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THE SOBOLEV–POINCARÉ INEQUALITY AND
THE $L_{q,p}$ -COHOMOLOGY OF TWISTED CYLINDERS

V.GOL'DSHTEIN, YA.A.KOPYLOV

ABSTRACT. We establish a vanishing result for the $L_{q,p}$ -cohomology ($q \geq p$) of a twisted cylinder, which is a generalization of a warped cylinder. The result is new even for warped cylinders. We base on the methods for proving the (p, q) -Sobolev–Poincaré inequality developed by L. Shartser.

Keywords: differential form, Sobolev–Poincaré inequality, $L_{q,p}$ -cohomology, twisted cylinder, homotopy operator.

1. INTRODUCTION

The $L_{q,p}$ -cohomology $H_{q,p}^k(M)$ of a Riemannian manifold (M, g) is, by definition, the quotient of the space of closed p -integrable differential k -forms by the exterior differentials of q -integrable k -forms. If $p = q$ then $L_{q,p}$ -cohomology is usually referred to simply as L_p -cohomology and the index p is used instead of p, p in all the notations.

A *twisted product* $X \times_h Y$ of two Riemannian manifolds (X, g_X) and (Y, g_Y) is the direct product manifold $X \times_g Y$ endowed with a Riemannian metric of the form

$$(1.1) \quad g := g_X + h^2(x, y)g_Y,$$

where $h : X \times Y \rightarrow \mathbb{R}$ is a smooth positive function (see [5]). If X is a half-interval $[a, b)$ then the twisted product $X \times_h Y$ is called a *twisted cylinder*.

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We refer to an m -dimensional Riemannian manifold (M, g_M) as an *asymptotic twisted product* (respectively, as an *asymptotic twisted cylinder*) if, outside an m -dimensional compact submanifold, it is bi-Lipschitz equivalent to a twisted product (respectively, to a twisted cylinder).

In this paper, we prove some vanishing results for the $L_{q,p}$ -cohomology of twisted cylinders $[a, b] \times_h N$ for a positive smooth function $h : [a, b] \times N \rightarrow \mathbb{R}$ in the case where the base N is a closed manifold and $p \geq q > 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N + 1)}$.

If in (1.1) the function h depends only on x then we obtain the familiar notion of a *warped product* (see [1]). Twisted products were the object of recent investigations [4, 6, 8, 9, 10, 16, 21]. The $L_{q,p}$ -cohomology of warped cylinders $[a, b] \times_h N$, i.e., of product manifolds $[a, b] \times N$ endowed with a warped product metric

$$g = dt^2 + h^2(t)g_N,$$

where g_N is the Riemannian metric of N and $h : [a, b] \rightarrow \mathbb{R}$ is a positive smooth function, was studied by Gol'dshtein, Kuz'minov, and Shvedov [11], Kuz'minov and Shvedov [19, 20] (for $p = q$), and Kopylov [17, 18] for $p, q \in [1, \infty)$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{\dim N + 1}$. In particular, in [18], the second author found a sufficient condition for the L_{qp} -cohomology of a warped cylinder to be nontrivial (and even infinite-dimensional) in terms of Hardy's inequality.

The main result of the paper (Theorem 7.1) states that the $L_{q,p}$ -cohomology $H_{q,p}^k(C_{a,b}^h N)$ of the twisted cylinder $C_{a,b}^h N$ with $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N + 1)}$ is zero provided that the de Rham cohomology $H_{\text{DR}}^k(N)$ of the base N is trivial and some integral conditions on the twisting function involving p , q and an auxiliary parameter $\bar{p}$ are fulfilled.

The paper is organized as follows: In Section 2, we recall some basic definitions concerning the $L_{q,p}$ -cohomology of Riemannian manifolds. Section 3 describes the representations of differential forms on a twisted cylinder obtained in [10] and analogous to the representations of forms on a warped product proposed by Gol'dshtein, Kuz'minov, and Shvedov in [12]. In Section 4, we develop a version of the weighted Sobolev–Poincaré inequality for convex sets in $\mathbb{R}^n$ by introducing a homotopy operator and consider some of its consequences; the exposition is based on the ideas of Shartser suggested in [22] and [23]. In Section 5, we consider a new homotopy operator A_α on differential forms defined on a convex domain in $\mathbb{R}^n$ and show that it guarantees the fulfillment of an inequality of Sobolev–Poincaré-type for $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. In Section 6, using the ideas of Shartser's article [23], we “glue” local homotopy operators on a twisted cylinder to obtain a global homotopy operator. In Section 7, we use this global homotopy operator for proving our above-mentioned main result on the triviality of the $L_{q,p}$ -cohomology of a twisted cylinder (Theorem 7.1), and in Section 8, we extend this theorem to asymptotic twisted cylinders (Theorem 8.2). Section 9 contains some examples.

2. BASIC DEFINITIONS

We recall the main definitions and notations.

Below we tacitly assume all manifolds to be oriented.

Let M be a smooth oriented Riemannian manifold. Denote by $\mathcal{D}^k(M) := C_0^\infty(M, \Lambda^k)$ the space of all smooth differential k -forms with compact support contained in $M \setminus \partial M$ denote by $L_{loc}^1(M, \Lambda^k)$ the space of locally integrable differential forms.

Denote by $L^p(M, \Lambda^k)$ the Banach space of locally integrable differential k -forms endowed with the norm $\|\theta\|_{L^p(M, \Lambda^k)} := \left(\int_M |\theta|^p dx\right)^{\frac{1}{p}} < \infty$ (as usual, we identify forms coinciding outside a set of measure zero). Of course, we can add a positive (smooth) weight $\sigma : M \rightarrow \mathbb{R}$ and thus integrate $|\theta|^p \sigma^p$ to obtain the weighted L^p -space $L^p(M, \Lambda^k, \sigma)$.

Definition 2.1. We call a differential $(k+1)$ -form $\theta \in L^1_{loc}(M, \Lambda^{k+1})$ the *weak exterior derivative* (or *differential*) of a differential k -form $\phi \in L^1_{loc}(M, \Lambda^k)$ and write $d\phi = \theta$ if

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M)$.

Remark 2.2. Note that the orientability of M is not substantial in this definition since one can take integrals over orientable domains on M instead of integrals over M .

We then introduce an analog of Sobolev spaces for differential k -forms, i.e., the space of q -integrable forms with p -integrable weak exterior derivative:

$$\Omega^k_{q,p}(M) = \{ \omega \in L^q(M, \Lambda^k) \mid d\omega \in L^p(M, \Lambda^{k+1}) \}.$$

This is a Banach space for the graph norm

$$\|\omega\|_{q,p} = \left(\|\omega\|_{L^q(M, \Lambda^k)}^2 + \|d\omega\|_{L^p(M, \Lambda^{k+1})}^2 \right)^{1/2}.$$

The space $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < q, p < \infty$. This can be proved using standard arguments of functional analysis.

We now define our basic ingredients (for three parameters r, q, p).

Definition 2.3. Put

- (a) $Z^k_{p,r}(M) = \text{Ker}[d : \Omega^k_{p,r}(M) \rightarrow L^r(M, \Lambda^{k+1})]$.
- (b) $B^k_{q,p}(M) = \text{Im}[d : \Omega^{k-1}_{q,p}(M) \rightarrow L^p(M, \Lambda^k)]$.

