structures

$$(3.28) \quad H^3(X, \mathbb{Q}) = H^1(C, R^2\pi_* \mathbb{Q}) \bigoplus \sum_{\delta \in \Delta \setminus \Delta^{\text{multiplicative}}} (i_{\Delta}f)_* H^1(Z_\delta, \mathbb{Q}).$$

§ 4. A PROOF OF THE THEOREM

4.1. For any point $\delta \in \Delta = \{\delta \in C \mid r_\delta > 0\}$ set

$$\begin{aligned} m_\delta &\stackrel{\text{def}}{=} \text{Card}(\mathcal{M}_\delta / \mathcal{M}_\delta^0), \\ m &\stackrel{\text{def}}{=} \prod_{\delta \in \Delta} m_\delta. \end{aligned}$$

Fix a prime number p , which does not divide the number m . Denote by $p_{X/C}^{m!} : X \dashrightarrow X$ a rational map, coinciding on the generic scheme fibre X_η of the structure morphism $\pi : X \rightarrow C$ with the isogeny of the multiplication by the number $p^{m!}$.

In virtue of the universal property of the Néron model [15, (1.1.2)] there is the canonical isomorphism

$$(4.1) \quad \text{End}_C(\mathcal{M}) \xrightarrow{\sim} \text{End}_{\kappa(\eta)}(X_\eta).$$

Consider a commutative diagram

$$(4.2) \quad \begin{array}{ccc} \tilde{X} & & \\ \downarrow \sigma & \searrow \nu & \\ X & \xrightarrow{p_{X/C}^{m!}} & X \end{array}$$

of a resolution of indeterminacies of the rational map $p_{X/C}^{m!}$. By Hironaka's results and by (4.1) we may assume that the morphism σ is the composite of monoidal transformations with non-singular centres, and $\sigma|_{\sigma^{-1}(\mathcal{M})} : \sigma^{-1}(\mathcal{M}) \rightarrow \mathcal{M}$ is the identity morphism.

Let

$$[p_{X/C}^{m!}]^* : H^*(X, \mathbb{Q}) \xrightarrow{x \mapsto \sigma_* \nu^*(x)} H^*(X, \mathbb{Q})$$

be the linear operator determined by the diagram (4.2).

4.2. Lemma. *The linear operator $[p_{X/C}^{m!}]^* : H^3(X, \mathbb{Q}) \rightarrow H^3(X, \mathbb{Q})$ preserves the decomposition*

$$H^3(X, \mathbb{Q}) = H^1(C, R^2\pi_*\mathbb{Q}) \bigoplus \sum_{\delta \in \Delta \setminus \Delta_{\text{multiplicative}}} (i_{\Delta}f)_* H^1(Z_{\delta}, \mathbb{Q}),$$

and it induces the multiplication by the number $p^{2m!}$ in the space

$$H^1(C, R^2\pi_*\mathbb{Q}) = \text{Ker}[H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3\pi_*\mathbb{Q})]$$

and the multiplication by the number $p^{m!}$ in the space

$$\sum_{\delta \in \Delta \setminus \Delta_{\text{multiplicative}}} (i_{\Delta}f)_* H^1(Z_{\delta}, \mathbb{Q}).$$

Proof. In virtue of the theorem on local invariant cycles there is the canonical identification

$$(4.3) \quad H^1(C, R^2\pi_*\mathbb{Q}) = H^1(C, j_* R^2\pi'_*\mathbb{Q});$$

similarly, for the structure morphism $\tilde{\pi} : \tilde{X} \rightarrow C$, we obtain the canonical identification

$$(4.4) \quad H^1(C, R^2\tilde{\pi}_*\mathbb{Q}) = H^1(C, j_* R^2\tilde{\pi}'_*\mathbb{Q}).$$

The commutative diagram of rational maps (4.2) yields the commutative diagram of C' -morphisms

$$\begin{array}{ccc} \tilde{X}' & & \\ \simeq \downarrow \sigma' & \searrow \nu' & \\ X' & \xrightarrow{p_{X'/C'}^{m!}} & X', \end{array}$$

so that, for any open subset $U \subset C'$, one can determine the commutative diagram

$$\begin{array}{ccc} H^2((\tilde{\pi}')^{-1}(U), \mathbb{Q}) & & \\ \simeq \uparrow \sigma'^* & \nwarrow \nu'^* & \\ H^2((\pi')^{-1}(U), \mathbb{Q}) & \xleftarrow{[p_{X'/C'}^{m!}]^*} & H^2((\pi')^{-1}(U), \mathbb{Q}), \end{array}$$

which in turn yields the commutative diagram of local systems

$$(4.5) \quad \begin{array}{ccc} R^2 \tilde{\pi}'_* \mathbb{Q} & & \\ \simeq \uparrow \sigma'^* & \nwarrow \nu'^* & \\ R^2 \pi'_* \mathbb{Q} & \xleftarrow{[p_{X'/C'}^{m!}]^*} & R^2 \pi'_* \mathbb{Q}. \end{array}$$

