E. M. Rudoy, Asymptotic modelling of bonded plates by a soft thin adhesive layer, *Sib. Elektron. Mat. Izv.*, 2020, Volume 17, 615–625

DOI: <https://doi.org/10.33048/semi.2020.17.040>

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use
<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 52.6.47.48

April 26, 2023, 16:31:37

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 17, стр. 615–625 (2020)

УДК 517.9

DOI 10.33048/semi.2020.17.040

MSC 74K20

ASYMPTOTIC MODELLING OF BONDED PLATES BY A SOFT
THIN ADHESIVE LAYER

E.M. RUDOV

ABSTRACT. In the present paper, a composite structure is considered. The structure is made of three homogeneous plates: two linear elastic adherents and a thin adhesive. It is assumed that elastic properties of the adhesive layer depend on its thickness ε as ε to the power of 3. Passage to the limit as ε goes to zero is justified and a limit model is found in which the influence of the thin adhesive layer is replaced by an interface condition between adherents. As a result, we have analog of the spring type condition in the plate theory. Moreover, a representation formula of the solution in the adhesive layer has been obtained.

Keywords: bonded structure, Kirchhoff-Love's plate, composite material, spring type interface condition, biharmonic equation.

1. INTRODUCTION

The characterization of interface conditions between bonded elastic media is a classical problem in solids mechanics (see, e.g., [1, 2, 3, 4]). This problem arises, when composite material should be modelled. Due to small thickness of a glue layer (or, so called, an adhesive) numerical computation of the solution of the corresponding boundary value problem can be very difficult because it requires fine meshes. In this situation, instead of the full model the approximate one with the interface condition between adherents is introduced. Many interface conditions are currently studied rather well from both mathematical and mechanical point of view for different models of solids mechanics: linear and nonlinear elasticity, piezoelectricity, magneto-electro-thermo-elasticity, delamination cracks (see, e.g., [5, 6, 7, 8, 9, 10, 11]).

RUDOV, E.M., ASYMPTOTIC MODELLING OF BONDED PLATES BY A SOFT THIN ADHESIVE LAYER.

© 2020 RUDOV E.M.

The work was supported by the Russian Foundation for Basic Research (grant 18-29-10007).

Received January, 21, 2020, published April, 17, 2020.

However, to our best knowledge, the problem of constructing an asymptotic expansion for bonded Kirchhoff-Love's plates has not been addressed in the existing literature.

In the present work, we consider a composite structure consisting of two plates glued together by a third one (adhesive layer) along some common interfaces. The structure is in equilibrium under the action of applied forces and equilibrium of each plates is described by biharmonic equations. The problem (case of pure bending) is formulated as a variational one. Namely, we consider a minimization problem of the energy functional over a set of admissible deflections of the composite plate in the space H^2 is considered (see, e.g., [12, 13]).

It is assumed that the elastic properties of the adhesive layer depend on its thickness ε as ε^3 . ε is the small parameter of the problem. But the elastic properties of the glued plates do not depend on ε and remain constants. The main goal of the present paper is to strictly mathematically justify the passage to the limit when ε tends to zero.

The result is a model in which the adhesive layer is replaced by conditions on the interface of two adherent plates. By analogy with elasticity (see, e.g., [4, 5, 14]) we obtain the so-called spring-type condition, which for plates means that the transverse forces acting on common interface are proportional to the jump of the deflections of the plates.

2. STATEMENT OF THE PROBLEM

Let $\Omega_{\pm} \subset \mathbb{R}^2$ be two disjoint domains with boundaries $\partial\Omega_{\pm}$. Denote by $S = \partial\Omega_- \cap \partial\Omega_+$ the common part of boundaries of domains $\Omega_{\pm}$ and assume that S is an interval, laying on x_2 -axis, i.e., $S = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \in (a, b)\}$, $a < b$; $\partial\Omega_{\pm} = \bar{\Gamma}_{D\pm} \cup \bar{\Gamma}_{N\pm}$, $\Gamma_{D\pm} \cap \Gamma_{N\pm} = \emptyset$, and $\bar{\Gamma}_{D\pm} \cap \bar{S} = \emptyset$.