The subspace $Z^k_{p,r}(M)$ does not depend on r and is a closed subspace in $L^p(M, \Lambda^k)$ (see Lemma [14, Lemma 2.4(i)]). This allows us to use the notation $Z^k_p(M)$ for all $Z^k_{p,r}(M)$. Note that $Z^k_p(M) \subset L^p(M, \Lambda^k)$ is always a closed subspace but that is in general not true for $B^k_{q,p}(M)$. Denote by $\overline{B}^k_{q,p}(M)$ its closure in the L^p -topology. Observe also that since $d \circ d = 0$, one has $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$. Thus,

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M, \Lambda^k).$$

Definition 2.4. Suppose that $1 \leq q, p \leq \infty$. The $L_{q,p}$ -cohomology of (M, g) is defined as the quotient

$$H^k_{q,p}(M) := Z^k_p(M) / B^k_{q,p}(M),$$

and the *reduced* $L_{q,p}$ -cohomology of (M, g) is, by definition, the space

$$\overline{H}^k_{q,p}(M) := Z^k_p(M) / \overline{B}^k_{q,p}(M).$$

Since $B_{p,q}^k$ is not always closed, the L_p -cohomology is in general a (non-Hausdorff) semi-normed space, while the reduced L_p -cohomology is a Banach space.

Below $|X|$ stands for the volume of a Riemannian manifold (X, g) .

It follows from the results of [13] that, under suitable assumptions on p, q , the $L_{q,p}$ -cohomology of a Riemannian manifold M can be expressed in terms of smooth forms.

Let $C^\infty(M, \Lambda^k)$ be the space of smooth k -forms on M .

Introduce the notations:

$$\begin{aligned} C^\infty L^p(M, \Lambda^k) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k); \\ C^\infty L^p(M, \Lambda^k, \sigma) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k, \sigma); \\ C^\infty \Omega_{q,p}^k(M) &:= C^\infty(M, \Lambda^k) \cap \Omega_{q,p}^k(M); \\ C^\infty H_{q,p}^k(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z_p^k(M)}{C^\infty(M, \Lambda^k) \cap B_{q,p}^k(M)}; \\ C^\infty \overline{H}_{q,p}^k(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z_p^k(M)}{C^\infty(M, \Lambda^k) \cap \overline{B}_{q,p}^k(M)}. \end{aligned}$$

Theorem 2.5. [13, Theorem 12.5 and 12.8, Corollary 12.9]. *Let (M, g) be a n -dimensional Riemannian manifold and suppose the fulfillment of one of the following conditions:*

- $p, q \in (1, \infty)$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$;
- $p, q \in [1, \infty)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.

Then the cohomology $H_{q,p}^(M)$ can be represented by smooth forms, and thus $H_{q,p}^*(M) = C^\infty H_{q,p}^*(M)$.*

More exactly, any closed form in $Z_p^k(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M, \Lambda^k) \cap Z_p^k(M)$ are cohomologous modulo $d\Omega_{q,p}^{k-1}(M)$ then they are cohomologous modulo $dC^\infty \Omega_{q,p}^{k-1}(M)$.

Similarly, any reduced cohomology class can be represented by a smooth form.

3. DIFFERENTIAL FORMS ON A TWISTED CYLINDER

From now on, $C_{a,b}^h N$ is the twisted cylinder $[a, b) \times_h N$, that is, the product of a half-interval $[a, b)$ and a closed smooth n -dimensional Riemannian manifold (N, g_N) equipped with the Riemannian metric $dt^2 + h^2(t, x)g_N$, where $h : [a, b) \times N \rightarrow \mathbb{R}$ is a smooth positive function.

Every differential form on $[a, b) \times N$ admits a unique representation of the form $\omega = \omega_A + dt \wedge \omega_B$, where the forms ω_0 and ω_1 do not contain dt (cf. [12]). It means that ω_0 and ω_1 can be viewed as one-parameter families $\omega_A(t)$ and $\omega_B(t)$, $t \in I$, of differential forms on N .

The modulus of a form ω of degree k on $C_{a,b}^h N$ is expressed via the moduli of $\omega_A(t)$ and $\omega_B(t)$ on N as follows:

$$(3.1) \quad |\omega(t, x)|_{C_{a,b}^h N} = [h^{-2k}(t, x)|\omega_A(t, x)|_N^2 + h^{-2(k+1)}(t, x)|\omega_B(t, x)|_N^2]^{1/2}$$

Consequently,
(3.2)

$$\|\omega\|_{L^p(C_{a,b}^h N, \Lambda^k)} = \left[\int_a^b \int_N \left(h^{2(\frac{n}{p}-k)}(t,x) |\omega_A(t,x)|_N^2 + h^{2(\frac{n}{p}-k+1)}(t,x) |\omega_B(t,x)|_N^2 \right)^{\frac{p}{2}} dx dt \right]^{\frac{1}{p}}.$$

Put

$$f_{k,p}(t) = \min_{x \in N} \{ h^{\frac{n}{p}-k}(t,x) \}$$

and

$$F_{k,p}(t) = \max_{x \in N} \{ h^{\frac{n}{p}-k}(t,x) \}.$$

4. THE WEIGHTED SOBOLEV–POINCARÉ INEQUALITY FOR CONVEX SETS IN $\mathbb{R}^n$

Denote by $\Omega_{loc}^*(M)$ the space all locally integrable differential forms with locally integrable weak differential.

Suppose that $D \subset \mathbb{R}^n$ is a convex set and $\psi_y : D \times [0,1] \rightarrow D$, $\psi_y(x,t) := tx + (1-t)y$, is the homotopy induced by the convex structure. For a k -form $\omega \in \Omega_{loc}^k(D)$ the pullback $\psi_y^* \omega$ can be written as

$$\psi_y^* \omega(x,t) = (\psi_y^* \omega)_0(x,t) + dt \wedge (\psi_y^* \omega)_1(x,t),$$

where $(\psi_y^* \omega)_0$ and $(\psi_y^* \omega)_1$ do not contain dt .

For each $y \in D$ define a homotopy operator

$$K_y : \Omega_{loc}^k(D) \rightarrow \Omega_{loc}^{k-1}(D)$$

as follows:

$$K_y \omega(x) := \int_0^1 (\psi_y^* \omega)_1(t) dt$$

It is easy to see that K_y takes smooth forms to smooth forms. It is proved in [15] that $K_y d\omega + dK_y \omega = \omega$. The following proposition is a generalization of results from [2] and Shartser's thesis [22] (see also [23]) to the weighted case and to unbounded convex domains.

Proposition 4.1. *Suppose that D is a convex set in $\mathbb{R}^n$, $q \geq p \geq 1$, and $\beta : D \rightarrow \mathbb{R}$ is a positive smooth function.*

If the inequality

$$C(k,p,q,n,\beta) := \int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt < \infty$$

holds then the inequality

$$\left\| \beta(x) \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^p(D,dy)} \right\|_{L^q(D,dx)} \leq C(k,p,q,n,\beta) \|d\omega\|_{L^p(D,\Lambda^{k+1})}.$$

is valid for every $\omega \in \Omega_{loc}^k(D)$ such that $d\omega \in L^p(D, \Lambda^{k+1})$. Here $\mathbf{1}_{tx+(1-t)D}$ is the characteristic function of the set $tx + (1-t)D$.