For any fibre X_s of the Abelian scheme $\pi' : X' \rightarrow C'$, the isogeny of the multiplication by the number $p^{m!}$ induces the multiplication by $p^{m!}$ in the space $H^1(X_s, \mathbb{Q})$ [2, Lemma 2A3, Section 2A11]. Consequently, the canonical map in the bottom row of the diagram (4.5) is the multiplication by the number $p^{2m!}$ in virtue of the equality $R^2 \pi'_* \mathbb{Q} = \wedge^2 R^1 \pi'_* \mathbb{Q}$, so that

$$(4.6) \quad \nu'^* = p^{2m!} \sigma'^*.$$

It is evident that the isomorphism $\sigma'^* : R^2 \pi'_* \mathbb{Q} \xrightarrow{\sim} R^2 \tilde{\pi}'_* \mathbb{Q}$ determines the isomorphism

$$j_* R^2 \pi'_* \mathbb{Q} \xrightarrow{\sim} j_* R^2 \tilde{\pi}'_* \mathbb{Q}$$

and the corresponding isomorphism of cohomology

$$H^1(C, j_* R^2 \pi'_* \mathbb{Q}) \xrightarrow{\sim} H^1(C, j_* R^2 \tilde{\pi}'_* \mathbb{Q}),$$

which in virtue of (4.3) - (4.4) views as $\sigma^* : H^1(C, R^2 \pi_* \mathbb{Q}) \xrightarrow{\sim} H^1(C, R^2 \tilde{\pi}_* \mathbb{Q})$. On the other hand, the map $\nu'^* : R^2 \pi'_* \mathbb{Q} \rightarrow R^2 \tilde{\pi}'_* \mathbb{Q}$ determines the map

$$j_* R^2 \pi'_* \mathbb{Q} \rightarrow j_* R^2 \tilde{\pi}'_* \mathbb{Q}$$

and the corresponding map of cohomology $H^1(C, j_* R^2 \pi'_* \mathbb{Q}) \rightarrow H^1(C, j_* R^2 \tilde{\pi}'_* \mathbb{Q})$, which in virtue of (4.3) - (4.4) and (4.6) views as

$$\nu^* = p^{2m!} \sigma^* : H^1(C, R^2 \pi_* \mathbb{Q}) \rightarrow H^1(C, R^2 \tilde{\pi}_* \mathbb{Q}).$$

The well known formula $\sigma_* \sigma^* = \text{id}_{H^3(X, \mathbb{Q})}$ shows that the linear operator

$$[p_{X/C}^{m!}]^* = \sigma_* \nu^* : H^3(X, \mathbb{Q}) \rightarrow H^3(X, \mathbb{Q})$$

induces the multiplication by the number $p^{2m!}$ in the subspace $H^1(C, R^2 \pi_* \mathbb{Q}) \subset H^3(X, \mathbb{Q})$.

It is known that, for any smooth complex projective variety W , the Picard variety

$$\text{Pic}^0(W) = H^{0,1}(W, \mathbb{C}) / \text{Im}[H^1(W, \mathbb{Z}) \rightarrow H^{0,1}(W, \mathbb{C})]$$

and the Albanese variety dual to it ([40, Ch. II, § 3]; [41, Ch. 2, the end of § 6]; [42, P. 171 - 172])

$$\text{Alb}(W) = H^0(W, \Omega_W^1)^\vee / (H_1(W, \mathbb{Z}) / \text{tors})$$

are stable under a transition to a smooth projective variety birationally equivalent to the variety W , because a monoidal transformation $f : W' \rightarrow W$ with a centre at a Zariski closed smooth subvariety $D \hookrightarrow W$ determines the canonical isomorphism of Hodge structures $H^1(W, \mathbb{Z}) \xrightarrow{f^*} H^1(W', \mathbb{Z})$ [42, Proposition 13.1] and a birational

map of projective non-singular complex varieties is the composite of projective blow-ups and projective contractions with smooth centres [43, Theorem 0.1.1].

Let $\delta \in \Delta \setminus \Delta^{\text{multiplicative}}$. Then the toric rank r_δ belongs to the set $\{1, \dots, d-1\}$. The fibre X_δ is a union of smooth $(d-1)$ -dimensional varieties $X_{\delta i}$ of multiplicity 1 with normal crossings, and the variety $X_{\delta i}$ is the closure of the irreducible component $\mathcal{M}_{\delta i}$ of the algebraic group $\mathcal{M}_\delta$ in the Zariski topology of the fibre X_δ . On the other hand,

$$\mathcal{M}_{\delta i} = a_{\delta i} \mathcal{M}_\delta^0$$

for some element $a_{\delta i} \in \mathcal{M}_{\delta i}$, therefore the variety $\mathcal{M}_{\delta i}$ is isomorphic to the connected component $\mathcal{M}_\delta^0$ of the neutral element of the group $\mathcal{M}_\delta$, which is included into an exact sequence of algebraic groups

$$(4.7) \quad 1 \rightarrow \text{Gm}^{r_\delta} \rightarrow \mathcal{M}_\delta^0 \xrightarrow{f_\delta} A_\delta \rightarrow 0,$$

where A_δ is an Abelian variety of a strictly positive dimension.

From now on we denote by $\text{alb}_{\delta i} : X_{\delta i} \rightarrow \text{Alb}(X_{\delta i})$ the Albanese map, which is determined uniquely up to a translation on the Abelian variety $\text{Alb}(X_{\delta i})$ [40, Ch. II, § 3, Theorem 11].