To put an adhesive rectangular layer of the thickness $2\varepsilon d$ between the domains $\Omega_{\pm}$, we shift the domains $\Omega_{\pm}$ along the x_1 -axis by $\pm\varepsilon d$, respectively, i.e.,

$$\Omega_{\pm}^{\varepsilon} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = x_1 \pm \varepsilon d, y_2 = x_2, (x_1, x_2) \in \Omega_{\pm}\}.$$

Here d is a global characteristic length (for example, diameter of the union $\Omega_- \cup \Omega_+$); $\varepsilon > 0$ is a small dimensionless parameter. Then an adhesive between Ω_- and Ω_+ is described by

$$\Omega_m^{\varepsilon} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (-\varepsilon d, \varepsilon d), y_2 \in (a, b)\}.$$

Let $S_{\pm}^{\varepsilon}$ denote the common interfaces between the adherents $\Omega_{\pm}^{\varepsilon}$ and the adhesive Ω_m^{ε} , i.e., $S_{\pm}^{\varepsilon} = \partial\Omega_m^{\varepsilon} \cap \partial\Omega_{\pm}^{\varepsilon}$, where

$$S_{\pm}^{\varepsilon} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = \pm\varepsilon d, y_2 \in (a, b)\}.$$

Let $\Omega^{\varepsilon} = \Omega_-^{\varepsilon} \cup \Omega_+^{\varepsilon} \cup S_-^{\varepsilon} \cup S_+^{\varepsilon} \cup \Omega_m^{\varepsilon}$ be a middle surface of the composite structure (a heterogeneous plate), consisting of middle surfaces $\Omega_{\pm}$ of two homogeneous plates, which are glued together by the thin plate with a middle surface Ω_m^{ε} . We assume that the thickness of all plates is equal to h (see Fig. 1).

Let $E_{\pm}$, E_m^{ε} and $k_{\pm}$, k_m be Young's modulus and Poisson's ratio corresponding to $\Omega_{\pm}$, Ω_m^{ε} , respectively, such that $E_{\pm}$, $k_{\pm}$, and k_m are constant, while Young's modulus E_m^{ε} depends on ε as follows

$$E_m^{\varepsilon} = \varepsilon^3 E_m,$$

where $E_m = \text{const}$. It means that we deal with a soft adhesive.

FIG. 1. Middle surface of the composite structure

We introduce the following notation for the bending stiffness

$$\mu_{\pm} = \frac{E_{\pm} h^3}{12(1 - k_{\pm}^2)}, \quad \mu_m^\varepsilon = \varepsilon^3 \mu_m$$

with $\mu_m = \frac{E_m h^3}{12(1 - k_m^2)}$.

Let u_ε be deflections of Ω^ε . We prescribe the homogeneous Dirichlet conditions on $\Gamma_{D\pm}^\varepsilon \subset \partial\Omega^\varepsilon$, where

$$\Gamma_{D\pm}^\varepsilon = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = x_1 \pm \varepsilon d, y_2 = x_2, (x_1, x_2) \in \Gamma_{D\pm}\}$$

. We assume that the composite plate is in equilibrium under acting of an external force $f \in L_2(\Omega^\varepsilon)$ such that $f = 0$ a.e. in Ω_m^ε .

Let us denote by μ^ε the bending stiffness and by k the Poisson ratio of the composite plate, where $\mu^\varepsilon = \mu_{\pm}$ and $k = k_{\pm}$ in $\Omega_{\pm}^\varepsilon$, and $\mu^\varepsilon = \mu_m^\varepsilon$ and $k = k_m$ in Ω_m^ε .

