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ALL TIGHT DESCRIPTIONS OF 3-PATHS IN PLANE GRAPHS
WITH GIRTH AT LEAST 8

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ABSTRACT. Lebesgue (1940) proved that every plane graph with minimum degree δ at least 3 and girth g (the length of a shortest cycle) at least 5 has a path on three vertices (3-path) of degree 3 each. A description is tight if no its parameter can be strengthened, and no triplet dropped.

Borodin et al. (2013) gave a tight description of 3-paths in plane graphs with $\delta \geq 3$ and $g \geq 3$, and another tight description was given by Borodin, Ivanova and Kostochka in 2017.

In 2015, we gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of tight descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we characterized (2018) all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$.

Recently, eleven tight descriptions of 3-paths were obtained for plane graphs with $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták, four of which descriptions are for $g \geq 9$. In 2018, Aksenov, Borodin and Ivanova proved nine new tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$ and showed that no other tight descriptions exist.

The purpose of this note is to give a complete list of tight descriptions of 3-paths in the plane graphs with $\delta = 2$ and $g \geq 8$.

Keywords: Plane graph, structure properties, tight description, 3-path, minimum degree, height, weight, girth.

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1. INTRODUCTION

Throughout the paper, G is a plane graph. Let $\delta(G)$ be the minimum vertex degree and let $w_k(G)$ be the minimum degree-sum of a path on k vertices in G . We will drop the argument when G is clear from context. The degree of a vertex v or a face f , that is the number of edges incident with v or f , is denoted by $d(v)$ or $d(f)$, respectively. A k -vertex is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k , respectively. Hence, a k^+ -vertex v satisfies $d(v) \geq k$, etc. An edge uv is an (i, j) -edge if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i, j, k) or (i, j, k) -path if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [29] proved that every G with $\delta = 5$ has a $(5, 6)$ -edge, and Franklin [17] strengthened this to the existence of at least two 6^- -neighbors of a 5^- -vertex; this implies that $w_3 \leq 17$, which bound is sharp. Recently, we proved [8] that there is also a $(5, 6, 6)$ -path, which is tight, and there are no tight descriptions of 3-paths for $\delta = 5$ other than $\{(6, 5, 6)\}$ and $\{(5, 6, 6)\}$. (A description of paths is *tight* if no its parameter can be strengthened and no term dropped).

In 1996, Jendrol' and Madaras [25] ensured for $\delta = 5$ that $w_4 \leq 23$, which is sharp. Recently, we found [9] the first tight description of 4-paths for $\delta = 5$, and then Batueva, Borodin and Ivanova [4] found all remaining nine such tight descriptions.

It follows from Lebesgue's [28] results in 1940 that each G with $\delta \geq 3$ satisfies $w_2 \leq 14$. For 3-connected plane graphs, Kotzig [27] proved a precise result: $w_2 \leq 13$.

In 1972, Erdős (see [18]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all plane graphs with $\delta \geq 3$. Barnette (see [18]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [5]. More generally, Borodin [6] proved that every G with $\delta \geq 3$ contains a $(3, 10)$ -, or $(4, 7)$ -, or $(5, 6)$ -edge, which description is tight.

In 1993, Ando, Iwasaki, Kaneko [3] proved that every 3-connected G satisfies $w_3 \leq 21$, which is sharp due to the Jendrol' construction in [19]. This was refined by Borodin [7] in 1997 as follows: every 3-connected G has: (i) either $w_3 \leq 18$ or a vertex of degree ≤ 15 adjacent to two 3-vertices, and (ii) either $w_3 \leq 17$ or $w_2 \leq 7$. Here, the bounds $w_3 \leq 21$ and $w_3 \leq 17$ were known to be tight long ago, and the sharpness of $w_3 \leq 18$ was recently confirmed by Borodin et al. [14].

In 1997, Jendrol' [20] gave an approximate description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(10, 3, 10)$, $(7, 4, 7)$, $(6, 5, 6)$, $(3, 4, 15)$, $(3, 6, 11)$, $(3, 8, 5)$, $(3, 10, 3)$, $(4, 4, 11)$, $(4, 5, 7)$, or $(4, 7, 5)$.

In 2013, Borodin et al. [14] gave the first tight description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, $(6, 5, 6)$. Another similar tight description for $\delta \geq 3$ and $g \geq 3$ was given by Borodin, Ivanova and Kostochka [15].

In 2015, we [10] gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we [12] characterized all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$. Other results on k -paths with $k \geq 3$ and $\delta \geq 3$ can be found in surveys Borodin–Ivanova [11], Cranston–West [16] and Jendrol'–Voss [26].

