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ISOMETRIES OF SPACES OF LOG-INTEGRABLE FUNCTIONS

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ABSTRACT. We consider the F -space $(L_{\log}(\Omega, \mu), \|\cdot\|_{\log})$ of log-integrable functions defined on measure space (Ω, μ) with finite measure. We prove that $(L_{\log}(\Omega_1, \mu_1), \|\cdot\|_{\log})$ and $(L_{\log}(\Omega_2, \mu_2), \|\cdot\|_{\log})$ are isometric if and only if there exists a measure preserving isomorphism from (Ω_1, μ_1) onto (Ω_2, μ_2) .

Keywords: F -spaces, isometries, Boolean algebras, measure preserving isomorphisms, log-integrable functions.

1. INTRODUCTION

The study of linear isometries of Banach function spaces was begun by S. Banach [2, Ch. XI], who gave a description of all linear isometries on the space $L_p[0, 1]$ with $p \neq 2$. J. Lamperti obtained a description of all isometries from $L_p(\Omega_1, \mu_1)$ into $L_p(\Omega_2, \mu_2)$, $1 \leq p < \infty$, $p \neq 2$, where (Ω_i, μ_i) is an arbitrary measure space with the finite measure, $i = 1, 2$ [9] (see also [7, Ch. 3, §3.2, Theorem 3.2.5]). One of the corollaries of such a description is the following

Theorem 1. *Let (Ω_i, μ_i) be a measure space with a finite measure, let ∇_i be a complete Boolean algebra of all classes of equal μ_i -almost everywhere sets, $i = 1, 2$, $1 \leq p < \infty$, $p \neq 2$. Then a Banach spaces $L_p(\Omega_1, \mu_1)$ and $L_p(\Omega_2, \mu_2)$ are isometric if and only if there exists a Boolean isomorphism $\varphi: \nabla_1 \rightarrow \nabla_2$.*

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An important metrizable analogue of Banach spaces $L_p(\Omega, \mu)$ is the F -space $L_{\log}(\Omega, \mu)$ of all log-integrable functions introduced in [4]. The F -space $L_{\log}(\Omega, \mu)$ is defined by the equality

$$L_{\log}(\Omega, \mu) = \{f \in L_0(\Omega, \mu) : \|f\|_{\log} = \int_{\Omega} \log(1 + |f|) d\mu < +\infty\},$$

where $L_0(\Omega, \mu)$ is the algebra of equivalence classes of almost everywhere (a.e.) finite real (complex) valued measurable functions on (Ω, μ) . It is known that $L_{\log}(\Omega, \mu)$ is a subalgebra of the algebra $L_0(\Omega, \mu)$. In addition, the F -norm $\|\cdot\|_{\log}$ defines a metric $\rho_{\log}(f, g) = \|f - g\|_{\log}$ such that $(L_{\log}(\Omega, \mu), \rho_{\log})$ is a complete metric topological vector space [4].

A natural problem is to find necessary and sufficient conditions ensuring that the F -spaces $L_{\log}(\Omega_1, \mu_1)$ and $L_{\log}(\Omega_2, \mu_2)$ are isometric. In Section 3 we give the following criterion for the existence of a surjective isometry for F -spaces of log-integrable functions.

Theorem 2. *F -spaces $L_{\log}(\Omega_1, \mu_1)$ and $L_{\log}(\Omega_2, \mu_2)$ are isometric if and only if there exists a Boolean isomorphism $\varphi : (\nabla_1, \mu_1) \rightarrow (\nabla_2, \mu_2)$.*

2. PRELIMINARIES

Let (Ω, μ) be a measure space with finite measure μ , and let $L_0(\Omega, \mu)$ (respectively, $L_{\infty}(\Omega, \mu)$) be the algebra of equivalence classes of real (complex) valued measurable (respectively, essentially bounded measurable) functions on (Ω, μ) . Let ∇ be the complete Boolean algebra of all classes $[A]$ of equal μ -a.e. sets. It is known that $\hat{\mu}([A]) = \mu(A)$ is a strictly positive finite measure on ∇ . Below, we denote the measure $\hat{\mu}$ by μ , the algebra $L_0(\Omega, \mu)$ (respectively, $L_{\infty}(\Omega, \mu)$) by $L_0(\nabla)$ (respectively, $L_{\infty}(\nabla)$) and the integral $\int_{\Omega} f d\mu$ by $\int_{\nabla} f d\mu$.