We will formulate the equilibrium problem of the composite plate with the middle surface Ω^ε as a variational problem. Let us define a Sobolev space

$$H^{2,0}(\Omega^\varepsilon) = \{v \in H^2(\Omega^\varepsilon) \mid v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_{D\pm}^\varepsilon\};$$

the energy functional

$$\Pi(v) = \frac{1}{2} B^\varepsilon(v, v) - l(v)$$

with

$$B^\varepsilon(v, w) = \int_{\Omega^\varepsilon} \mu^\varepsilon (v_{,11} w_{,11} + v_{,22} w_{,22} + k(v_{,11} w_{,22} + v_{,22} w_{,11}) + 2(1 - k)v_{,12} w_{,12}) dy,$$

$$l(v) = \int_{\Omega^\varepsilon} f v dy.$$

Then the equilibrium problem is as follows: find a function $u_\varepsilon \in H^{2,0}(\Omega^\varepsilon)$ such that

$$(1) \quad \Pi(u_\varepsilon) = \inf_{v \in H^{2,0}(\Omega^\varepsilon)} \Pi(v),$$

or, equivalently,

$$(2) \quad B^\varepsilon(u^\varepsilon, v) = l(v) \quad \forall v \in H^{2,0}(\Omega^\varepsilon).$$

3. DECOMPOSITION OF PROBLEM (1)

Let us reformulate problem (1) in an equivalent form. Namely, we decompose it into three subproblems defined in domains $\Omega_\pm^\varepsilon$, Ω_m^ε and connected along the common interfaces $S_\pm^\varepsilon$. Let us introduce a set

$$K_\varepsilon = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega_-^\varepsilon) \times H^{2,0}(\Omega_+^\varepsilon) \times H^2(\Omega_m^\varepsilon) \mid v_\pm = v_m, \frac{\partial v_\pm}{\partial n} = \frac{\partial v_m}{\partial n} \text{ on } S_\pm^\varepsilon, \}$$

where

$$H^{2,0}(\Omega_\pm^\varepsilon) = \{v_\pm \in H^2(\Omega_\pm^\varepsilon) \mid v_\pm = \frac{\partial v_\pm}{\partial n} = 0 \text{ on } \Gamma_{D\pm}^\varepsilon\};$$

and define bilinear forms

$$b_\pm^\varepsilon(v_\pm, u_\pm) = \mu_\pm \int_{\Omega_\pm^\varepsilon} (v_{,11}w_{,11} + v_{,22}w_{,22} + k_\pm(v_{,11}w_{,22} + v_{,22}w_{,11}) + 2(1 - k_\pm)v_{,12}w_{,12}) dy,$$

$$b_m^\varepsilon(v_m, u_m) = \mu_m \int_{\Omega_m^\varepsilon} (v_{,11}w_{,11} + v_{,22}w_{,22} + k_m(v_{,11}w_{,22} + v_{,22}w_{,11}) + 2(1 - k_m)v_{,12}w_{,12}) dy.$$

Then the problem (2) can be reformulate equivalently as follows: find a triplet $(u_{\varepsilon-}, u_{\varepsilon+}, v_{\varepsilon m}) \in K_\varepsilon$ such that

$$(3) \quad b_-^\varepsilon(u_{\varepsilon-}, v_-) + b_+^\varepsilon(u_{\varepsilon+}, v_+) + \varepsilon^3 b_m^\varepsilon(u_{\varepsilon m}, v_m) = l_-^\varepsilon(v_-) + l_+^\varepsilon(v_+) \quad \forall (v_-, v_+, v_m) \in K_\varepsilon,$$

where

$$l_\pm^\varepsilon(v_\pm) = \int_{\Omega_\pm^\varepsilon} f_{\varepsilon\pm} v_\pm dy,$$

and $f_{\varepsilon\pm}$ is the restriction of the function f on domains $\Omega_\pm^\varepsilon$, respectively.

4. RESCALING AND ASYMPTOTIC EXPANSIONS

We change coordinates in each domains $\Omega_\pm^\varepsilon$ and Ω_m^ε to obtain domains which are independent of ε . Namely, we consider the following coordinate transformations:

$$(4) \quad x_1 = y_1 \mp \varepsilon d, \quad x_2 = y_2, \quad (x_1, x_2) \in \Omega_\pm, \quad (y_1, y_2) \in \Omega_\pm^\varepsilon,$$

$$(5) \quad z_1 = \frac{y_1}{\varepsilon}, \quad z_2 = y_2, \quad (z_1, z_2) \in \Omega_m, \quad (y_1, y_2) \in \Omega_m^\varepsilon,$$

where

$$\Omega_m = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \in (-d, d), \quad z_2 \in (a, b)\}.$$

Denote by

$$S_m^\pm = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 = \pm d, \quad z_2 \in (a, b)\},$$

$$S_m^a = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \in (-d, d), \quad z_2 = a\},$$

$$S_m^b = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \in (-d, d) \ z_2 = b\}$$

parts of $\partial\Omega_m$ of the domain Ω_m , i.e., $\partial\Omega_m = \overline{S^- \cup S^+ \cup S_m^a \cup S_m^b}$.