Proof. By the definition of K_y , we have

$$\left\| \beta(x) \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^p(D,dy)} \right\|_{L^q(D,dx)} = \left\| \beta(x) \left\| \int_0^1 \frac{(\psi_y^* d\omega)_1(x,t)}{|x-y|} dt \right\|_{L^p(D,dy)} \right\|_{L^q(D,dx)}$$

$$\begin{aligned}
&\leq \int_0^1 \left\| \beta(x) \left\| \frac{(\psi_y^* d\omega)_1(x, t)}{|x-y|} \right\|_{L^p(D, dy)} \right\|_{L^q(D, dx)} dt \\
&\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{|(\psi_y^* d\omega)_1(x, t)|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt.
\end{aligned}$$

As usual, we identify the tangent space to $\mathbb{R}^n$ at any of its points with $\mathbb{R}^n$. By easy calculations,

$$|(\psi_y^* d\omega)_1(x, t)| \leq |x-y| t^k |d\omega(\psi_y(x, t))|.$$

Therefore,

$$\begin{aligned}
&\int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{|(\psi_y^* d\omega)_1(x, t)|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt \\
&\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} |d\omega(\psi_y(x, t))|^p dy \right]^{q/p} dx \right\}^{1/q} dt \\
&= \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} |d\omega(tx + (1-t)y)|^p dy \right]^{q/p} dx \right\}^{1/q} dt := I.
\end{aligned}$$

The change of variables $z = tx + (1-t)y$ in the inner integral yields

$$I = \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p dz \right]^{q/p} dx \right\}^{1/q} t^k (1-t)^{-n/p} dt$$

Since D is convex, the set $tx + (1-t)D$ is contained in D for all $x \in D$ and $t \in [0, 1]$. Using Minkowski's integral inequality, we infer

$$\begin{aligned}
&\left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p dz \right]^{q/p} dx \right\}^{1/q} \\
&= \left\{ \int_D \beta^q(x) \left[\int_D \mathbf{1}_{tx+(1-t)D}(z) |d\omega(z)|^p dz \right]^{q/p} dx \right\}^{1/q} \\
&= \left\{ \left(\int_D \left[\int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) |d\omega(z)|^p dz \right]^{q/p} dx \right)^{p/q} \right\}^{1/p} \\
&= \left\{ \left\| \int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) |d\omega(z)|^p dz \right\|_{L^{q/p}(D, dx)} \right\}^{1/p} \\
&\leq \left\{ \int_D \left\| \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) |d\omega(z)|^p \right\|_{L^{q/p}(D, dx)} dz \right\}^{1/p}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) |d\omega(z)|^q dx \right)^{p/q} dz \right\}^{1/p} \\
&= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{p/q} |d\omega(z)|^p dz \right\}^{1/p} \\
&\leq \left(\sup_{z \in D} \int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{1/q} \left(\int_D |d\omega(z)|^p dz \right)^{1/p} \\
&= \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} \|d\omega\|_{L^p(D, \Lambda^{k+1})}.
\end{aligned}$$

The proposition follows. $\square$

Estimate

$$C(k, p, q, n, \beta) = \int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-n/p} dt$$

in particular cases.

Corollary 4.2. Suppose that D is a convex set of finite measure in $\mathbb{R}^n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, and the weight $\beta(x) \equiv 1$. Then

$$C(k, p, q, n, 1) \leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt.$$

Remark 4.3. It is easy to see that the integral of the corollary exists because of the conditions imposed on p and q .

Proof. Using the change of variables $u = tx$, we obtain

$$\begin{aligned}
&\int_0^1 \sup_{z \in D} \|\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-n/p} dt \\
&= \int_0^1 \sup_{z \in D} \|\mathbf{1}_{u+(1-t)D}(z)\|_{L^q(tD, du)} t^{k-n/q} (1-t)^{-n/p} dt.
\end{aligned}$$

Note that $|tD \cap \{u + (1-t)D\}| \leq |D| \min(t^n, (1-t)^n)$. It follows that

$$\begin{aligned}
&\|\mathbf{1}_{u+(1-t)D}(z)\|_{L^q(tD, du)} \leq |D|^{1/q} \min(t^{n/q}, (1-t)^{n/q}); \\
C(k, p, q, n, 1) &\leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt
\end{aligned}$$

$\square$

Corollary 4.4. Suppose that U is a convex set of finite measure $|U|$ in $\mathbb{R}^n$, $D = [a, b) \times U$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, and $\beta : [a, b) \rightarrow \mathbb{R}$ is an integrable positive function. If $\|\beta\|_{L^q([a, b))} < \infty$ then

$$C(k, p, q, n, \beta) \leq |U|^{1/q} \|\beta\|_{L^q([a, b))}.$$

Proof. If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a, b)$ and $w \in U$. Using the special type of the weight $\beta(x) := \beta(\tau)$ and representing $z \in D$ as $z = (\eta, \zeta)$ with $\eta \in [a, b)$ and $\zeta \in U$, we obtain

$$\int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-\frac{n+1}{p}} dt$$

$$\leq \int_0^1 \sup_{a \leq \eta < b} \left(\int_a^b \beta^q(\tau) \mathbf{1}_{t\tau+(1-t)[a,b]}(\eta) d\tau \right)^{\frac{1}{q}} \sup_{\zeta \in U} \left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k (1-t)^{-\frac{n+1}{p}} dt,$$

where $x = (\tau, w)$.

Using the change of variables $u = tw$ and the estimate

$$|tU \cap \{u + (1-t)U\}| \leq |U| \min(t^n, (1-t)^n),$$

we finally get

$$\begin{aligned} & \int_0^1 \sup_{a \leq \eta < b} \left(\int_a^b \beta^q(\tau) \mathbf{1}_{t\tau+(1-t)[a,b]}(\eta) d\tau \right)^{\frac{1}{q}} \sup_{\zeta \in U} \left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k (1-t)^{-\frac{n+1}{p}} dt \\ & \leq |U|^{1/q} \|\beta\|_{L^q([a,b])} \int_0^1 t^{k-n/q} (1-t)^{-(n+1)/p} \min(t^{n/q}, (1-t)^{n/q}) dt \end{aligned}$$

The conditions on p and q imply the finiteness of the last integral. $\square$

Corollary 4.4 is a key ingredient in the proof of our main result, Theorem 7.1. Unfortunately, for being able to “separate” the variable t , we have to impose the stronger constraint $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1} - \frac{1}{q(n+1)}$ than the condition $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1}$ given by Proposition 4.1.

5. A NEW HOMOTOPY OPERATOR FOR $q \geq p$. THE CASE OF A CONVEX DOMAIN IN $\mathbb{R}^n$

In the previous section, we considered the homotopy operator on Ω_{loc}^* of the form

$$A_\alpha = \int_D \alpha(y) K_y \omega(x) dy$$

for a convex set D in $\mathbb{R}^n$. We will need to modify A for obtaining some estimates.

Consider the same operator K_y as in the previous section:

$$\psi_y(x, t) = tx + (1-t)y, \quad K_y \omega(x) = \int_0^1 (\psi_y)_1^* \omega dt.$$

Recall that $dK_y \omega + K_y d\omega = \omega$. Choose a smooth positive function $\alpha : D \rightarrow \mathbb{R}$ such that $\int_D \alpha(x) dx = 1$ and put

$$A_\alpha \omega(x) := \int_D \alpha(y) K_y \omega(x) dy, \quad \omega \in \Omega_{\text{loc}}^*.$$

By a straightforward calculation,

$$dA_\alpha \omega = d \left(\int_D \alpha(y) K_y \omega(x) dy \right) = \int_D \alpha(y) d_x K_y \omega(x) dy;$$

$$A_\alpha d\omega = \int_D \alpha(y) K_y d\omega(x) dy;$$

$$dA_\alpha \omega + A_\alpha d\omega = \int_D \alpha(y) [d_x K_y \omega(x) + K_y d\omega(x)] dy = \int_D \alpha(y) \omega(x) dy = \omega.$$

In particular, if $d\omega = 0$ then

$$dA_\alpha \omega = \omega.$$

The definition of A_α easily implies the following

Proposition 5.1. *The homotopy operator A_α takes smooth forms to smooth forms.*

Definition 5.2. Call a smooth positive function $\alpha : D \rightarrow \mathbb{R}$ an *admissible weight* for a convex domain $D \subset \mathbb{R}^n$ and $p \geq 1$ if

$$\int_D \alpha(x) dx = 1; \quad \|\alpha\|_{L^{p'}(D)} < \infty; \quad \|\alpha(y)|y|\|_{L^{p'}(D)} < \infty.$$