Following [4], we consider in $L_0(\nabla)$ the subalgebra

$$L_{\log}(\nabla, \mu) = \{f \in L_0(\nabla) : \int_{\nabla} \log(1 + |f|) d\mu < +\infty\}$$

of log-integrable measurable functions, and for each $f \in L_{\log}(\nabla, \mu)$, we set

$$\|f\|_{\log} = \int_{\nabla} \log(1 + |f|) d\mu.$$

A non-negative function $\|\cdot\|_{\log} : L_{\log}(\nabla, \mu) \rightarrow [0, \infty)$ is a F -norm on the linear space $L_{\log}(\nabla, \mu)$, that is ,

- (i). $\|f\|_{\log} > 0$ for all $0 \neq f \in L_{\log}(\nabla, \mu)$;
 - (ii). $\|\alpha f\|_{\log} \leq \|f\|_{\log}$ for all $f \in L_{\log}(\nabla, \mu)$ and number α with $|\alpha| \leq 1$;
 - (iii). $\lim_{\alpha \rightarrow 0} \|\alpha f\|_{\log} = 0$ for all $f \in L_{\log}(\nabla, \mu)$;
 - (iv). $\|f + g\|_{\log} \leq \|f\|_{\log} + \|g\|_{\log}$ for all $f, g \in L_{\log}(\nabla, \mu)$,
- (see, for example, [8, Ch. 1, § 2])

It is known that the set $L_{\log}(\nabla, \mu)$ is a complete metric topological vector space with respect to the metric $\rho(f, g) = \|f - g\|_{\log}$ [4]. In addition, the inclusion $L_p(\nabla, \mu) \subset L_{\log}(\nabla, \mu)$ is true for all $0 < p < \infty$.

If $f_n \in L_{\log}(\nabla, \mu)$, $|f_n| \leq g \in L_{\log}(\nabla, \mu)$, $n = 1, 2, \dots$, and $f_n \rightarrow f \in L_0(\nabla)$ μ -a.e. then $\log(1 + |f|) \leq \log(1 + g) \in L_1(\nabla, \mu)$, $\log(1 + |f_n - f|) \rightarrow 0$, μ -a.e. and $\log(1 + |f_n - f|) \leq \log(1 + 2g) \in L_1(\nabla, \mu)$. Consequently, $f \in L_{\log}(\nabla, \mu)$, and by Lebesgue's Theorem (see, for example, [5, Ch. V, Sec. 26, Theorem D]), $\|f_n - f\|_{\log} \rightarrow 0$ as $n \rightarrow \infty$.

Let (∇_1, μ_1) , (∇_2, μ_2) be complete Boolean algebras with a strictly positive finite measures, and let $\varphi : \nabla_1 \rightarrow \nabla_2$ be a Boolean isomorphism. By [3, Theorem 2.3] there exists a unique isomorphism Φ from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$ such that $\Phi(e) = \varphi(e)$ for all $e \in \nabla_1$. It is clear that the function $\lambda(\varphi(e))$, $e \in \nabla_1$, is a strictly positive finite measure on Boolean algebra ∇_2 , and by [1, Proposition 3] we get $\Phi(L_{\log}(\nabla_1, \mu_1)) = L_{\log}(\nabla_2, \lambda)$.

Denote by $\frac{d\lambda}{d\mu_2}$ the Radon-Nikodym derivative of measure λ with respect to the measure μ_2 . It is well known that $0 \leq \frac{d\lambda}{d\mu_2} \in L_0(\nabla_2)$, and $f \in L_1(\nabla_2, \lambda)$ if and only if $(f \cdot \frac{d\lambda}{d\mu_2}) \in L_1(\nabla, \mu_2)$, in addition, $\int_{\nabla_2} f d\lambda = \int_{\nabla_2} f \cdot (\frac{d\lambda}{d\mu_2}) d\mu_2$.

Using [1, Proposition 3] we get the following

Proposition 1. $\Phi(L_{\log}(\nabla_1, \mu_1)) = L_{\log}(\nabla_2, \mu_2) \iff \frac{d\lambda}{d\mu_2}, \frac{d\mu_2}{d\lambda} \in L_{\infty}(\nabla_2)$.