The advantage of decomposing the problem is that we have after coordinate transformations domains which are independent of ε unlike, for example, [4], where adherents was shifted from the adhesive by the distance εd .

Note that the coordinate transformations (4) and (5) are smooth and one-to-one for all $\varepsilon > 0$. It implies one-to-one correspondence between $H^{2,0}(\Omega_\pm^\varepsilon)$, $H^2(\Omega_m^\varepsilon)$ and $H^{2,0}(\Omega_\pm)$, $H^2(\Omega_m)$, respectively. Moreover, the set of admissible deflections K_ε is one-to-one transformed into a set K^ε , where

$$K^\varepsilon = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m) \mid v_\pm|_S = v_m|_{S_m^\pm}, \\ v_{\pm,1}|_S = \frac{1}{\varepsilon} v_{m,1}|_{S_m^\pm}\}.$$

Hereinafter, we assume that for any functions $v_\pm(x)$, $x \in \Omega_\pm$, and $v_m(z)$, $z \in \Omega_m$, equality $v_\pm|_S = v_m|_{S_m^\pm}$ means that

$$v_\pm(0, x_2) = v_m(\pm d, z_2), \quad x_2 = z_2 \in (a, b).$$

Apply coordinate transformations to integrals in (3). As a result we have that a triple $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon) \in K^\varepsilon$ is a solution of the following variational equality

$$(6) \quad b_-(u_-^\varepsilon, v_-) + b_+(u_+^\varepsilon, v_+) + B_m^\varepsilon(u_m^\varepsilon, v_m) = l_-(v_-) + l_+(v_+) \\ \forall V = (v_-, v_+, v_m) \in K^\varepsilon$$

where

$$u_\pm^\varepsilon(x_1, x_2) = u_{\varepsilon\pm}(x_1 \pm \varepsilon d, x_2), \quad u_m^\varepsilon(z_1, z_2) = u_{\varepsilon m}(\varepsilon z_1, z_2),$$

$$B_m^\varepsilon(v, w) = \mu_m \int_{\Omega_m} (v_{,11} w_{,11} + \varepsilon^4 v_{,22} w_{,22} + \varepsilon^2 k_m (v_{,11} w_{,22} + v_{,22} w_{,11}) + \\ + 2(1 - k_m) \varepsilon^2 v_{,12} w_{,12}) \, dz,$$

$$l_\pm(v_\pm) = \int_{\Omega_\pm} f_\pm^\varepsilon v_\pm \, dx, \quad f_\pm^\varepsilon(x_1, x_2) = f_\pm(x_1 \pm \varepsilon d, x_2).$$

We will need several auxiliary statements.

Lemma 1. *For any function $v_m \in H^1(\Omega_m)$ the following inequalities*

$$(7) \quad \|v_m\|_{L_2(\Omega_m)}^2 \leq C \left(\|v_{m,1}\|_{L_2(\Omega_m)}^2 + \|v_m\|_{L_2(S_m^\pm)}^2 \right)$$

hold, where a constant C does not depend on v_m .

Proof. This results from some direct integrations along x_1 -axis in the domain Ω_m . $\square$

Corollary 1. *For any function $v_m \in H^2(\Omega_m)$ the followings inequalities*

$$(8) \quad \|v_{m,1}\|_{L_2(\Omega_m)}^2 \leq C \left(\|v_{m,11}\|_{L_2(\Omega_m)}^2 + \|v_{m,1}\|_{L_2(S_m^\pm)}^2 \right),$$

$$(9) \quad \|v_m\|_{L_2(\Omega_m)}^2 \leq C \left(\|v_{m,11}\|_{L_2(\Omega_m)}^2 + \|v_{m,1}\|_{L_2(S_m^\pm)}^2 + \|v_m\|_{L_2(S_m^\pm)}^2 \right)$$

hold, where a constant C does not depend on v_m .