One may assume that $X_{\delta 1} = \overline{\mathcal{M}_\delta^0}$. It is known that the canonical rational map $F_\delta : X_{\delta 1} \dashrightarrow A_\delta$, determined by the extension (4.7), in reality is regular [40, Ch. II, § 1, Theorem 2]. It follows from (4.7) that fibres of the morphism f_δ are isomorphic to the torus Gm^{r_δ} . Therefore, for any morphism $\Phi : X_{\delta 1} \rightarrow A$ into an arbitrary Abelian variety A , the restriction of the morphism Φ to any fibre of the morphism f_δ is a constant map [40, Ch. II, § 1, P. 25, Corollary]. Thus the morphism $\Phi|_{\mathcal{M}_\delta^0}$ is decomposed as $\mathcal{M}_\delta^0 \xrightarrow{f_\delta} A_\delta \rightarrow A$ and, consequently, the morphism Φ is the composite $\overline{\mathcal{M}_\delta^0} \xrightarrow{F_\delta} A_\delta \rightarrow A$. Therefore $\text{Alb}(X_{\delta 1}) = A_\delta$. Since the Albanese variety is stable under a transition to a birationally equivalent variety $X_{\delta i}$ of the variety $X_{\delta 1}$, then

$$(4.8) \quad \forall i \quad \text{Alb}(X_{\delta i}) = A_\delta.$$

Let X^{sm} be the set of all points $x \in X$ such that the structure morphism π is smooth at x . It is evident that the special fibre X_δ^{sm} is a disjoint union of semi-Abelian schemes $\mathcal{M}_{\delta i}$ which are isomorphic to the variety $\mathcal{M}_\delta^0$ [17, Section 4.4]. Since reductions are stable in Grothendieck's sense, then, for any ramified covering $\tilde{C} \rightarrow C$, the connected component of the neutral element of the special fibre $\mathcal{M}_\delta$ of the Néron model $\mathcal{M} \rightarrow C$ is isomorphic to the connected component of the neutral element of the special fibre $\tilde{\mathcal{M}}_\delta$ of the Néron model $\tilde{\mathcal{M}} \rightarrow \tilde{C}$ [15, Corollaries 3.3, 3.9]; in particular, all irreducible components $\tilde{X}_{\delta j}$ of the special fibre of the Künnemann compactification of the Néron model $\tilde{\mathcal{M}} \rightarrow \tilde{C}$ are birationally equivalent to the variety $X_{\delta i}$. Consequently, the equalities (4.8) are preserved under the base change $\tilde{C} \rightarrow C$.

The canonical surjection $f_\delta : \mathcal{M}_\delta^0 \rightarrow A_\delta$ admits a prolongation to a *surjective morphism* $F_\delta : X_{\delta 1} \rightarrow A_\delta$, which in turn yields an injection $F_\delta^* : H^1(A_\delta, \mathbb{Q}) \hookrightarrow H^1(X_{\delta 1}, \mathbb{Q})$ [2, Proposition 1.2.4], so that in reality it follows from (4.8) that the injection F_δ^* is an *isomorphism*

$$(4.9) \quad H^1(A_\delta, \mathbb{Q}) \xrightarrow{\sim} H^1(X_{\delta 1}, \mathbb{Q}).$$

Since the space $H^1(X_{\delta i}, \mathbb{Q})$ is stable under a transition to a variety $X_{\delta 1}$ birationally equivalent to the variety $X_{\delta i}$, then (4.8) - (4.9) and the exact sequence

$$0 \rightarrow H^1(\text{Alb}(X_{\delta i}), \mathbb{Q}) \rightarrow H^1(X_{\delta i}, \mathbb{Q}) \rightarrow H^0(X_{\delta i}, R^1 \text{alb}_{\delta i*} \mathbb{Q}),$$

determined by the Leray spectral sequence

$$E_2^{p,q}(\text{alb}_{\delta i}) = H^p(\text{Alb}(X_{\delta i}), R^q \text{alb}_{\delta i*} \mathbb{Q})$$

of the Albanese map $\text{alb}_{\delta i}$, yield the canonical isomorphism of rational Hodge structures

$$(4.10) \quad \text{alb}_{\delta i}^* : H^1(\text{Alb}(X_{\delta i}), \mathbb{Q}) \xrightarrow{\sim} H^1(X_{\delta i}, \mathbb{Q}).$$

Finitely, the map $\text{alb}_{\delta i}$ is included into the commutative diagram of rational maps

$$(4.11) \quad \begin{array}{ccc} X_{\delta 1} & \xrightarrow{\text{alb}_{\delta 1}} & \text{Alb}(X_{\delta 1}) \\ \uparrow \times a_{\delta i}^{-1} & & \simeq \uparrow (\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_* \\ | & & \\ X_{\delta i} & \xrightarrow{\text{alb}_{\delta i}} & \text{Alb}(X_{\delta i}), \end{array}$$

where for all $x_{\delta i} \in \mathcal{M}_{\delta i}$ we have:

$$\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}](x_{\delta i}) = (\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_* \text{alb}_{\delta i}(x_{\delta i}) + c_{\delta 1},$$