If an isomorphism $\varphi : \nabla_1 \rightarrow \nabla_2$ preserves the measure, that is, $\mu_2(\varphi(e)) = \mu_1(e)$ for all $e \in \nabla_1$, then using the equality $\Phi(\log(1 + |f|)) = \log(1 + \Phi(|f|))$, $f \in L_0(\nabla_1)$, [1, Proposition 3] we get

$$\begin{aligned} \|\Phi(f)\|_{\log} &= \int_{\nabla_2} \log(1 + |\Phi(f)|) d\mu_2 = \int_{\nabla_2} \Phi(\log(1 + |f|)) d\mu_2 \\ &= \int_{\nabla_1} \log(1 + |f|) d\mu_1 = \|f\|_{\log} \quad \text{for all } f \in L_{\log}(\nabla_1, \mu_1). \end{aligned}$$

Therefore, using Proposition 1, we get the following

Theorem 3. *If $\varphi : \nabla_1 \rightarrow \nabla_2$ is a measure-preserving isomorphism then the restriction $\Phi|_{L_{\log}(\nabla_1, \mu_1)}$ is a surjective positive linear isometry from $L_{\log}(\nabla_1, \mu_1)$ onto $L_{\log}(\nabla_2, \mu_2)$.*

3. ISOMETRIES OF F -SPACES OF $\log$ -INTEGRABLE FUNCTIONS

Below we give a description of all linear isometries from $L_{\log}(\nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$.

We need the following disjointness property of linear isometries from $L_{\log}(\nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$.

Proposition 2. *If $U : L_{\log}(\nabla_1, \mu_1) \rightarrow L_{\log}(\nabla_2, \mu_2)$ is a linear isometry, then $U(f) \cdot U(g) = 0$ for any $f, g \in L_{\log}(\nabla_1, \mu_1)$ with $f \cdot g = 0$.*

Доказательство. Let $\mathbf{1}$ be the unity in the Boolean algebra ∇_1 and let $e = s(f) = \mathbf{1} - \sup\{p \in \nabla_1 : p \cdot f = 0\}$ be the support of function $f \in L_{\log}(\nabla_1, \mu_1)$. Since $f \cdot g = 0$ it follows that $e \cdot q = 0$, where $q = s(g)$. Consequently,

$$\int_{\nabla_2} \log(1 + |U(f) + U(g)|) d\mu_2 = \|U(f) + U(g)\|_{\log} = \|U(f + g)\|_{\log} = \|f + g\|_{\log} =$$

$$\begin{aligned}
&= \int_{\nabla_1} \log(1 + |f + g|) \, d\mu_1 = \int_{\nabla_1} \log(1 + |f| + |g|) \, d\mu_1 = \\
&= \int_{e \cdot \nabla_1} \log(1 + |f|) \, d\mu_1 + \int_{q \cdot \nabla_1} \log(1 + |g|) \, d\mu_1 = \|f\|_{\log} + \|g\|_{\log} = \\
&= \|U(f)\|_{\log} + \|U(g)\|_{\log} = \int_{\nabla_2} \log(1 + |U(f)|) \, d\mu_2 + \int_{\nabla_2} \log(1 + |U(g)|) \, d\mu_2 = \\
&= \int_{\nabla_2} \log(1 + |U(f)| + |U(g)| + |U(f)| \cdot |U(g)|) \, d\mu_2,
\end{aligned}$$

which is impossible in the case $\mu_2(\{|U(f)| \cdot |U(g)| \neq 0\}) > 0$.

Therefore, $|U(f)| \cdot |U(g)| = 0$. In particular, $U(f) \cdot U(g) = 0$. $\square$

Using Proposition 2 we obtain the following description of linear isometries from $L_{\log}(\nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$.

Theorem 4. *Let $U : L_{\log}(\nabla_1, \mu_1) \rightarrow L_{\log}(\nabla_2, \mu_2)$ be a linear isometry. Then there exists an injective σ -additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1}) \cdot \Phi(f)$ for each $f \in L_{\log}(\nabla_1, \mu_1)$. In addition,*

$$|U(\mathbf{1})| = -1 + 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}, \quad (1)$$

where ν is a measure on ∇_1 defined by the equality $\nu(e) = \mu_2(\Phi(e))$, $e \in \nabla_1$.