Aksenov, Borodin and Ivanova [1] proved precise upper bounds for w_3 in several natural classes of plane graphs with $\delta = 2$ and $5 \leq g \leq 7$ and disproved a conjecture by Jendrol' and Maceková [21] concerning the case $g = 5$.

Recently, eleven tight descriptions of 3-paths were obtained for $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták [21–24], four of which descriptions are for $g \geq 9$ (for details, see Theorems 1 and 2 below).

Aksenov, Borodin and Ivanova [2] gave the following complete list of tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$.

Theorem 1 ([2]). *There exist precisely these tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth g at least 9:*

- (A) $g \geq 16$: $\{(2, 2, 2)\}$ (folklore);
- (B) $11 \leq g \leq 15$:
 $\{(2, 2, 3)\}$ ([22]) and
 $\{(2, 3, 2)\}$;
- (C) $g = 10$:
 $\{(2, 2, 3), (2, 3, 2)\}$ ([21], the tightness shown in [22]),
 $\{(2, 4, 2)\}$ ([22]),
 $\{(2, 3, 3)\}$, $\{(2, 2, 4), (3, 2, 3)\}$, and
 $\{(3, 2, 4)\}$;
- (D) $g = 9$:
 $\{(2, 2, 5), (2, 3, 2)\}$ ([23]),
 $\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 2, 5), (3, 2, 3)\}$, $\{(2, 5, 3)\}$, $\{(2, 3, 5)\}$, and
 $\{(3, 2, 5)\}$.

In [13] we described all tight descriptions of 3-paths centered at 2-vertices whenever $g \geq 6$.

The purpose of this note is to completely resolve the case $g \geq 8$, as follows.

Theorem 2. *There exist these and only these five tight descriptions of 3-paths in plane graphs with minimum degree 2 and girth at least 8:*

- (i) $\{(2, 2, 5), (2, 3, 2)\}$ ([22]);
- (ii) $\{(2, 5, 2), (2, 2, 3)\}$;
- (iii) $\{(3, 2, 5)\}$ ([13]);
- (iv) $\{(2, 3, 5)\}$;
- (v) $\{(2, 5, 3)\}$.

2. PROVING THEOREM 2

Since $\{(2, 2, 5), (2, 3, 2)\}$ is a tight description ([22]), it follows that $\{(2, 3, 5)\}$ is also a description, since both a $(2, 2, 5)$ -path and a $(2, 3, 2)$ -path are also $(2, 3, 5)$ -paths. So, we first prove that $\{(2, 5, 2), (2, 2, 3)\}$ is a description. This implies that $\{(2, 5, 3)\}$ is also a description, since both a $(2, 5, 2)$ -path and a $(2, 2, 3)$ -path are also $(2, 5, 3)$ -paths.

Next, we show that all descriptions $\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 3, 5)\}$ and $\{(2, 5, 3)\}$ are tight.

Finally, we show that there are no tight descriptions for $g \geq 8$ other than those five listed in Theorem 2.

2.1. Proving that $\{(2, 5, 2), (2, 2, 3)\}$ is a description. We need the following refinement of the already mentioned Borodin's [6] tight description $\{(3, 10), (4, 7), (5, 6)\}$ of 2-paths (that is, edges) in plane graphs with $\delta \geq 3$.

Lemma 1 ([13]). *Every plane graph with minimum degree at least 3 has at least one of the following:*

- (a) *a 3-face incident with a $(3, 10)$ -, or $(4, 7)$ -, or $(5, 6)$ -edge;*
- (b) *a 4-face incident either with two 3-vertices and another 5^- -vertex or with a 3-vertex, two 4-vertices and the forth vertex of degree at most 5;*
- (c) *a 5-face incident with four 3-vertices and the fifth vertex of degree at most 5, where all parameters are best possible.*

Suppose on the contrary that G does not obey the description $\{(2, 5, 2), (2, 2, 3)\}$.

We consider the graph G^* with $\delta(G^*) \geq 3$ obtained from G by contracting all 2-vertices and look at its 5^- -faces f^* implied by Lemma 1. Note that $g(G^*) \geq 3$ since $g(G) \geq 8$ and due to the absence $(2, 2, 2)$ -paths in G .

If $f^* = v_1 v_2 v_3$ with $d(v_1) \leq 5$, then then the boundary $\partial(f)$ of the pre-image f of f^* in G must have at most two 2-vertices at the pair of edges since $g(G) \geq 8$, which is impossible due to the absence $(2, 2, 3)$ -paths in G . On the other hand, both edges $v_1 v_2$ and $v_1 v_3$ cannot contain a 2-vertices due to the absence of $(2, 5, 2)$ -paths in G . This implies $d(f) \leq 3 + 2 + 1 < 8$, a contradiction.