$c_{\delta 1} \in \text{Alb}(X_{\delta 1})$, $(\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*$ is the canonical *isomorphism* of Abelian varieties [40, Ch. II, § 3, P. 41], determined in virtue of the universal property of the Albanese map by a *regular* [40, Ch. II, § 1, Theorem 2] morphism

$$\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}] : X_{\delta i} \rightarrow \text{Alb}(X_{\delta 1}).$$

From (4.11) we obtain the commutative diagram of isomorphisms

$$(4.12) \quad \begin{array}{ccc} H^1(X_{\delta 1}, \mathbb{Q}) & \xleftarrow{\text{alb}_{\delta 1}^*} & H^1(\text{Alb}(X_{\delta 1}), \mathbb{Q}) \\ \simeq \downarrow [\times a_{\delta i}^{-1}]^* & & \simeq \downarrow ((\text{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*)^* \\ H^1(X_{\delta i}, \mathbb{Q}) & \xleftarrow{\text{alb}_{\delta i}^*} & H^1(\text{Alb}(X_{\delta i}), \mathbb{Q}), \end{array}$$

which determines the isomorphism $[\times a_{\delta i}^{-1}]^*$ and it is independent of a choice of a point $c_{\delta 1}$ (because the translation $z \mapsto z + c_{\delta 1}$ on the Albanese variety $\text{Alb}(X_{\delta 1})$ acts identically on the space $H^1(\text{Alb}(X_{\delta 1}), \mathbb{C})$, generated by differentials $dz_i, \overline{dz_i}$ ($i = 1, \dots, \dim_{\mathbb{C}} \text{Alb}(X_{\delta 1})$) in terms of some lattice $L_{\delta 1} \hookrightarrow \mathbb{C}^{\dim_{\mathbb{C}} \text{Alb}(X_{\delta 1})}$ and canonical global coordinates z_i ($i = 1, \dots, \dim_{\mathbb{C}} \text{Alb}(X_{\delta 1})$) on the variety $\mathbb{C}^{\dim_{\mathbb{C}} \text{Alb}(X_{\delta 1})} / L_{\delta 1} = \text{Alb}(X_{\delta 1})$).

Consider the commutative diagrams with exact rows

$$(4.13) \quad \begin{array}{ccccccc} 1 \rightarrow & \text{Gm}^{r_{\delta}} & \rightarrow & \mathcal{M}_{\delta}^0 & \rightarrow & A_{\delta} & \rightarrow 0 \\ & \downarrow p_{X/C}^{m!} |_{\text{Gm}^{r_{\delta}}} & & \downarrow p_{X/C}^{m!} |_{\mathcal{M}_{\delta}^0} & & \downarrow \times p^{m!} & \\ 1 \rightarrow & \text{Gm}^{r_{\delta}} & \rightarrow & \mathcal{M}_{\delta}^0 & \rightarrow & A_{\delta} & \rightarrow 0; \end{array}$$

$$(4.14) \quad \begin{array}{ccccccc} 1 \rightarrow & \mathcal{M}_{\delta}^0 & \rightarrow & \mathcal{M}_{\delta} & \rightarrow & \mathcal{M}_{\delta} / \mathcal{M}_{\delta}^0 & \rightarrow 0 \\ & \downarrow p_{X/C}^{m!} |_{\mathcal{M}_{\delta}^0} & & \downarrow p_{X/C}^{m!} |_{\mathcal{M}_{\delta}} & & \downarrow \times p^{m!} & \\ 1 \rightarrow & \mathcal{M}_{\delta}^0 & \rightarrow & \mathcal{M}_{\delta} & \rightarrow & \mathcal{M}_{\delta} / \mathcal{M}_{\delta}^0 & \rightarrow 0, \end{array}$$

where $\mathcal{M}_{\delta} / \mathcal{M}_{\delta}^0$ is a finite group (of order m_{δ}) of *connected components* of the algebraic group $\mathcal{M}_{\delta}$ [15, Section (1.1.5)]. The evident surjectivity of canonical maps $p_{X/C}^{m!} |_{\text{Gm}^{r_{\delta}}}, A_{\delta} \xrightarrow{\times p^{m!}} A_{\delta}$ and the corresponding to the diagram (4.13) exact

sequence of the snake-like diagram [34, § 1, Proposition 2] show that the canonical map $p_{X/C}^{m!}|_{\mathcal{M}_\delta^0}$ is surjective. On the other hand, the multiplication by the invertible in the ring $\mathbb{Z}/m_\delta\mathbb{Z}$ element $p \bmod m_\delta$ yields a permutation of elements of the finite group $\mathcal{M}_\delta/\mathcal{M}_\delta^0$. Consequently, by Lagrange's theorem, the multiplication by the element $p^{m!} \bmod m_\delta$ is the *identity bijection* of the set $\mathcal{M}_\delta/\mathcal{M}_\delta^0$. Therefore from the commutativity of the diagram (4.14), from the exactness of the corresponding sequence of the snake-like diagram and from the surjectivity of the morphism $p_{X/C}^{m!}|_{\mathcal{M}_\delta^0}$ it follows that the morphism $p_{X/C}^{m!}|_{\mathcal{M}_\delta}$ is surjective, and

$$(4.15) \quad (\forall i) \quad p_{X/C}^{m!}(\mathcal{M}_{\delta i}) = \mathcal{M}_{\delta i}.$$

Consequently, there exists a commutative diagram

$$(4.16) \quad \begin{array}{ccc} & W_{\delta i} & \\ & \downarrow \sigma_{\delta i} & \searrow \nu_{\delta i} \\ \overline{\mathcal{M}_{\delta i}} & = & X_{\delta i} \xrightarrow{p_{X/C}^{m!}|_{X_{\delta i}}} X_{\delta i}, \end{array}$$

of a resolution of indeterminacies of the rational map $p_{X/C}^{m!}|_{X_{\delta i}}$, where the morphism $\sigma_{\delta i}$ is the composite of monoidal transformations with non-singular centres, lying over the variety $X_{\delta i} \setminus \mathcal{M}_\delta$ [43, Theorem 0.1.1].