Доказательство. We set $\varphi(e) = s(U(e))$, $e \in \nabla_1$, $p_1 = \varphi(\mathbf{1}_{\nabla_1})$. Using the proof of Theorem 3.2.5 [7] we obtain that the map φ is an injective σ -additive Boolean homomorphism from ∇_1 into $p_1 \cdot \nabla_2$. Consequently, there exists a unique injective σ -additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $\Phi(e) = \varphi(e)$ for all $e \in \nabla_1$ (see proof of Theorem 2.3 in [3]). In addition, the restriction $J = \Phi|_{L_{\infty}(\nabla_1)}$ is a $\|\cdot\|_{\infty}$ -continuous injective homomorphism from $L_{\infty}(\nabla_1)$ into $L_{\infty}(\nabla_2)$. Using the equalities

$$\begin{aligned}
U(e) &= U(\mathbf{1} - (\mathbf{1} - e)) \cdot s(U(e)) = \\
&= U(\mathbf{1}) \cdot \varphi(e) - U(\mathbf{1} - e) \cdot \varphi(\mathbf{1} - e) \cdot \varphi(e) = U(\mathbf{1}) \cdot J(e).
\end{aligned}$$

and $\|\cdot\|_{\infty}$ -continuous of homomorphism J we obtain that

$$U(f) = U(\mathbf{1}) \cdot J(f) = U(\mathbf{1}) \cdot \Phi(f) \text{ for all } f \in L_{\infty}(\nabla_1). \quad (2)$$

We show now that the equality (2) is true for all $f \in L_{\log}(\nabla_1, \mu_1)$. It suffices to verify equality (2) for any $0 \leq f \in L_{\log}(\nabla_1, \mu_1)$.

Choose a sequence $0 \leq f_n \in L_{\infty}(\nabla_1)$ such that $f_n \uparrow f$. Then

$$\begin{aligned}
&\|U(\mathbf{1}) \cdot \Phi(f_n) - U(f)\|_{\log} = \|U(f_n) - U(f)\|_{\log} = \\
&\|U(f_n - f)\|_{\log} = \|f_n - f\|_{\log} \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \quad (3)$$

Since $\Phi(f_n) \uparrow \Phi(f)$, $\Phi(f_n) \in L_{\infty}(\nabla_2)$, it follows that $U(\mathbf{1}) \cdot \Phi(f_n) \in L_{\log}(\nabla_2, \mu_2)$ for all $n = 1, 2, \dots$, and $U(\mathbf{1}) \cdot \Phi(f_n) \rightarrow U(\mathbf{1}) \cdot \Phi(f)$ μ_2 -a.e. In addition, the sequence $\{U(\mathbf{1}) \cdot \Phi(f_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete metric space $(L_{\log}(\nabla_2, \mu_2), \|\cdot\|_{\log})$. Consequently, $\|U(\mathbf{1}) \cdot \Phi(f_n) - g\|_{\log} \rightarrow 0$ for some $g \in L_{\log}(\nabla_2, \mu_2)$. By (3) we have that

$$U(f) = g = (\mu_2 - a.e.) \lim_{n \rightarrow \infty} U(\mathbf{1}) \cdot \Phi(f_n) = U(\mathbf{1}) \cdot \Phi(f).$$

We show now that $|U(\mathbf{1})| = -1 + 2^{\Phi(\frac{d\mu_1}{d\nu})}$. It is clear that $\nu(e) = \mu_2(\Phi(e))$, $e \in \nabla_1$, is a strictly positive finite measure on the Boolean algebra ∇_1 . Since

$$\int_{\nabla_2} \Phi(f) d\mu_2 = \int_{\nabla_1} f d\nu \quad \text{for all } f \in L_1(\nabla_1, \nu),$$

it follows for each $e \in \nabla_1$ that

$$\begin{aligned} \int_{e \cdot \nabla_1} \log 2 \, d\mu_1 &= \|e\|_{\log} = \|U(e)\|_{\log} = \int_{\nabla_2} \log(1 + |U(\mathbf{1})| \cdot \Phi(e)) \, d\mu_2 = \\ &= \int_{\nabla_2} \Phi(\log(1 + \Phi^{-1}(|U(\mathbf{1})|)) \cdot e) \, d\mu_2 = \int_{\nabla_1} \log(1 + \Phi^{-1}(|U(\mathbf{1})|) \cdot e) \, d\nu = \\ &= \int_{e \cdot \nabla_1} \log(1 + \Phi^{-1}(|U(\mathbf{1})|)) \cdot \frac{d\nu}{d\mu_1} \, d\mu_1. \end{aligned}$$