For $p \geq 1$, we as usual put

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

Theorem 5.3. Suppose that $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ is a convex set, $\beta : D \rightarrow \mathbb{R}$ is a positive smooth function, and $\alpha : D \rightarrow \mathbb{R}$ is an admissible weight. If

$$C_1(k, p, q, n, \beta) := \int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-n/p} dt < \infty;$$

$$C_2(k, p, q, n, \beta) := \int_0^1 \sup_{z \in D} \| |x| \beta(x) \mathbf{1}_{tx+(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-n/p} dt < \infty$$

then for any $\omega \in C^\infty L^p(D, \Lambda^k)$ we have

$$\|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1}, \beta)} \leq C(k, p, q, \alpha, \beta, n) \|\omega\|_{L^p(D, \Lambda^k)}$$

where

$$C(k, p, q, \alpha, \beta, n) = \|\alpha(y)|y|\|_{L^{p'}(D)} C_1(k, p, q, n, \beta) + \|\alpha\|_{L^{p'}(D)} C_2(k, p, q, n, \beta).$$

Proof. Put $\xi := A_\alpha \omega$. If $p > 1$ then, by Hölder's inequality, we infer

$$\begin{aligned} \|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1}, \beta)} &= \left\| \beta(x) \int_D \alpha(y) K_y \omega(x) dy \right\|_{L^q(D, \Lambda^{k-1}, dx)} \\ &\leq \left\| \beta(x) \left\| \frac{K_y \omega(x)}{|x-y|} \right\|_{L^p(D, \Lambda^{k-1}, dy)} \right\|_{L^q(D, \Lambda^{k-1}, dx)} \left\| \alpha(y)|x-y| \right\|_{L^{p'}(D, dy)}. \end{aligned}$$

The above estimate also obviously holds for $p = 1$.

By the triangle inequality,

$$\|\alpha(y)|x-y|\|_{L^{p'}(D, dy)} \leq |x| \|\alpha(y)\|_{L^{p'}(D, dy)} + \|\alpha(y)|y|\|_{L^{p'}(D, dy)}.$$

Therefore,

$$\begin{aligned} \|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1})} &\leq \|\alpha(y)|y|\|_{L^{p'}(D, dy)} \left\| \beta(x) \left\| \frac{K_y \omega(x)}{|x-y|} \right\|_{L^p(D, \Lambda^{k-1}, dy)} \right\|_{L^q(D, \Lambda^{k-1}, dx)} \\ &\quad + \|\alpha(y)\|_{L^{p'}(D, dy)} \left\| \beta(x)|x| \left\| \frac{K_y \omega(x)}{|x-y|} \right\|_{L^p(D, \Lambda^{k-1}, dy)} \right\|_{L^q(D, \Lambda^{k-1}, dx)}. \end{aligned}$$

By Proposition 4.1,

$$\begin{aligned} \left\| \beta(x) \left\| \frac{K_y \omega(x)}{|x-y|} \right\|_{L^p(D, \Lambda^{k-1}, dy)} \right\|_{L^q(D, \Lambda^{k-1}, dx)} &\leq C_1(k, p, q, n, \beta) \|\omega\|_{L^p(D, \Lambda^k)}; \\ \left\| \beta(x)|x| \left\| \frac{K_y \omega(x)}{|x-y|} \right\|_{L^p(D, \Lambda^{k-1}, dy)} \right\|_{L^q(D, \Lambda^{k-1}, dx)} &\leq C_2(k, p, q, n, \beta) \|\omega\|_{L^p(D, \Lambda^k)}. \end{aligned}$$

The theorem is proved. $\square$

Corollary 5.4. *Suppose that $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ is a convex set, $\alpha : [a, b] \rightarrow \mathbb{R}$ is an admissible weight, $\beta, \gamma : D \rightarrow \mathbb{R}$ are positive smooth functions. If the conditions*

$$\begin{aligned} C_1(k, \bar{p}, q, n, \beta) &:= \int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-n/\bar{p}} dt < \infty; \\ C_2(k, \bar{p}, q, n, \beta) &:= \int_0^1 \sup_{z \in D} \| |x| \beta(x) \mathbf{1}_{tx+(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-n/\bar{p}} dt < \infty; \\ Q(k, \bar{p}, p, \gamma) &:= \|\gamma^{-1}\|_{L^{p\bar{p}/(p-\bar{p})}(D)} < \infty \end{aligned}$$

are fulfilled for some $\bar{p}$, $1 \leq \bar{p} \leq p$ (for $\bar{p} = p$, we put $\frac{p\bar{p}}{p-\bar{p}} = \infty$), then the inequality

$$\|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1}, \beta)} \leq C(k, p, q, \alpha, \beta, \gamma, n) \|\omega\|_{L^p(D, \Lambda^k, \gamma)},$$

where

$$C(k, p, q, \alpha, \beta, \gamma, n) = Q(k, \bar{p}, p, \gamma) C(k, \bar{p}, q, n, \alpha, \beta),$$

holds for any $\omega \in C^\infty L^p(D, \Lambda^k)$.

Proof. By Theorem 5.3,

$$\|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1}, \beta)} \leq C(k, \bar{p}, q, n, \alpha, \beta) \|\omega\|_{L^{\bar{p}}(D, \Lambda^k)}.$$

If $\bar{p} < p$ then, using Hölder's inequality, we have

$$(5.1) \quad \|\omega\|_{L^{\bar{p}}(D, \Lambda^k)} \leq \|\gamma \omega\|_{L^p(D, \Lambda^k)} \|\gamma^{-1}\|_{L^{p\bar{p}/(p-\bar{p})}(D)}.$$

Inequality (5.1) also holds for $\bar{p} = p$.

The corollary follows. $\square$

Corollary 5.5. *Suppose that $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, U is a bounded convex set in $\mathbb{R}^n$, $D = [a, b] \times U$, $\alpha : [a, b] \rightarrow \mathbb{R}$ is an admissible weight, and $\beta, \gamma : [a, b] \rightarrow \mathbb{R}$ are positive smooth functions. If the conditions $\|\beta\|_{L^q([a, b])} < \infty$, $\|\tau \beta(\tau)\|_{L^q([a, b])} < \infty$, and $\|\gamma^{-1}\|_{L^{p\bar{p}/(p-\bar{p})}([a, b])} < \infty$ are fulfilled for some $\bar{p}$, $1 \leq \bar{p} \leq p$ (for $\bar{p} = p$, we put $\frac{p\bar{p}}{p-\bar{p}} = \infty$), then the inequality*

$$\|A_\alpha \omega\|_{L^q(D, \Lambda^{k-1}, \beta)} \leq \text{const} \|\omega\|_{L^p(D, \Lambda^k, \gamma)}$$

with some constant depending $k, p, q, n, \alpha, \beta$, and γ holds for any $\omega \in C^\infty L^p(D, \Lambda^k, \gamma)$.

Proof. Suppose that a number $\bar{p} \leq p$ satisfies the conditions of the corollary.