Proof. Inequality (8) results from inequality (7) for function $v_{m,1} \in H^1(\Omega_m)$. In turn, (9) follows from (7) and (8). $\square$

Corollary 2. *For any function $V = (v_-, v_+, v_m) \in K^\varepsilon$ and $\varepsilon \in (0, 1)$ the followings inequalities*

$$(10) \quad \|v_{m,1}\|_{L_2(\Omega_m)}^2 + \|v_m\|_{L_2(\Omega_m)}^2 \leq C \left(\|v_{m,1}\|_{L_2(\Omega_m)}^2 + \|v_\pm\|_{H^{2,0}(\Omega_\pm)}^2 \right)$$

hold, where a constant C does not depend on V and $\varepsilon > 0$.

Proof. Adding (8) and (9) and taking into account the continuity of the trace operator, we obtain (10) for all $\varepsilon \in (0, 1)$. $\square$

Now we introduce a set

$$K = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m) \mid v_{m,1}|_{S_m^\pm} = 0, v_m|_{S_m^\pm} = v_\pm|_S\}$$

and prove the following lemma.

Lemma 2. *For any function $V = (v_-, v_+, v_m) \in K$ there exists $V^\varepsilon = (v_-^\varepsilon, v_+^\varepsilon, v_m^\varepsilon) \in K^\varepsilon$ such that*

$$(11) \quad V^\varepsilon \rightarrow V \quad \text{strongly in } H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m).$$

Proof. Let us construct an extension of functions defined on $S_m^\pm$ in the nonsmooth domain Ω_m . Take any bounded domain $\tilde{\Omega}_m$ with a smooth boundary $\partial\tilde{\Omega}_m$ such that $\Omega_m \subset \tilde{\Omega}_m$ and $\partial\tilde{\Omega}_m \setminus \partial(\tilde{\Omega}_m \setminus \Omega_m) = S_m^- \cup S_m^+$.

Take a function $v_\pm \in H^{2,0}(\Omega_\pm)$ and fix $\varepsilon > 0$. Further we extend restrictions of functions $\pm v_{\pm,1} \in H^{1/2}(S_m^\pm)$ on whole the boundary $\partial\tilde{\Omega}_m$ (for instance, solving the mixed boundary value problem for the Laplace operator with the Dirichlet conditions $\pm v_{\pm,1}$ on $S_m^\pm$ and homogeneous Neumann conditions on $\partial\tilde{\Omega}_m \setminus (S_m^- \cup S_m^+)$). Denote this extension by $q \in H^{1/2}(\partial\tilde{\Omega}_m)$.

Next, in the smooth domain $\tilde{\Omega}_m$ we consider рассмотрим the following mixed boundary value problem for bilaplacian:

$$\begin{aligned} \Delta^2 p &= 0 \quad \text{in } \tilde{\Omega}_m, \\ p &= 0 \quad \text{on } \partial\tilde{\Omega}_m, \\ \frac{\partial p}{\partial n} &= q \quad \text{on } \partial\tilde{\Omega}_m, \end{aligned}$$

where n is a unit normal vector to $\partial\tilde{\Omega}_m$. Due to the definition q note that

$$\frac{\partial p}{\partial n} = \pm p_{,1} = \pm v_{\pm,1} \quad \text{on } S_m^\pm.$$

At last, denote by $\varphi_m \in H^2(\Omega_m)$ a restriction of p to the domain Ω_m . Note that the function φ_m satisfies the following conditions:

$$(12) \quad \varphi_m = 0 \quad \text{a.e. on } S_m^\pm,$$

$$(13) \quad \varphi_{m,1} = v_{\pm,1} \quad \text{a.e. on } S_m^\pm.$$

Finally, for the function $V \in K$ we put

$$V^\varepsilon = (v_-, v_+, v_m + \varepsilon \varphi_m).$$

It is obviously that due to (12), (13) the function V^ε belongs to the set K^ε and convergence (11) holds. $\square$