Consequently,

$$\int_{\nabla_1} (\log 2 - \log((1 + \Phi^{-1}(|U(\mathbf{1})|))^{\frac{d\nu}{d\mu_1}}) \cdot e) \, d\mu_1$$

for all $e \in \nabla_1$. Thus

$$\log 2 - \log((1 + \Phi^{-1}(|U(\mathbf{1})|))^{\frac{d\nu}{d\mu_1}}) = 0,$$

that is,

$$2 = (1 + \Phi^{-1}(|U(\mathbf{1})|))^{\frac{d\nu}{d\mu_1}}.$$

This means that

$$\Phi^{-1}(|U(\mathbf{1})|) = -1 + 2^{\left(\frac{d\nu}{d\mu_1}\right)^{-1}} = -1 + 2^{\frac{d\mu_1}{d\nu}},$$

and

$$|U(\mathbf{1})| = \Phi(-\mathbf{1}) + \Phi\left(2^{\left(\frac{d\mu_1}{d\nu}\right)}\right) = -\mathbf{1} + \Phi\left(2^{\left(\frac{d\mu_1}{d\nu}\right)}\right).$$

It remains to show that $\Phi\left(2^{\left(\frac{d\mu_1}{d\nu}\right)}\right) = 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}$.

Since the restriction $J = \Phi|_{L_\infty(\nabla_1)}$ is a $\|\cdot\|_\infty$ -continuous injective homomorphism from C^* -algebra $L_\infty(\nabla_1)$ into $L_\infty(\nabla_2)$, it follows that for any $g \in L_\infty(\nabla_1)$ and every continuous real function $u : [0, +\infty) \rightarrow \mathbb{R}$ the equality

$$J(u \circ |g|) = u \circ J(|g|)$$

holds. Consequently,

$$\Phi(2^{|g|}) = J(2^{|g|}) = 2^{J(|g|)} = 2^{\Phi(|g|)}$$

for all $g \in L_\infty(\nabla_1)$. If $0 \leq f \in L_0(\nabla_1)$, then setting $g_n = f \cdot \{0 \leq f \leq n\}$, we obtain that $0 \leq g_n \in L_\infty(\nabla_1)$, $n = 1, 2, \dots$, $g_n \uparrow f$, $\Phi(g_n) \uparrow \Phi(f)$, $2^{g_n} \uparrow 2^f$ and $\Phi(2^{g_n}) \uparrow \Phi(2^f)$.

Since $2^{\Phi(g_n)} = \Phi(2^{g_n})$, it follows that $\Phi(2^f) = 2^{\Phi(f)}$. Consequently,

$$\Phi\left(2^{\left(\frac{d\mu_1}{d\nu}\right)}\right) = 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}, \quad \text{and} \quad |U(\mathbf{1})| = -\mathbf{1} + 2^{\Phi\left(\frac{d\mu_1}{d\nu}\right)}.$$

□

Remark that in [6] the version of Theorem 4 without (1) is obtained for any F -spaces and for positive isometries with disjointness property.

The following Corollary refines Theorem 4 for surjective linear isometries.

Corollary 1. *Let $U : L_{\log}(\nabla_1, \mu_1) \rightarrow L_{\log}(\nabla_2, \mu_2)$ be a surjective linear isometry. Then there exists an isomorphism Φ from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1}_{\nabla_1}) \cdot \Phi(f)$ for each $f \in L_{\log}(\nabla_1, \mu_1)$.*