For any element $x_{\delta 1} \in \mathcal{M}_{\delta 1}$ we have:

$$\begin{aligned} a_{\delta i} x_{\delta 1} &\in \mathcal{M}_{\delta i}; \\ p_{X/C}^{m!}|_{X_{\delta i}}(a_{\delta i} x_{\delta 1}) &= (a_{\delta i} x_{\delta 1})^{p^{m!}} = a_{\delta i}^{p^{m!}} x_{\delta 1}^{p^{m!}} = p_{X/C}^{m!}|_{X_{\delta i}}(a_{\delta i}) p_{X/C}^{m!}|_{X_{\delta 1}}(x_{\delta 1}) \in \mathcal{M}_{\delta i}, \end{aligned}$$

and it follows from (4.15) that

$$p_{X/C}^{m!}|_{X_{\delta i}}(a_{\delta i}) = a_{\delta i} b_{\delta 1}$$

for some element $b_{\delta 1} \in \mathcal{M}_{\delta 1}$. Consequently, in accordance with (4.8), the diagram (4.16) is extendable to the commutative diagram of rational maps

$$\begin{array}{ccccc} & W_{\delta i} & & & \\ & \downarrow \sigma_{\delta i} & & \searrow \nu_{\delta i} & \\ \overline{\mathcal{M}_{\delta i}} & = & X_{\delta i} & \xrightarrow{p_{X/C}^{m!}|_{X_{\delta i}}} & X_{\delta i} \\ & & \downarrow \times a_{\delta i}^{-1} & & \downarrow \times a_{\delta i}^{-1} b_{\delta 1}^{-1} \\ \overline{\mathcal{M}_{\delta 1}} & = & X_{\delta 1} & \xrightarrow{p_{X/C}^{m!}|_{X_{\delta 1}}} & X_{\delta 1} \\ & & \downarrow \text{alb}_{\delta 1} & & \downarrow \text{alb}_{\delta 1} \\ & & \text{Alb}(X_{\delta 1}) & & \text{Alb}(X_{\delta 1}) \\ & & \parallel & & \parallel \\ & & A_\delta & \xrightarrow{z \mapsto p^{m!} z} & A_\delta, \end{array}$$

which in virtue of (4.8), (4.10) and (4.12) yields the commutative diagram of *isomorphisms* of rational Hodge structures

$$\begin{array}{ccccc}
 & H^1(W_{\delta i}, \mathbb{Q}) & & & \\
 & \uparrow [\sigma_{\delta i}]^* & \nwarrow [\nu_{\delta i}]^* & & \\
 H^1(\overline{\mathcal{M}_{\delta i}}, \mathbb{Q}) & = & H^1(X_{\delta i}, \mathbb{Q}) & \xleftarrow{[p_{X/C}^{m_1}|_{X_{\delta i}}]^*} & H^1(X_{\delta i}, \mathbb{Q}) \\
 & \uparrow [\times a_{\delta i}^{-1}]^* & & & \uparrow [\times a_{\delta i}^{-1} b_{\delta 1}^{-1}]^* \\
 (4.17) \quad H^1(\overline{\mathcal{M}_{\delta 1}}, \mathbb{Q}) & = & H^1(X_{\delta 1}, \mathbb{Q}) & \xleftarrow{[p_{X/C}^{m_1}|_{X_{\delta 1}}]^*} & H^1(X_{\delta 1}, \mathbb{Q}) \\
 & \uparrow \text{alb}_{\delta 1}^* & & & \uparrow \text{alb}_{\delta 1}^* \\
 & H^1(A_{\delta}, \mathbb{Q}) & \xleftarrow{\times p^{m_1}} & & H^1(A_{\delta}, \mathbb{Q}).
 \end{array}$$

Consider the *morphism* $\varphi_{\delta i} : X_{\delta i} \rightarrow A_{\delta}$, which is the composite of rational maps

$$X_{\delta i} \xrightarrow{\times a_{\delta i}^{-1}} X_{\delta 1} \xrightarrow{\text{alb}_{\delta 1}} A_{\delta}.$$