Lemma 3. Assume that for some $i, j = 1, 2$ the following convergences

$$\begin{aligned} u_m^\varepsilon &\rightarrow u_m \quad \text{weakly in } L_2(\Omega_m), \\ u_{m,i}^\varepsilon &\rightarrow p_i \quad \text{weakly in } L_2(\Omega_m), \\ u_{m,ij}^\varepsilon &\rightarrow q_{ij} \quad \text{weakly in } L_2(\Omega_m) \end{aligned}$$

hold as $\varepsilon \rightarrow 0$. Then $p_i = u_{m,i}$, $q_{ij} = u_{m,ij}$ a.e. in Ω_m .

Proof. Let us consider a relation

$$\int_{\Omega_m} u_m^\varepsilon \varphi_{,ij} dz = \int_{\Omega_m} u_{m,ij}^\varepsilon \varphi dz$$

which is valid for all $\varphi \in C_0^\infty(\Omega_m)$. Passing to the limit in this relation as $\varepsilon \rightarrow 0$ we have the following one

$$\int_{\Omega_m} u_m \varphi_{,ij} dz = \int_{\Omega_m} q_{ij} \varphi dz \quad \forall \varphi \in C_0^\infty(\Omega_m).$$

It implies that $q_{ij} = u_{m,ij}$ a.e. in Ω_m . Using similar reasoning, we can show that $p_i = u_{m,i}$ a.e. in Ω_m . $\square$

Lemma 4. Assume that for $\alpha > 0$, $\beta > 0$, and for $i, j = 1, 2$ the following convergences

$$\begin{aligned} u_m^\varepsilon &\rightarrow u_m \quad \text{weakly in } L_2(\Omega_m), \\ \varepsilon^\alpha u_{m,i}^\varepsilon &\rightarrow p_i \quad \text{weakly in } L_2(\Omega_m), \\ \varepsilon^\beta u_{m,ij}^\varepsilon &\rightarrow q_{ij} \quad \text{weakly in } L_2(\Omega_m) \end{aligned}$$

hold as $\varepsilon \rightarrow 0$. Then $p_i = 0$, $q_{ij} = 0$ a.e. in Ω_m .

Proof. Again let us consider the relation

$$\int_{\Omega_m} u_m^\varepsilon \varphi_{,ij} dz = \int_{\Omega_m} u_{m,ij}^\varepsilon \varphi dz$$

which is valid for all $\varphi \in C_0^\infty(\Omega_m)$. Multiplying this relation by ε^β and passing to the limit as $\varepsilon \rightarrow 0$ we have the following relation

$$0 = \int_{\Omega_m} q \varphi dz \quad \forall \varphi \in C_0^\infty(\Omega_m).$$

It implies that $q = 0$ a.e. in Ω_m . Using similar reasoning, we can show that $p = 0$ a.e. in Ω_m . $\square$

5. LIMIT PROBLEM

Now we are ready to justify the passage to the limit as $\varepsilon \rightarrow 0$. Let us substitute $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon)$ in (6) as test functions. As a result, we get the following estimate:

$$\begin{aligned} (14) \quad &\|u_-^\varepsilon\|_{H^{2,0}(\Omega_-)}^2 + \|u_+^\varepsilon\|_{H^{2,0}(\Omega_+)}^2 + \|u_{m,11}^\varepsilon\|_{L_2(\Omega_m)}^2 + \\ &+ \|\varepsilon u_{m,12}^\varepsilon\|_{L_2(\Omega_m)}^2 + \|\varepsilon^2 u_{m,22}^\varepsilon\|_{L_2(\Omega_m)}^2 \leq C, \end{aligned}$$

where by C we denote, as usual, a positive constant which is independent of ε . Moreover, due to (14) and Corollary 1 we have

$$(15) \quad \|u_m^\varepsilon\|_{L_2(\Omega_m)}^2 + \|u_{m,1}^\varepsilon\|_{L_2(\Omega_m)}^2 \leq C.$$