If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a, b]$ and $w \in U$. By Corollary 4.4, since $\frac{1}{\bar{p}} - \frac{1}{q} < \frac{q-1}{q(n+1)}$ and $\|\beta\|_{L^q([a, b])} < \infty$, we have

$$\begin{aligned} & \int_0^1 \sup_{z \in D} \|\beta(\tau) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \\ & \leq |U|^{1/q} \|\beta\|_{L^q([a, b])} \int_0^1 t^{k-n/q} (1-t)^{-(n+1)/p} \min(t^{n/q}, (1-t)^{n/q}) dt. \end{aligned}$$

On the other hand, since $\|\tau \beta(\tau)\|_{L^q([a, b])} < \infty$, we have by Corollary 4.4:

$$\begin{aligned} & \int_0^1 \sup_{z \in D} \| |x| \beta(\tau) \mathbf{1}_{tx+(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \\ & = \int_0^1 \sup_{z \in D} \left\| \sqrt{\tau^2 + w^2} \beta(\tau) \mathbf{1}_{tx+(1-t)D}(z) \right\|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{2} \int_0^1 \sup_{z \in D} \|(|\tau| + |w|)\beta(\tau) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \\
&\leq \sqrt{2} \int_0^1 \sup_{z \in D} \|\tau\beta(\tau) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \\
&\quad + \sqrt{2} \int_0^1 \sup_{z \in D} \|w\beta(\tau) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \\
&\leq \sqrt{2} |U|^{1/q} \|\tau\beta(\tau)\|_{L^q([a,b])} \int_0^1 t^{k-n/q} (1-t)^{-(n+1)/\bar{p}} \min(t^{n/q}, (1-t)^{n/q}) dt \\
&\quad + \sqrt{2} \sup_{w \in U} |w| \|\beta\|_{L^q([a,b])} \int_0^1 t^{k-n/q} (1-t)^{-(n+1)/\bar{p}} \min(t^{n/q}, (1-t)^{n/q}) dt < \infty.
\end{aligned}$$

The relations $\|\tau\beta(\tau)\|_{L^q([a,b])} < \infty$ and $\|x\beta(\tau)\|_{L^q(D)} < \infty$ enable us to apply Corollary 5.4 and obtain the desired assertion. $\square$

6. GLOBALIZATION: THE SOBOLEV–POINCARÉ INEQUALITY ON A CYLINDER

Here we globalize the Sobolev–Poincaré inequality to cylinders. The main assertion of the section is

Theorem 6.1. *Suppose that M is the cylinder $[a, b) \times N$, where N is a closed manifold of dimension n , $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, and $\beta, \gamma : [a, b) \rightarrow \mathbb{R}$ be positive smooth functions. Let ω be an exact k -form in $C^\infty L^p(M, \Lambda^k, \gamma)$. If the conditions $\|\beta\|_{L^q([a,b])} < \infty$, $\|t\beta(t)\|_{L^q([a,b])} < \infty$, and $\|\gamma^{-1}\|_{L^{p\bar{p}/(p-\bar{p})}([a,b])} < \infty$ are fulfilled for some $\bar{p}$, $1 \leq \bar{p} \leq p$ (for $\bar{p} = p$, we put $\frac{p\bar{p}}{p-\bar{p}} = \infty$), then there exists a $(k-1)$ -form $\xi \in C^\infty L^q(M, \Lambda^{k-1}, \beta)$ such that*

$$(6.1) \quad d\xi = \omega \quad \text{and} \quad \|\xi\|_{L^q(M, \Lambda^{k-1}, \beta)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^k, \gamma)}.$$

Let $\tilde{\mathcal{U}} = \{\tilde{U}_x\}$, $x \in N$, be a coordinate open cover of the base N . At each point $x \in N$, consider a geodesic ball U_x that is geodesically convex (small balls are geodesically convex, see [7, Proposition 4.2]) and such that its closure (a compact set) is contained in $\tilde{U}_x$. Then $\mathcal{U}' = \{\tilde{U}_x\}$ is an open cover of N . Extract a finite subcover $\mathcal{U} = \{U_i\}$, $i = 1, \dots, l$, from $\mathcal{U}'$. Since $\mathcal{U}$ consists of geodesic balls, it is a *good cover*, i.e., all finite intersections $U_I = U_{i_0} \cap \dots \cap U_{i_{s-1}}$, $I = (i_0, \dots, i_{s-1})$, are bi-Lipschitz diffeomorphic to convex open sets with compact closure in $\mathbb{R}^n$. With such a cover $\mathcal{U}$, we associate the corresponding cover $\mathcal{V} = \{V_i = [a, b) \times U_i\}$, $i = 1, \dots, l$, of M and put $V_I = V_{i_0} \cap \dots \cap V_{i_{s-1}}$ for $I = (i_0, \dots, i_{s-1})$. Then each intersection V_I is bi-Lipschitz diffeomorphic to a cylinder of the form $[a, b) \times U_{\mathbb{R}^n}$, where $U_{\mathbb{R}^n}$ is a convex set with compact closure in $\mathbb{R}^n$. By analogy with [23], we put

$$K^{k,0} := C^\infty(M, \Lambda^k); \quad K^{k,s} := \bigoplus_{i_0 < \dots < i_{s-1}} C^\infty(V_I, \Lambda^k).$$

Given $\varkappa \in K^{r,s}$, denote by $\varkappa_I$, $I = (i_0, \dots, i_{s-1})$, $i_0 < \dots < i_s$, the components of $\varkappa$. Define a coboundary operator $\delta : K^{k,s} \rightarrow K^{k,s+1}$ as follows:

$$(\delta \varkappa)_J = \left(\sum_{r=0}^s (-1)^r \varkappa_{j_0 \dots \hat{j}_r \dots j_s} \right) \Big|_{V_J}, \quad J = (j_0, \dots, j_s).$$

Let $L^q(K^{k,s})$ be the space of elements $\varkappa \in K^{k,s}$ with the finite norm

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0 < \dots < i_{s-1}} \|\varkappa_I\|_{L^q(V_I, \Lambda^k, \beta)}.$$

As usual, if $\varkappa \in K^{k,s}$ has components $\varkappa_I$, $I = (i_0, \dots, i_{s-1})$, $i_0 < \dots < i_s$, and ν is a permutation of the set $\{0, \dots, s-1\}$ then $\alpha_{\nu(I)} = \alpha_I \text{sign } \nu$.

The following proposition is a modification for our case of [23, Proposition 3.6], which is in turn an adaptation of [3, Propositions 8.3 and 8.5].

Proposition 6.2. *$(K^{k,\bullet}, \delta)$ is an exact complex. Moreover, if $\lambda \in L^q(K^{k,s+1}, \beta)$ satisfies $\delta\lambda = 0$ then there exists $\varkappa \in L^q(K^{k,s}, \beta)$ such that $\lambda = \delta\varkappa$ and*

- $\|\varkappa\|_{L^q(K^{k,s},\beta)} \leq \text{const} \|\lambda\|_{L^q(K^{k,s+1},\beta)}$
- $\|d\varkappa\|_{L^q(K^{k+1,s},\beta)} \leq \text{const} (\|\lambda\|_{L^q(K^{k,s+1},\beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1},\beta)}).$

Proof. The fact that $(K^{k,\bullet}, \delta)$ is an exact complex was established in [3, Propositions 8.3 and 8.5] but we will give the standard argument for completeness. If $\varkappa \in L^q(K^{k,s}, \beta)$ then

$$\begin{aligned} (\delta(\delta\varkappa))_{i_0 \dots i_{s+1}} &= \sum_r (-1)^i (\delta\varkappa)_{i_0 \dots \hat{i}_r \dots i_{s+1}} \\ &= \sum_{l < r} (-1)^r (-1)^l \varkappa_{i_0 \dots \hat{i}_l \dots \hat{i}_r \dots i_{s+1}} + \sum_{l < r} (-1)^r (-1)^{l-1} \varkappa_{i_0 \dots \hat{i}_l \dots \hat{i}_r \dots i_{s+1}} = 0. \end{aligned}$$

Suppose that $\lambda \in L^q(K^{k,s+1}, \beta)$ is such that $\delta\lambda = 0$. Let $\tilde{\rho}_j$ be a partition of unity subordinate to the cover $\{U_i\}$ of N . Then the functions $\rho_j : M \rightarrow \mathbb{R}$, $\rho_j(t, x) = \tilde{\rho}_j(x)$ for all $(t, x) \in M = [a, b) \times N$, constitute a partition of unity subordinate to the cover $\{V_i\}$ of M . Put

$$(6.2) \quad \varkappa_{i_0 \dots i_{s-1}} := \sum_j \rho_j \lambda_{j i_0 \dots i_{s-1}}.$$

Show that $\delta\varkappa = \lambda$.