It is evident that the composite of rational maps

$$X_{\delta i} \xrightarrow{\times a_{\delta i}^{-1} b_{\delta 1}^{-1}} X_{\delta 1} \xrightarrow{\text{alb}_{\delta 1}} A_{\delta}$$

is a morphism and it takes the form

$$X_{\delta i} \xrightarrow{\varphi_{\delta i}} A_{\delta} \xrightarrow{z \mapsto z - f_{\delta}(b_{\delta 1})} A_{\delta},$$

where the map f_{δ} is determined by the exact sequence (4.7) of algebraic groups. On the other hand, using the commentary to the formula (4.12), it is easy to see that the translation $A_{\delta} \xrightarrow{z \mapsto z - f_{\delta}(b_{\delta 1})} A_{\delta}$ induces the *identity map* of the space $H^1(A_{\delta}, \mathbb{Q})$, therefore it follows from the commutativity of the diagram (4.17) and from (4.8) - (4.9) that there exists the commutative diagram

$$\begin{array}{ccc}
 H^1(W_{\delta i}, \mathbb{Q}) & & \\
 \simeq \uparrow [\sigma_{\delta i}]^* & \nwarrow [\nu_{\delta i}]^* & \\
 H^1(X_{\delta i}, \mathbb{Q}) & & H^1(X_{\delta i}, \mathbb{Q}) \\
 \simeq \uparrow \varphi_{\delta i}^* & & \simeq \uparrow \varphi_{\delta i}^* \\
 H^1(A_{\delta}, \mathbb{Q}) & \xleftarrow{\times p^{m_1}} & H^1(A_{\delta}, \mathbb{Q}),
 \end{array}$$

so that $[\nu_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = p^{m_1}[\sigma_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})}$. Consequently,

$$(4.18) \quad [\sigma_{\delta i}]_*[\nu_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = p^{m_1}[\sigma_{\delta i}]_*[\sigma_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = p^{m_1}|_{H^1(X_{\delta i}, \mathbb{Q})}$$

in virtue of the well-known equality $[\sigma_{\delta i}]_*[\sigma_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = \text{id}_{H^1(X_{\delta i}, \mathbb{Q})}$.

It is well known that there is a long exact sequence of cohomology with compact supports [29, Ch. III, § 1, Remark 1.30]

$$\dots \rightarrow H_c^{2d-3}(X \setminus \pi^{-1}(\Delta), \mathbb{Q}) \rightarrow H_c^{2d-3}(X, \mathbb{Q}) \rightarrow H_c^{2d-3}(\pi^{-1}(\Delta), \mathbb{Q}) \rightarrow \dots,$$

therefore the Poincaré duality [29, Ch. VI, § 11, Corollary 11.2] yields the exact sequence [24, Corollary (8.2.8)]

$$(4.19) \quad \oplus_{\delta \in \Delta, i=1, \dots, m_{\delta}} H^1(X_{\delta i}, \mathbb{Q}) = H^1(Z, \mathbb{Q}) \xrightarrow{(i_{\Delta} f)_*} H^3(X, \mathbb{Q}) \rightarrow H^3(X', \mathbb{Q}).$$

From now on we identify the divisor $X_{\delta i}$ with the zero section of a normal bundle $\mathcal{N}_{X_{\delta i}/X}$. There is a class $t \in H^2(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i}, \mathbb{Q})$, called the *Thom class* of the normal bundle $\mathcal{N}_{X_{\delta i}/X}$, characterized by the property that it restricts to the

chosen generator of the 1-dimensional space $H^2([\mathcal{N}_{X_{\delta i}/X}]_x, [\mathcal{N}_{X_{\delta i}/X}]_x \setminus x, \mathbb{Q})$ for all $x \in X_{\delta i}$. It determines *the Thom isomorphism*

$$H^k(X_{\delta i}, \mathbb{Q}) \xrightarrow{\sim} H^{k+2}(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i}, \mathbb{Q}),$$

which is given by $\alpha \mapsto \alpha \smile t$ ([44], P. 3) via the identification of cohomology $H^*(X_{\delta i}, \mathbb{Q}) \xrightarrow{\sim} H^*(\mathcal{N}_{X_{\delta i}/X}, \mathbb{Q})$ determined by the canonical map $\mathcal{N}_{X_{\delta i}/X} \rightarrow X_{\delta i}$.

It is known that, for the embedding $\iota_{X_{\delta i}/X} : X_{\delta i} \hookrightarrow X$ of a divisor $X_{\delta i}$, the *Gysin map* of cohomology $\iota_{X_{\delta i}/X*} : H^k(X_{\delta i}, \mathbb{Q}) \rightarrow H^{k+2}(X, \mathbb{Q})$ is the composite ([44], P. 11, Section (iv))

$$\begin{aligned} H^k(X_{\delta i}, \mathbb{Q}) &\xrightarrow[\sim]{t} H^{k+2}(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i}, \mathbb{Q}) \\ &= H^{k+2}(X, X \setminus X_{\delta i}, \mathbb{Q}) \rightarrow H^{k+2}(X, \mathbb{Q}). \end{aligned}$$

On the other hand, the image of the element $1 \in H^0(X_{\delta i}, \mathbb{Q})$ with respect to the Gysin map $H^0(X_{\delta i}, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$ coincides with the cohomology class $\text{cl}_X(X_{\delta i}) \in H^2(X, \mathbb{Q})$, so one may identify the Thom class of the normal bundle $\mathcal{N}_{X_{\delta i}/X}$ with the class $\text{cl}_X(X_{\delta i})$ [29, Ch. VI, § 6]. Therefore the Gysin map $\iota_{X_{\delta i}/X*}$ is defined by the formula [45, Definition 14]

$$(4.20) \quad \alpha \mapsto \alpha \smile \text{cl}_X(X_{\delta i}).$$

In accordance with (4.18) we have the equalities of operators

$$(4.21) \quad [p_{X/C}^{m!}|_{X_{\delta i}}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = [\sigma_{\delta i}]_*[\nu_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = p^{m!}|_{H^1(X_{\delta i}, \mathbb{Q})}.$$