The inequalities (14), (15), and Corollary 2 imply that there exists subsequences and functions $u_{\pm}$, u_m , p , q such that

$$(16) \quad \begin{aligned} u_{\pm}^{\varepsilon} &\rightarrow u_{\pm} \quad \text{weakly in } H^{2,0}(\Omega_{\pm}), \\ u_m^{\varepsilon} &\rightarrow u_m \quad \text{weakly in } L_2(\Omega_m), \\ u_{m,1}^{\varepsilon} &\rightarrow u_{m,1} \quad \text{weakly in } L_2(\Omega_m), \\ u_{m,11}^{\varepsilon} &\rightarrow u_{m,11} \quad \text{weakly in } L_2(\Omega_m), \\ \varepsilon u_{m,12}^{\varepsilon} &\rightarrow p \quad \text{weakly in } L_2(\Omega_m), \\ \varepsilon^2 u_{m,22}^{\varepsilon} &\rightarrow q \quad \text{weakly in } L_2(\Omega_m). \end{aligned}$$

Note that from definition of the set K^{ε} it follows

$$(17) \quad u_{m,1} = 0 \quad \text{a.e. on } S_m^{\pm},$$

$$(18) \quad u_m|_{S_m^{\pm}} = u_{\pm}|_S.$$

Theorem 1. *Let $U = (u_-, u_+, u_m)$ be the limit function from (16). Then*

$$u_{m,11}(z_1, z_2) = -\frac{3(u_+(0, z_2) - u_-(0, z_2))}{2d^3} z_1 \quad \text{a.e. in } \Omega_m.$$

Proof. Take $\varphi \in C_0^{\infty}(\Omega_m)$ and substitute $(0, 0, \varphi) \in K^{\varepsilon}$ in (6) as a test function. As a result, we have

$$\int_{\Omega_m} u_{m,11} \varphi_{,11} dz = 0 \quad \forall \varphi \in C_0^{\infty}(\Omega_m).$$

It means that there exist two function $\alpha(z_2)$ and $\beta(z_2)$ such that

$$(19) \quad u_{m,11}(z_1, z_2) = \alpha(z_2)z_1 + \beta(z_2) \quad \text{a.e. in } \Omega_m.$$

Find functions α and β . On the one hand, due to (17) we have

$$\int_{-d}^d u_{m,11}(z_1, z_2) dz_1 = 0 \quad \text{a.e. on } (a, b).$$

On the other hand, from (19) we get

$$\int_{-d}^d (\alpha(z_2)z_1 + \beta(z_2)) dz_1 = 2d\beta(z_2) \quad \text{a.e. on } (a, b).$$

Thus, we get

$$\beta(z_2) = 0 \quad \text{a.e. on } (a, b).$$

In virtue of (17) we have

$$\int_{-d}^d u_{m,11}^2(z_1, z_2) dz_1 = \int_{-d}^d \alpha(z_2)z_1 u_{m,11}(z_1, z_2) dz_1 = -\alpha(z_2)(u_m(d, z_2) - u_m(-d, z_2))$$

a.e. on (a, b) . In the same time

$$\int_{-d}^d u_{m,11}^2(z_1, z_2) dz_1 = \int_{-d}^d (\alpha(z_2)z_1)^2 dz_1 = \frac{2}{3}\alpha^2(z_2)d^3$$

a.e. on (a, b) .

Thus, the following relation

$$\frac{2}{3}\alpha^2(z_2)d^3 = -\alpha(z_2)(u_m(d, z_2) - u_m(-d, z_2))$$

holds.

Let us assume that for some point $\bar{z}_2 \in (a, b)$ the function $\alpha(\bar{z}_2)$ is not equal to zero; then due to (18)

$$\alpha(\bar{z}_2) = -\frac{3(u_+(0, \bar{z}_2) - u_-(0, \bar{z}_2))}{2d^3}$$

Now suppose that there exists a set $Z_2 \subset (a, b)$ of nonzero measure such that $\alpha(z_2) = 0$ for all $z_2 \in Z_2$. Then $u_{m,11}(z_1, z_2) = 0$ a.e. in $(-d, d) \times Z_2$. It means that there exist functions $\gamma(z_2)$ and $\delta(z_2)$ such that $u_m(z_1, z_2) = \gamma(z_2)z_1 + \delta(z_2)$ a.e. in $(-d, d) \times Z_2$. Due to (17), $\gamma(z_2) = 0$ a.e. in Z_2 . In virtue of (18), $0 = u_m(d, z_2) - u_m(-d, z_2) = u_+(0, z_2) - u_-(0, z_2)$. Thus, we have the following condition: if $\alpha(z_2) = 0$ then $u_+(0, z_2) - u_-(0, z_2) = 0$. $\square$