We have

$$(\delta\varkappa)_{i_0 \dots i_s} = \sum_r (-1)^r \varkappa_{i_0 \dots \hat{i}_r \dots i_s} = \sum_{r,j} (-1)^r \rho_j \lambda_{j i_0 \dots \hat{i}_r \dots i_s}.$$

Since λ is a cocycle,

$$(\delta\lambda)_{j i_0 \dots i_s} = \lambda_{i_0 \dots i_s} + \sum_r (-1)^{r+1} \lambda_{j i_0 \dots \hat{i}_r \dots i_s} = 0$$

Hence,

$$(\delta\varkappa)_{i_0 \dots i_s} = \sum_j \rho_j \sum_r (-1)^r \lambda_{j i_0 \dots \hat{i}_r \dots i_s} = \sum_j \rho_j \lambda_{i_0 \dots i_s} = \lambda_{i_0 \dots i_s}.$$

Thus, $(K^{k,\bullet}, \delta)$ is indeed an exact complex.

The element $\varkappa$ defined by (6.2) admits the estimates of the norms mentioned in the proposition.

Indeed, we infer

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0 < \dots < i_{s-1}} \left\| \sum_j \rho_j \lambda_{j i_0 \dots i_{s-1}} \right\|_{L^q(U_I, \Lambda^k, \beta)}$$

$$\begin{aligned}
&\leq \sum_{i_0 < \dots < i_{s-1}} \sum_j \|\rho_j \lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_I, \Lambda^k, \beta)} \\
&\leq \sum_{i_0 < \dots < i_{s-1}} \sum_j \|\lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_{j,I}, \Lambda^k, \beta)} \leq \|\lambda\|_{L^q(K^{k,s+1}, \beta)},
\end{aligned}$$

which gives the first estimate of the proposition.

Let us prove the second estimate. We have

$$d\lambda_{i_0 \dots i_{s-1}} = \sum_j (d\rho_j \wedge \lambda_{j i_0 \dots i_{s-1}} + \rho_j d\lambda_{j i_0 \dots i_{s-1}}).$$

Therefore,

$$\begin{aligned}
\|\lambda\|_{L^q(K^{k+1,s}, \beta)} &= \sum_{i_0 < \dots < i_{s-1}} \left\| \sum_j d\rho_j \wedge \lambda_{j i_0 \dots i_{s-1}} + \rho_j d\lambda_{j i_0 \dots i_{s-1}} \right\|_{L^q(U_I, \Lambda^{k+1}, \beta)} \\
&\leq \sum_{i_0 < \dots < i_{s-1}} \sum_j (\|d\rho_j \wedge \lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_I, \Lambda^{k+1}, \beta)} + \|\rho_j d\lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_I, \Lambda^{k+1}, \beta)}) \\
&\leq \text{const} \sum_{i_0 < \dots < i_{s-1}} \sum_j (\|\lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_I, \Lambda^k, \beta)} + \|d\lambda_{j i_0 \dots i_{s-1}}\|_{L^q(U_I, \Lambda^{k+1}, \beta)}) \\
&= \text{const} (\|\lambda\|_{L^q(K^{k,s+1}, \beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1}, \beta)}).
\end{aligned}$$

□

Now, applying the general scheme of [23], we first construct some elements $\xi^s \in L^q(K^{k-s-1,s+1}, \beta)$ and then elements $x^s \in L^q(K^{k-s-1,s}, \beta)$ such that $\xi = x^0 \in C^\infty L^q(M, \Lambda^{k-1}, \beta)$ is an element satisfying the claim of Theorem 6.1.

Construction of the elements $\xi^s \in L^q(K^{k-s-1,s+1}, \beta)$.

Put $\xi^{-1} = \omega$ and define (by induction) ξ^s by setting its component $(\xi^s)_I$ to be a solution to the equation

$$(6.3) \quad d\xi_I^s = (\delta\xi^{s-1})_I$$

in V_I , $I = (i_0, \dots, i_s)$ such that

$$(6.4) \quad \|\xi_I^s\|_{L^q(V_I, \Lambda^{k-s-1}, \beta)} \leq \text{const} \|(\delta\xi^{s-1})_I\|_{L^q(V_I, \Lambda^{k-s}, \beta)}$$

for $0 \leq s \leq k-1$.

Note that such a solution always exists due to the local Sobolev–Poincaré inequality (Corollary 5.5) since V_I is bi-Lipschitz diffeomorphic to a cylinder over a convex subset in $\mathbb{R}^n$ with compact closure.

We have the following estimate of the weighted q -norm of ξ^s :

Proposition 6.3. *If $I = (i_0, \dots, i_s)$ then*

$$\|\xi_I^s\|_{L^q(V_I, \Lambda^{k-s-1}, \beta)} \leq \text{const} \|\omega\|_{L^q(M, \Lambda^k, \gamma)}.$$

Proof. Use induction on s . For $s = 0$, the assertion follows from the local Sobolev–Poincaré inequality. Let now $s > 0$. We infer

$$\begin{aligned}
\|\xi_I^s\|_{L^q(V_I, \Lambda^{k-s-1}, \beta)} &\leq \text{const} \|(\delta\xi^{s-1})_I\|_{L^q(V_I, \Lambda^{k-s}, \beta)} \\
&\leq \text{const} \sum_{r=0}^s \|\xi_{i_0 \dots \hat{i}_r \dots i_{s-1}}^{s-1}\|_{L^q(V_I, \Lambda^{k-s}, \beta)}
\end{aligned}$$

$$\begin{aligned}
&\leq \text{const} \sum_{r=0}^s \|\xi_{i_0 \dots \hat{i}_r \dots i_{s-1}}^{s-1}\|_{L^q(V_{i_0 \dots \hat{i}_r \dots i_{s-1}}, \Lambda^{k-s}, \beta)} \\
&\leq \text{const} \sum_{r=0}^s \|\omega\|_{L^p(M, \Lambda^k, \gamma)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^k, \gamma)}
\end{aligned}$$

□

Note that ξ^{k-1} is a collection of 0-forms satisfying the condition $d\xi^{k-1} = 0$. Thus, the functions $(\delta\xi^{k-1})_I$ are constants on each set V_I , $I = (i_0, \dots, i_k)$. The global constant functions $(\delta\xi^{k-1})_I$ on M belong to $L^q(M, \beta)$ due to the hypotheses on β .

The following assertion is Theorem 3.10 in [23]:

Lemma 6.4. *There exists $c \in K^{0,k}$ with constant components c_I , $I = (i_0, \dots, i_{k-1})$, such that*

$$(\delta c)_I = \sum_{r=0}^k (-1)^r c_{i_0 \dots \hat{i}_r \dots i_k} (\delta\xi^{k-1})_I, \quad I = (i_0, \dots, i_k).$$

In addition, there exist numbers $b_{I,L} \in \mathbb{R}$, $I = (i_0, \dots, i_{k-1})$, $L = (i_0, \dots, i_k)$, such that

$$c_I = \sum_L b_{I,L} (\delta\xi^{k-1})_L,$$

where $b_{i,L}$ depend on the chosen cover $\mathcal{U}$ of N .