On the other hand, returning to the isomorphism (4.1), we claim that *the morphism* $p_{X/C}^{m!} : \mathcal{M} \rightarrow \mathcal{M}$ is étale ([29], Ch. I, § 3, Theorem 3.20), because it is non-ramified ([29], Ch. I, § 3, Proposition 3.2) in virtue of *the smoothness* of its *finite* fibres, which are group schemes over the field of characteristic zero ([46], Lecture 25, Theorem 1), and the canonical morphism of rings $\mathcal{O}_{\mathcal{M}, p_{X/C}^{m!}|_{\mathcal{M}}(y)} \rightarrow \mathcal{O}_{\mathcal{M}, y}$ is injective for all $y \in \mathcal{M}$, because it is included into the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}, p_{X/C}^{m!}|_{\mathcal{M}}(y)} & \rightarrow & \mathcal{O}_{\mathcal{M}, y} \\ \bigcap & & \bigcap \\ \kappa(\mathcal{M}) & \hookrightarrow & \kappa(\mathcal{M}), \end{array}$$

where $\kappa(\mathcal{M})$ is the field of rational functions on the variety $\mathcal{M}$ and the inclusion of fields $\kappa(\mathcal{M}) \hookrightarrow \kappa(\mathcal{M})$ is determined by the dominant morphism

$$p_{X/C}^{m!} : \mathcal{M} \rightarrow \mathcal{M}.$$

As a result, we get from (4.15) the equalities of *smooth* divisors

$$(4.22) \quad [p_{X/C}^{m!}]^{-1}(\mathcal{M}_{\delta i}) = [p_{X/C}^{m!}]^*(\mathcal{M}_{\delta i}) = \mathcal{M}_{\delta i}.$$

It is evident that there is the decomposition of the groups of divisors

$$(4.23) \quad \text{Div}(\tilde{X}) = \sigma^*(\text{Div}(X)) \oplus \text{Ker}(\sigma_*),$$

where the group $\text{Ker}(\sigma_*)$ is generated by divisors, which are contractible by the morphism σ , and the map $\sigma_* \sigma^*$ is the identity [47, Ch. III, § 3, Section 3.5].

Since the codimension of the set of points of indeterminacy of the rational map $p_{X/C}^{m!}$ is greater than 1, then it follows from the definition of the strict preimage

$X_{\delta_i}^{\text{strict}}$ of the divisor $X_{\delta_i} = \overline{\mathcal{M}_{\delta_i}}$ [47, Ch. III, § 3, Section 3.3] and from (4.22), (4.23) that

$$\begin{aligned} \nu^*(X_{\delta_i}) &\in X_{\delta_i}^{\text{strict}} + \text{Ker}(\sigma_*); \\ [p_{X/C}^{m!}]^*(X_{\delta_i}) &= \sigma_*\nu^*(X_{\delta_i}) = \sigma_*(X_{\delta_i}^{\text{strict}}) = X_{\delta_i}. \end{aligned}$$

Hence one has the equality

$$[p_{X/C}^{m!}]^*|_{H^2(X, \mathbb{Q})}(\text{cl}_X(X_{\delta_i})) = \text{cl}_X(X_{\delta_i}),$$

therefore it follows from (4.20), (4.21) and from the functoriality of constructions under consideration that, for any element $\alpha \in H^1(X_{\delta_i}, \mathbb{Q})$, there are the equalities

$$\begin{aligned} [p_{X/C}^{m!}]^*(\iota_{X_{\delta_i}/X*}(\alpha)) &= [p_{X/C}^{m!}]^*|_{H^3(X, \mathbb{Q})}(\alpha \smile \text{cl}_X(X_{\delta_i})) \\ &= [p_{X/C}^{m!}|_{X_{\delta_i}}]^*|_{H^1(X_{\delta_i}, \mathbb{Q})}(\alpha) \smile [p_{X/C}^{m!}]^*|_{H^2(X, \mathbb{Q})}(\text{cl}_X(X_{\delta_i})) \\ (4.24) \quad &= p^{m!} \alpha \smile \text{cl}_X(X_{\delta_i}) = p^{m!} \iota_{X_{\delta_i}/X*}(\alpha). \end{aligned}$$

Consequently, the lemma follows from (3.23), (4.19) and (4.24).

4.3. By definition of the direct image of cohomology we have [38, Formula (1.2)]:

$$(4.25) \quad \text{pr}_{2*}(H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X, \mathbb{Q})) = 0 \quad \text{for all } i \neq 2d.$$

Let $u_{3,3}, u_{3,3^\perp}, u_{3^\perp,3}, u_{3^\perp,3^\perp}, h$ be the components of the algebraic correspondence $u = (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})] \smile^2]$ in direct summands

$$\begin{aligned} &K_{3X} \otimes_{\mathbb{Q}} K_{3X}; \dots; K_{3X}^\perp \otimes_{\mathbb{Q}} K_{3X}^\perp; \\ H &\stackrel{\text{def}}{=} \oplus_{p+q=6, p \neq 3} H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X, \mathbb{Q}) \end{aligned}$$

of the Künneth decomposition of the space $H^6(X \times X, \mathbb{Q})$.