Доказательство. Since $U^{-1} : L_{\log}(\nabla_2, \mu_2) \rightarrow L_{\log}(\nabla_1, \mu_1)$ is a linear isometry it follows from Theorem 4 that there exists an injective σ -additive homomorphism Ψ from algebra $L_0(\nabla_2)$ into algebra $L_0(\nabla_1)$ such that $U^{-1}(g) = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(h)$ for each $h \in L_{\log}(\nabla_2, \mu_2)$. Therefore

$$f = U^{-1}(U(f)) = U^{-1}(U(\mathbf{1}_{\nabla_1}) \cdot \Phi(f)) = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(U(\mathbf{1}_{\nabla_1})) \cdot \Psi(\Phi(f))$$

for all $f \in L_{\log}(\nabla_1, \mu_1)$. In particular, $\mathbf{1}_{\nabla_1} = U^{-1}(\mathbf{1}_{\nabla_2}) \cdot \Psi(U(\mathbf{1}_{\nabla_1}))$. Thus $f = \Psi(\Phi(f))$ for each $f \in L_{\log}(\nabla_1, \mu_1)$. This means that $\Psi = \Phi^{-1}$ and Φ is an isomorphism from algebra $L_0(\nabla_1)$ onto algebra $L_0(\nabla_2)$. $\square$

We also need the following useful corollary from Theorem 4

Corollary 2. *Let $U(f) = U(\mathbf{1}) \cdot \Phi(f)$ be a linear isometry from $L_{\log}(\nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$ (see Theorem 4), let $0 \neq e \in \nabla_1$, $q = \Phi(e) \in \nabla_2$. Then $U_e(f) = U(f)$, $f \in L_{\log}(e \cdot \nabla_1, \mu_1)$, is a linear isometry from $L_{\log}(e \cdot \nabla_1, \mu_1)$ into $L_{\log}(q \cdot \nabla_2, \mu_2)$.*

Доказательство. It is clear that

$$U_e(f) = \Phi(e) \cdot U(\mathbf{1}) \cdot \Phi(f) = q \cdot U(\mathbf{1}) \cdot \Phi(f), \quad f \in L_{\log}(e \cdot \nabla_1, \mu_1),$$

is a linear map from $L_{\log}(e \cdot \nabla_1, \mu_1)$ into $L_{\log}(\nabla_2, \mu_2)$.

If $f = e \cdot f \in L_{\log}(e \cdot \nabla_1, \mu_1) \subset L_{\log}(\nabla_1, \mu_1)$ then

$$\begin{aligned} U(f) \cdot (\Phi(\mathbf{1}) - q) &= U(\mathbf{1}) \cdot \Phi(e \cdot f) \cdot (\Phi(\mathbf{1}) - \Phi(e)) = \\ &= U(\mathbf{1}) \cdot \Phi(f) \cdot (\Phi(e \cdot (\mathbf{1} - e))) = 0. \end{aligned}$$

Consequently, $U_e(f) \in L_{\log}(q \cdot \nabla_2, \mu_2)$ for all $f \in L_{\log}(e \cdot \nabla_1, \mu_1)$. In addition,

$$\begin{aligned} \|U_e(f)\|_{\log} &= \|U(f) \cdot \Phi(e)\|_{\log} = \|U(\mathbf{1}) \cdot \Phi(f) \cdot q\|_{\log} = \\ &= \int_{\nabla_2} \log(1 + q \cdot |U(f)|) \, d\mu_2 = \int_{\nabla_2} \log(1 + |U(f)|) \, d\mu_2 = \\ &= \|U(f)\|_{\log} = \|f\|_{\log} \quad \text{for all } f \in L_{\log}(e \cdot \nabla_1, \mu_1). \end{aligned}$$

$\square$

Let $\nabla = \nabla_1 = \nabla_2$, $\mu = \mu_1$, $\nu = \mu_2$, $h = \frac{d\nu}{d\mu}$. The following Theorem gives a sufficient condition for to be non-isometric the F -spaces $L_{\log}(\nabla, \mu)$ and $L_{\log}(\nabla, \nu)$.

Theorem 5. *Let ∇ be a complete Boolean algebra, let μ and ν be a strictly positive finite measures on ∇ , and let $\mu(\mathbf{1}) \neq \nu(\mathbf{1})$. Then F -spaces $L_{\log}(\nabla, \mu)$ and $L_{\log}(\nabla, \nu)$ are not isometric.*

Доказательство. Setting $h = \frac{d\nu}{d\mu}$, $t = \frac{\nu(\mathbf{1})}{\mu(\mathbf{1})}$, we have that $\int_{\nabla} h d\mu = \nu(\mathbf{1}) = t \cdot \mu(\mathbf{1})$, and

$$\int_{\nabla} \log(1 + \lambda)^h d\mu = \log(1 + \lambda) \int_{\nabla} h d\mu = \int_{\nabla} \log(1 + \lambda)^t d\mu. \quad (4)$$

for all $\lambda > 0$.