We have

Proposition 6.5. *The constants c_I of Lemma 6.4 satisfy the estimate*

$$\|c_I\|_{L^q(V_I, \beta)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^k, \gamma)}$$

Proof. By Lemma 6.4, each c_I is representable as $c_I = \sum_L b_{I,L} (\delta\xi^{k-1})_L$. Hence,

$$\|c_I\|_{L^q(V_I, \beta)} \leq \sum_L |b_{I,L}| \|(\delta\xi^{k-1})_L\|_{L^q(V_I, \beta)}.$$

Since $(\delta\xi^{k-1})_L$ is a globally defined constant function on M as in the proof of Proposition 6.3, we have

$$\begin{aligned}
\|(\delta\xi^{k-1})_L\|_{L^q(V_I, \beta)} &= \frac{\|\beta\|_{L^q([a,b])} (\text{vol}(U_I))^{1/q}}{\|\beta\|_{L^q([a,b])} (\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L, \beta)} \\
&= \frac{(\text{vol}(U_I))^{1/q}}{(\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L, \beta)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^k, \gamma)}.
\end{aligned}$$

This gives the estimate of the proposition. □

Construction of the elements $x^s \in L^q(K^{k-s-1, s}, \beta)$.

Let us now glue all the forms ξ^s , $s=0, \dots, k-1$, into a global form ξ satisfying (6.1). Construct by induction elements $x^s \in L^q(K^{k-s-1, s}, \beta)$, $s=k-1, \dots, 1, 0$, such that $\xi = x^0$ is a desired form on M .

Put $\tilde{\xi}_I^{k-1} = \xi_I^{k-1} - c_I$, where c_I is as in Lemma 6.4, $I = (i_0, \dots, i_{k-1})$. We have $d\tilde{\xi}_I^{k-1} = d\xi_I^{k-1}$ and $\delta\tilde{\xi}_I^{k-1} = 0$. By Proposition 6.2, there exists $x^{k-1} \in L^q(K^{0, k-1}, \beta)$ such that $\delta x^{k-1} = \tilde{\xi}^{k-1}$ and

$$\|dx^{k-1}\|_{L^q(K^{1, k-1}, \beta)} \leq \text{const} \|\tilde{\xi}^{k-1}\|_{L^q(K^{0, k}, \beta)},$$

$$\|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \leq \text{const} \left(\|\tilde{\xi}^{k-1}\|_{L^q(K^{0,k},\beta)} + \|d\tilde{\xi}^{k-1}\|_{L^q(K^{1,k},\beta)} \right).$$

Propositions 6.3 and 6.5 yield

$$(6.5) \quad \|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}$$

and

$$(6.6) \quad \begin{aligned} \|dx^{k-1}\|_{L^q(K^{1,k-1},\beta)} &\leq \text{const} (\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\delta\xi^{k-2}\|_{L^q(K^{1,k},\beta)}) \\ &\leq \text{const} (\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\xi^{k-2}\|_{L^q(K^{1,k-1},\beta)}) \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned}$$

Suppose that $x^{k-(r-1)}$ is already constructed. By Proposition 6.2, there exists x^{k-r} such that

$$\delta x^{k-r} = \xi^{k-r} - dx^{k-r+1},$$

where

$$(6.7) \quad \|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \leq \text{const} \|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}$$

and

$$(6.8) \quad \begin{aligned} \|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} &\leq \text{const} (\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|d\xi^{k-r}\|_{L^q(K^{r,k-r+1},\beta)}) \\ &\leq \text{const} (\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \\ &\quad + \|\delta\xi^{k-r-1}\|_{L^q(K^{r,k-r+1},\beta)}) \\ &\leq \text{const} (\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}). \end{aligned}$$

Here the last inequality stems from the fact that

$$\delta(\xi^{k-r} - dx^{k-r+1}) = \delta\delta x^{k-r} = 0.$$

The above considerations imply the following

Proposition 6.6. *The forms x^s admit the estimates:*

- (1) $\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}$;
- (2) $\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}$.

Proof. Use induction on r . For $r = 1$, (1) and (2) are just estimates (6.5) and (6.6). Assume that $r > 1$. For proving estimate (2), observe that, by the induction hypothesis and (6.8),

$$\begin{aligned} \|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} &\leq \text{const} (\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}) \\ &\leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned}$$

Now, Proposition 6.3 and estimates (6.7) and (2) yield

$$\begin{aligned} \|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} &\leq \text{const} \|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \\ &\leq \text{const} (\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}) \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned}$$

□

Finally, put $\xi = x^0$. Then $d\xi = \omega$. Indeed, we have

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

Since $\delta(\omega - dx^0)_i = (\omega - dx^0)|_{V_i}$, we infer that $\omega = dx_0$ on M . By Proposition 6.6,

$$\|\xi\|_{L^q(M,\Lambda^{k-1},\beta)} = \|x^0\|_{L^q(K^{k-1,0},\beta)} \leq \text{const} \|\omega\|_{L^p(M,\Lambda^k,\gamma)}.$$

Theorem 6.1 is completely proved.

7. $L_{q,p}$ -COHOMOLOGY OF A TWISTED CYLINDER

Theorem 7.1. *Suppose that N is a closed manifold of dimension n , $H_{\text{DR}}^k(N) = 0$, $q \geq p \geq 1$, and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$. If*

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b])} < \infty, \quad \|t \max(F_{k-2,q}, F_{k-1,q})(t)\|_{L^q([a,b])} < \infty,$$

and

$$\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^{\frac{p\bar{p}}{p-\bar{p}}}([a,b])} < \infty$$

for some $\bar{p}$, $1 \leq \bar{p} \leq p$ (for $\bar{p} = p$, we put $\frac{p\bar{p}}{p-\bar{p}} = \infty$), then $H_{q,p}^k(C_{a,b}^h N) = 0$.

Proof. Let $\bar{M}$ be the cylinder $[a,b] \times N$ with the usual product metric. By the Künneth formula for the de Rham cohomology, we have

$$H_{\text{DR}}^k(\bar{M}) = H_{\text{DR}}^k(N) = 0.$$

Let $\omega \in C^\infty L^p(C_{a,b}^h N, \Lambda^k)$. Using expression (3.2) for the norm and the definition of $f_{l,p}$, we infer

$$\begin{aligned} (7.1) \quad \|\omega\|_{L^p(\bar{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))} &= \left[\int_a^b \{\min(f_{k-1,p}(t), f_{k,p}(t))\}^p \int_N (|\omega_A(t, x)|_N^2 + |\omega_B(t, x)|_N^2)^{\frac{p}{2}} dx dt \right]^{\frac{1}{p}} \\ &\leq \left[\int_a^b \int_N (h^{2(\frac{n}{p}-k)}(t, x) |\omega_A(t, x)|_N^2 + h^{2(\frac{n}{p}-k+1)}(t, x) |\omega_B(t, x)|_N^2)^{\frac{p}{2}} dx dt \right]^{\frac{1}{p}} \\ &= \|\omega\|_{L^p(C_{a,b}^h N, \Lambda^k)}. \end{aligned}$$

Thus, $\omega \in C^\infty L^p(\bar{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))$. Since the de Rham cohomology $H_{\text{DR}}^k(\bar{M})$ is trivial, ω is exact, and we can apply Theorem 6.1, by which there exists $\xi \in C^\infty L^q(\bar{M}, \Lambda^k, \max(F_{k-2,q}(t), F_{k-1,q}(t)))$ with

$$(7.2) \quad \|\xi\|_{L^q(\bar{M}, \Lambda^{k-1}, \max(F_{k-2,q}, F_{k-1,q}))} \leq \text{const} \|\omega\|_{L^p(\bar{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))}.$$

For this form ξ , we have

$$\begin{aligned} (7.3) \quad \|\xi\|_{L^q(C_{a,b}^h N, \Lambda^{k-1})} &= \left[\int_a^b \int_N (h^{2(\frac{n}{q}-k+1)}(t, x) |\xi_A(t, x)|_N^2 + h^{2(\frac{n}{q}-k+2)}(t, x) |\xi_B(t, x)|_N^2)^{\frac{q}{2}} dx dt \right]^{\frac{1}{q}} \\ &\leq \left[\int_a^b \{\max(F_{k-2,q}(t), F_{k-1,q}(t))\}^q \int_N (|\xi_A(t, x)|_N^2 + |\xi_B(t, x)|_N^2)^{\frac{q}{2}} dx dt \right]^{\frac{1}{q}} \\ &= \|\xi\|_{L^q(\bar{M}, \Lambda^{k-1}, \max(F_{k-2,q}, F_{k-1,q}))}. \end{aligned}$$