4.4. Acting on the algebraic class u by the operator

$$[p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^* = \sigma_*\nu^* \otimes_{\mathbb{Q}} \sigma_*\nu^*,$$

we obtain in virtue of Lemma 4.2 the algebraic class

$$([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(u) = p^{4 \cdot m!} \cdot u_{3,3} + p^{3 \cdot m!} \cdot u_{3,3^\perp} + p^{3 \cdot m!} \cdot u_{3^\perp,3} + p^{2 \cdot m!} \cdot u_{3^\perp,3^\perp} + h_1,$$

where $h_1 = ([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(h) \in H$ (because the operator

$$\sigma_*\nu^* \otimes_{\mathbb{Q}} \sigma_*\nu^* = (\sigma \times \sigma)_*(\nu \times \nu)^*$$

transforms the subspace $H \subset H^6(X \times X, \mathbb{Q})$ into H and it transforms algebraic classes into algebraic cohomology classes [2, Proposition 1.3.7].

Subtracting from this class the element $p^{3 \cdot m!} \cdot u$, we obtain an algebraic class $(p^{4 \cdot m!} - p^{3 \cdot m!}) \cdot u_{3,3} + (p^{2 \cdot m!} - p^{3 \cdot m!}) \cdot u_{3^\perp,3^\perp} + (h_1 - p^{3 \cdot m!} \cdot h)$. Thus, for some element $h_2 \in H$, the class $p^{m!} \cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2$ is algebraic. It is evident that, for some element $h_3 \in H$, the class

$$([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(p^{m!} \cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2) = p^{5 \cdot m!} \cdot u_{3,3} - p^{2 \cdot m!} \cdot u_{3^\perp,3^\perp} + h_3$$

is algebraic. Subtracting from it the algebraic class

$$p^{4 \cdot m!} (p^{m!} \cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2) = p^{5 \cdot m!} \cdot u_{3,3} - p^{4 \cdot m!} \cdot u_{3^\perp,3^\perp} + p^{4 \cdot m!} \cdot h_2,$$

we obtain an algebraic class $(p^{4 \cdot m!} - p^{2 \cdot m!}) \cdot u_{3^\perp,3^\perp} + (h_3 - p^{4 \cdot m!} \cdot h_2)$. Therefore, for some element $h_4 \in H$, the class $u_{3^\perp,3^\perp} + h_4$ is algebraic. Consequently, for some element $h_5 \in H$, the class $u_{3,3} + h_5$ is algebraic. As a result, for some element

$h_6 \in H$, the class $u_{3,3^\perp} + u_{3^\perp,3} + h_6$ is algebraic. Acting on this class by the operator $[p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [1_{X/C}^*]$, we obtain an algebraic class of the form

$$p^{2 \cdot m!} \cdot u_{3,3^\perp} + p^{m!} \cdot u_{3^\perp,3} + h_7 \quad (h_7 \in H),$$

therefore, for some elements $h_8, h_9 \in H$, the classes $u_{3,3^\perp} + h_8$ and $u_{3^\perp,3} + h_9$ are algebraic. Thus, for some element $h_{10} \in H$, the class $u_{3,3} + u_{3,3^\perp} + h_{10}$ is *algebraic*.

Note that the correspondence $u_{3^\perp,3} + u_{3^\perp,3^\perp}$ annihilates the space $K_{(2d-3)X}$ in virtue of (3.16); besides, elements $h, h_{10} \in H$ annihilate this space according to the formula (4.25). Therefore in virtue of (3.9) the algebraic correspondence $u_{3,3} + u_{3,3^\perp} + h_{10}$ yields an isomorphism

$$(4.26) \quad K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \smile (u_{3,3} + u_{3,3^\perp} + h_{10}))} K_{3X},$$

and by (3.14) and (4.25) it is obvious that the correspondence $u_{3,3} + u_{3,3^\perp} + h_{10}$ annihilates the subspace

$$\text{cl}_X(H)^{\smile d-3} \smile K_{3X}^\perp = K_{(2d-3)X}^\perp \subset H^{2d-3}(X, \mathbb{Q}).$$

On the other hand, by (3.16), (3.18), (3.19) and Lemma 3.8 the *algebraic* correspondence $\wp(K_{3X}^\perp)$ annihilates the subspace $K_{(2d-3)X} \subset H^{2d-3}(X, \mathbb{Q})$ and it determines an isomorphism

$$K_{(2d-3)X}^\perp \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \smile \wp(K_{3X}^\perp))} K_{3X}^\perp,$$

therefore by (3.15), (3.17), (3.18) and (4.26) an algebraic correspondence

$$u_{3,3} + u_{3,3^\perp} + h_{10} + \wp(K_{3X}^\perp)$$

yields an isomorphism

$$H^{2d-3}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \smile (u_{3,3} + u_{3,3^\perp} + h_{10} + \wp(K_{3X}^\perp)))} H^3(X, \mathbb{Q}).$$

The theorem is proved.

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