Suppose that $t > 1$, and let U be a surjective linear isometry U from $L_{\log}(\nabla, \nu)$ onto $L_{\log}(\nabla, \mu)$. Since $\lim_{\lambda \rightarrow +\infty} \frac{(1+\lambda)^t}{1+\lambda} = +\infty$, it follows that there exists $\lambda > 1$ such that

$$\int_{\nabla} \log \frac{(1 + \lambda)^t}{\lambda} d\mu > \int_{\nabla} \log \frac{(1 + \lambda)^t}{1 + \lambda} d\mu > \|U(\mathbf{1})\|_{\log}.$$

Then

$$\int_{\nabla} \log(1 + \lambda)^t d\mu - \int_{\nabla} \log \lambda d\mu > \|U(\mathbf{1})\|_{\log}$$

and

$$\begin{aligned} \int_{\nabla} \log(1 + \lambda)^t d\mu &> \int_{\nabla} \log \lambda d\mu + \|U(\mathbf{1})\|_{\log} = \\ &= \int_{\nabla} \log \lambda d\mu + \int_{\nabla} \log(1 + |U(\mathbf{1})|) d\mu > \\ &> \int_{\nabla} \log \lambda d\mu + \int_{\nabla} \log\left(\frac{1}{\lambda} + |U(\mathbf{1})|\right) d\mu. \end{aligned} \quad (5)$$

Using (4) and (5), we get

$$\begin{aligned} \|\lambda \cdot \mathbf{1}\|_{\log} &= \int_{\nabla} \log(1 + \lambda \cdot \mathbf{1}) d\nu = \int_{\nabla} h \cdot \log(1 + \lambda \cdot \mathbf{1}) d\mu = \\ &= \int_{\nabla} \log(1 + \lambda \cdot \mathbf{1})^h d\mu = \int_{\nabla} \log(1 + \lambda)^t d\mu > \int_{\nabla} \log \lambda d\mu + \int_{\nabla} \log\left(\frac{1}{\lambda} + |U(\mathbf{1})|\right) d\mu = \\ &= \int_{\nabla} \log(1 + |U(\lambda \cdot \mathbf{1})|) d\mu = \|U(\lambda \cdot \mathbf{1})\|_{\log}. \end{aligned}$$

Consequently, $\|\lambda \cdot \mathbf{1}\|_{\log} \neq \|U(\lambda \cdot \mathbf{1})\|_{\log}$, that is, the map U is not an isometry.

If $0 < t < 1$ then changing the measures μ and ν , we obtain that $\frac{\mu(\mathbf{1})}{\nu(\mathbf{1})} > 1$, and by above proof there is no a surjective linear isometry V from $L_{\log}(\nabla, \mu)$ onto $L_{\log}(\nabla, \nu)$. $\square$

Now we can refine Theorem 4.

Theorem 6. *Let $U : L_{\log}(\nabla_1, \mu_1) \rightarrow L_{\log}(\nabla_2, \mu_2)$ be a linear isometry. Then there exists an injective σ -additive homomorphism Φ from algebra $L_0(\nabla_1)$ into algebra $L_0(\nabla_2)$ such that $U(f) = U(\mathbf{1}) \cdot \Phi(f)$ for each $f \in L_{\log}(\nabla_1, \mu_1)$, in addition, $\mu_2(\Phi(e)) = \mu_1(e)$ for all $e \in \nabla_1$.*

Доказательство. Assume by contradiction that there exists $0 \neq e \in \nabla_1$ such that $\mu_2(\Phi(e)) \neq \mu_1(e)$, that is,

$$\mu_1(\mathbf{1}_{e \cdot \nabla_1}) \neq \mu_2(\mathbf{1}_{\Phi(e) \cdot \nabla_2}).$$

By Corollary 2, the map U_e is a linear surjective isometry from $L_{\log}(e \cdot \nabla_1, \mu_1)$ onto $L_{\log}(\Phi(e) \cdot \nabla_2, \mu_2)$, which is impossible by Theorem 5. $\square$

Using Theorems 3, 6 and Corollary 1, we have the following criterion for the existence of isometries between of the F -spaces $L_{\log}(\nabla_1, \mu_1)$ and $L_{\log}(\nabla_2, \mu_2)$.