Now suppose $d(f^*) = v_1 v_2 v_3 v_4$. It follows from Lemma 1(b) that f^* must have a 3-vertex, say v_1 , opposite to a 5^- -vertex, v_3 , in $\partial(f)$. It is not hard to see that to avoid $(2, 5, 2)$ -paths and $(2, 2, 3)$ -paths, our f can have at most one 2-vertex at the pair of edges $v_1 v_2, v_1 v_4$ and at most two 2-vertices at $v_2 v_3, v_3 v_4$. However, then $d(f) \leq 4 + 1 + 1 < 8$, a contradiction.

Finally, suppose $d(f^*) = v_1 \dots v_5$ with $d(v_1) = \dots = d(v_4) = 3$ and $d(v_5) \leq 5$. Now at most one 2-vertex may be put on every edge in $\partial(f)$ and, moreover, at most one 2-vertex can appear at any two consecutive edges in $\partial(f)$. This implies $d(f) \leq 5 + 2 \times 1 < 8$, a contradiction.

2.2. Proving the tightness of $\{(2, 5, 2), (2, 2, 3)\}$, $\{(2, 3, 5)\}$ and $\{(2, 5, 3)\}$. To show the tightness of $\{(2, 5, 2), (2, 2, 3)\}$, we have to prove that neither $\{(2, 4, 2), (2, 2, 3)\}$ nor $\{(2, 5, 2), (2, 2, 2)\}$ is a description. To reject the former option, it suffices to put two 2-vertices on every edge of the icosahedron and note that the graph H_1 obtained has girth 9 but no $(4, 4, 4)$ -paths, since each its 3-path goes through 5-vertex.

To reject the latter, we reproduce the graph obtained in Jendrol' et al. (see Fig. 5 in [22]). Take concentric cycles $W_8 = w_1 \dots w_8$, $XY_{16} = x_1 y_1 \dots x_8 y_8$, $Z_8 = z_1 \dots z_8$, and add a path with two internal 2-vertices between w_i to x_i and also between y_i and z_i whenever $1 \leq i \leq 8$. It remains to observe that H_2 obtained has no $(2, 5, 2)$ -paths (and, in particular, no $(2, 2, 2)$ -paths) and $g(H_2) = 8$.

The tightness of $\{(2, 5, 3)\}$ follows similarly from the same graphs H_1 and H_2 . Indeed, we cannot strengthen $\{(2, 5, 3)\}$ to $\{(2, 4, 3)\}$, since each 3-path in H_1 goes through 5-vertex, and to $\{(2, 5, 2)\}$, since H_2 has no vertex adjacent to two 2-vertices.

Finally, to see the tightness of $\{(2, 3, 5)\}$, it suffices to observe that $\{(2, 3, 4)\}$ is invalid due to H_1 , while $\{(2, 2, 5)\}$ fails to describe the graph H_3 that is a result of deleting all edges joining 10-vertices to 10-vertices from the triangulation obtained from the icosahedron by putting a 3-vertex into each face and joining it to the

boundary vertices of this face. Note that H_3 is bipartite, with every edge joining a 3-vertex to a 5-vertex.

2.3. Proving the non-existence of tight descriptions other than those five in Theorem 2. Suppose $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g \geq 8$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$. It follows from the graph H_1 above that D must have an entry, say y_1 or z_1 , not smaller than 5.

CASE 1. $y_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 5^+, 2)$ since $\{(2, 5, 3)\}$ is already a tight description. It follows from H_2 (which has no $(2, 5, 2)$ -paths) that, say, $(x_2, y_2, z_2) = (2^+, 2^+, 3^+)$.

Since $\{(2, 5, 2), (2, 2, 3)\}$ is known to be a tight description, it follows that $D = \{(2, 5, 2), (2, 2, 3)\}$.

CASE 2. $z_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 2, 5^+)$ since $\{(3, 2, 5)\}$ and $\{(2, 3, 5)\}$ are tight descriptions. Let H_4 be a graph obtained from the dodecahedron by putting a 2-vertex on each edge. Clearly, H_4 satisfies $g(H_4) = 10$ and has only $(3, 2, 3)$ - and $(2, 3, 2)$ -paths. This implies that D must have a term, say (x_2, y_2, z_2) , such that either $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$ or $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$.

In the first case, we have $D = \{(2, 2, 5), (3, 2, 3)\}$ since $\{(2, 2, 5), (3, 2, 3)\}$ is known to be a tight description. In the second case, we similarly deduce that $D = \{(2, 2, 5), (2, 3, 2)\}$, as desired.

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