Combining (7.1), (7.2), and (7.3), we obtain

$$\|\xi\|_{L^q(C_{a,b}^h N, \Lambda^{k-1})} \leq \text{const} \|\omega\|_{L^p(C_{a,b}^h N, \Lambda^k)}.$$

Thus, $C^\infty H_{q,p}^k(C_{a,b}^h N) = 0$, and hence, by Theorem 2.5, also $H_{q,p}^k(C_{a,b}^h N) = 0$. $\square$

Remark 7.2. As was observed in the introduction, in [18], the second author established sufficient conditions for the L_{qp} -cohomology of a warped cylinder to be infinite-dimensional in terms of “almost duality” and Hardy’s inequality. In [18], the base manifold must have nontrivial L_p - and L_q -cohomology.

8. $L_{q,p}$ -COHOMOLOGY OF AN ASYMPTOTIC TWISTED CYLINDER

Recall the following definition, given in [10]:

Definition 8.1. We refer to a pair (M, X) consisting of an m -dimensional manifold M and an m -dimensional compact submanifold X with boundary as an *asymptotic twisted cylinder* $AC_{a,b}^h \partial X$ if $M \setminus (X \setminus \partial X)$ is bi-Lipschitz diffeomorphically equivalent to the twisted cylinder $C_{a,b}^h \partial X$.

For asymptotic twisted cylinders, Theorem 7.1 gives:

Theorem 8.2. Let $(M, X) = AC_{a,b}^h \partial X$ be an asymptotic twisted cylinder with $\dim M = \dim X = m = n + 1$. Assume that $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{qm}$, and $H_{\text{DR}}^k(\partial X) = 0$. If

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b])} < \infty, \quad \|t \max(F_{k-2,q}, F_{k-1,q})(t)\|_{L^q([a,b])} < \infty,$$

and

$$\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^{\frac{p\bar{p}}{p-\bar{p}}}([a,b])} < \infty$$

for some $\bar{p}$, $1 \leq \bar{p} \leq p$ (for $\bar{p} = p$, we put $\frac{p\bar{p}}{p-\bar{p}} = \infty$), then $H_{q,p}^k(M) = 0$.

Proof. Since bi-Lipschitz diffeomorphisms preserve L_{p_1} and L_{p_2} and extension by zero gives a topological isomorphism between the spaces $W_{p_1,p_2}^*(C_{a,b}^h \partial X)$ and $W_{p_1,p_2}^*(M)$ for all p_1, p_2 , we have a topological isomorphism

$$H_{p_1,p_2}^*(M) \cong H_{p_1,p_2}^*(C_{a,b}^h \partial X)$$

for all p_1, p_2 . The theorem now follows from Theorem 7.1. $\square$

9. EXAMPLES

Let us analyze the conditions of the last theorems for comparatively simple cases. Suppose that N is the n -dimensional sphere S^n . Then $H_{\text{DR}}^k(N) = 0$ for any $k \neq 0, n$. By the hypotheses of the theorems, $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$. Put

$$s(t) := \max_{x \in S^n} h(t, x) \quad \text{and} \quad g(t) := \min_{x \in S^n} h(t, x).$$

Then, by definition,

$$I_{1,q,k} := \max(F_{k-2,q}, F_{k-1,q}) = \max(s^{\frac{n}{q}-k+2}, s^{\frac{n}{q}-k+1}),$$

$$I_{2,q,k}(t) := t \max(F_{k-2,q}, F_{k-1,q})(t) = t \max(s^{\frac{n}{q}-k+2}(t), s^{\frac{n}{q}-k+1}(t))$$

and

$$I_{3,p,k} := \{\min(f_{k-1,p}, f_{k,p})\}^{-1} = \{\min(g^{\frac{n}{p}-k+1}, g^{\frac{n}{p}-k})\}^{-1}.$$

By the hypotheses of the theorems, we must check the integrability of these three functions in the corresponding degrees under the above-mentioned restrictions on p and q .

Suppose for simplicity that $s(t)$ and $g(t)$ are smooth increasing functions tending to ∞ as $t \rightarrow b - 0$. Denote the maximal integrability intervals for s^u and g^v by $(-\infty, \alpha)$ and $(-\infty, \beta)$, i.e. s^u is integrable on $[a, b)$ for every $u < \alpha$ and is

not integrable for every $u > \alpha$ and similarly for g^v . Let also α_1 be the supremum of μ such that $ts^\mu(t)$ is integrable on $[a, b)$.

For this case $I_{1,q,k} = s^{\frac{n}{q}-k+2}$, $I_{2,q,k}(t) = ts^{\frac{n}{q}-k+2}(t)$, and $I_{3,p,k} = g^{k-\frac{n}{p}}$.

The conditions of the theorems are fulfilled if

$$\frac{n}{q} - k + 2 < \min(\alpha, \alpha_1), \quad \frac{n}{p} - k > -\beta.$$

Note that these inequalities cannot hold simultaneously if $b = \infty$. In this case, α , α_1 , and β are all negative, whence $\frac{n}{p} - k > \frac{n}{q} - k + 2$. We thus have $\frac{1}{p} - \frac{1}{q} > \frac{2}{n}$, which contradicts the hypotheses.

Examine more closely the case of $0 \leq a < b < \infty$. The function t is bounded, and hence $\alpha = \alpha_1$. Therefore, the inequalities for $I_{1,q,k}$ and $I_{3,p,k}$ can be combined into one inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \leq \frac{1}{p} < \frac{k-\beta}{n}.$$

It means that the additional condition $k-2+\alpha \leq k-\beta$, i.e., $\alpha + \beta \leq 2$, must be fulfilled.

The last condition is $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, i.e., $p \leq q < \frac{np}{n+1-p}$.

Summarizing, we conclude that for known integrability limits α and β , we need to check two simple conditions for p and q :

$$\alpha + \beta \leq 2, \quad p < q < \frac{np}{n+1-p}$$

and the inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \leq \frac{1}{p} < \frac{k-\beta}{n}.$$

for the degree k .

Under these conditions, the cohomology of the twisted cylinder $C_{[a,b]}^h S^n$ vanishes.

For example, if $s(t) = g(t) = (b-t)^{-2}$ then $\alpha = \beta = 1/2$. For $p = 2$ we have $2 \leq q < 2\frac{n}{n-1}$.

The last inequality yields

$$(9.1) \quad \frac{k-3/2}{n} < \frac{1}{q} \leq \frac{1}{2} < \frac{k-1/2}{n}.$$

Let q be an arbitrary number in $(2, \frac{8}{3})$. Then the second inequality $q < \frac{2n}{n-1}$ gives us the constraint $n < q/(q-2)$. Since $q/(q-2) < 4$, we can take $n = 4$. We have $1/2 < (2k-1)/8$, i.e., $k > 2$. For $k = 3$, the leftmost inequality gives us the valid condition $3/8 < 1/q$. Note that if $q = 2$ then always $q < \frac{2n}{n-1}$. If n is even and $k = \frac{n}{2} + 1$ then all inequalities in (9.1) are fulfilled. Thus, we have

$$H_{q,2}^3(C_{[a,b]}^h S^4) = 0 \quad \text{if } q \in \left[2, \frac{8}{3}\right)$$

and

$$H_{2,2}^{l+1}(C_{[a,b]}^h S^{2l}) = 0, \quad l \geq 2.$$

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