Theorem 7. *The F -spaces $L_{\log}(\nabla_1, \mu_1)$ and $L_{\log}(\nabla_2, \mu_2)$ are isometric if and only if there exists a measure-preserving Boolean isomorphism $\varphi : (\nabla_1, \mu_1) \rightarrow (\nabla_2, \mu_2)$.*

Now, we provide a criterion for two F -spaces $L_{\log}(\nabla_1, \mu_1)$ and $L_{\log}(\nabla_2, \mu_2)$ to be isometric using the passports of the Boolean algebras ∇_1 and ∇_2 .

Let ∇ be a non-atomic complete Boolean algebra and let μ be a strictly positive finite measure on ∇ . By $\tau(e \cdot \nabla)$ denote the minimal cardinality of a set that is dense in the Boolean algebra $e \cdot \nabla$ with respect to the order topology ((o) -topology). A non-atomic complete Boolean algebra ∇ is said to be homogeneous if $\tau(e \cdot \nabla) = \tau(g \cdot \nabla)$ for any nonzero $e, g \in \nabla$. The cardinality $\tau(\nabla)$ is called the weight of a homogeneous Boolean algebra ∇ (see, for example [10, Ch. VII]).

Since $\mu(\mathbf{1}) < \infty$ it follows that ∇ is a direct product of homogeneous Boolean algebras $e_n \cdot \nabla$, where $e_n \cdot e_m = 0$, $n \neq m$, $\tau_n = \tau(e_n \cdot \nabla) < \tau_{n+1}$, $n, m = 1, 2, \dots$ [10, Ch. VII, § 2]).

Set $\alpha_n = \mu(e_n)$. The matrix $\begin{pmatrix} \tau_1 & \tau_2 & \cdots \\ \alpha_1 & \alpha_2 & \cdots \end{pmatrix}$ is called the passport of Boolean algebra (∇, μ) . The following theorem gives a classification of Boolean algebras with finite measures [10, Ch. VII, § 2].

Theorem 8. *Let μ_i be a strictly positive finite measure on the non-atomic complete Boolean algebra ∇_i , and let $\begin{pmatrix} \tau_1^{(i)} & \tau_2^{(i)} & \cdots \\ \alpha_1^{(i)} & \alpha_2^{(i)} & \cdots \end{pmatrix}$ be the passport of (∇_i, μ_i) , $i = 1, 2$. Then the following conditions are equivalent:*

- (i). *There exists a measure-preserving isomorphism $\varphi : (\nabla_1, \mu_1) \rightarrow (\nabla_2, \mu_2)$;*
- (ii). *$\tau_n^{(1)} = \tau_n^{(2)}$ and $\alpha_n^{(1)} = \alpha_n^{(2)}$ for all $n = 1, 2, \dots$*

Using Theorems 7 and 8 we obtain the following criterion for existence of isometries between of the F -spaces $L_{\log}(\nabla_1, \mu_1)$ and $L_{\log}(\nabla_2, \mu_2)$.

Corollary 3. *Let (∇_i, μ_i) be the same as in Theorem 8. Then the F -spaces $L_{\log}(\nabla_1, \mu_1)$ and $L_{\log}(\nabla_2, \mu_2)$ are isometric if and only if $\tau_n^{(1)} = \tau_n^{(2)}$ and $\alpha_n^{(1)} = \alpha_n^{(2)}$ for all $n = 1, 2, \dots$, where $\begin{pmatrix} \tau_1^{(i)} & \tau_2^{(i)} & \cdots \\ \alpha_1^{(i)} & \alpha_2^{(i)} & \cdots \end{pmatrix}$ is the passport of Boolean algebra (∇_i, μ_i) .*

Corollary 4. *(cf. Theorem 5). Let ∇ be a homogeneous Boolean algebra, let μ, ν be a strictly positive finite measures on ∇ . Then the F -spaces $L_{\log}(\nabla, \mu)$ and $L_{\log}(\nabla, \nu)$ are isometric if and only if $\mu(\mathbf{1}) = \nu(\mathbf{1})$.*

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