Introduce a space K_0 , where

$$K_0 = \{V = (v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times L_2(\Omega_m) \mid \\ v_{m,1} \in L_2(\Omega_m), v_{m,11} \in L_2(\Omega_m), v_{m,1} = 0 \text{ a.e. on } S_m^\pm, v_m|_{S_m^\pm} = v_\pm|_S\},$$

endowed with a norm

$$\|V\|_{K_0}^2 = \|v_-\|_{H^{2,0}(\Omega_-)}^2 + \|v_+\|_{H^{2,0}(\Omega_+)}^2 + \|v_m\|_{L_2(\Omega_m)}^2 + \|v_{m,1}\|_{L_2(\Omega_m)}^2 + \|v_{m,11}\|_{L_2(\Omega_m)}^2.$$

Note that in virtue of Corollary 2 the norm $\|\cdot\|_{K_0}$ is equivalent to $|\cdot|_{K_0}$, where

$$|V|_{K_0}^2 = \|v_-\|_{H^{2,0}(\Omega_-)}^2 + \|v_+\|_{H^{2,0}(\Omega_+)}^2 + \|v_{m,11}\|_{L_2(\Omega_m)}^2.$$

Theorem 2. Let $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon)$ be the solution of (6). Let $U = (u_-, u_+, u_m) \in K_0$ be a solution the following variational equality:

$$(20) \quad b_-(u_-, v_-) + b_+(u_+, v_+) + \mu_m \int_{\Omega_m} u_{m,11} v_{m,11} dz = \int_{\Omega_-} f_- v_- dz + \int_{\Omega_+} f_+ v_+ dz$$

for all $V \in K_0$. Then

$$U^\varepsilon \rightarrow U \quad \text{weakly in } K_0.$$

Proof. Let $V \in K$ be an arbitrary function. Due to the Lemma 2 for $\varepsilon > 0$ there exists $V^\varepsilon \in K^\varepsilon$ strongly convergences to V in $H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m)$. Substitute V^ε in the variational equality (6) and pass to the limit as $\varepsilon \rightarrow 0$. Taking into account (16) and the fact that K is dense in K_0 , we get (20). $\square$

Now we reformulate problem (20) in an equivalent form, which does not contain the function u_m and find the representation formula for u_m . Namely, the following theorem holds.

Theorem 3. Let $U = (u_-, u_+, u_m)$ be a solution of (20). Then (u_-, u_+) is a solution to the following variational problem:

$$(21) \quad b_-(u_-, v_-) + b_+(u_+, v_+) + \frac{3\mu_m}{2d^3} \int_S (u_+ - u_-)(v_+ - v_-) ds = \\ = \int_{\Omega_-} f_- v_- dz + \int_{\Omega_+} f_+ v_+ dz$$

for all $(u_-, u_+) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+)$. Moreover, the following formula

$$(22) \quad u_m(z_1, z_2) = -\frac{u_+(0, z_2) - u_-(0, z_2)}{4d^3} z_1^3 + \frac{3(u_+(0, z_2) - u_-(0, z_2))}{4d} z_1 + \\ + \frac{u_+(0, z_2) + u_-(0, z_2)}{2} \quad \text{a.e. in } \Omega_m$$

holds.

Proof. Since $U \in K_0$, due to the Theorem 1 for any $v \in K_0$ we have

$$\int_{\Omega_m} u_{m,11} v_{m,11} dz = \frac{3}{2d^3} \int_S (u_+(0, z_2) - u_-(0, z_2))(v_+(0, z_2) - v_-(0, z_2)) dz_2.$$

The formula (22) it follows from Theorem 1 and properties (17) and (18). $\square$

Assuming that the solution (u_-, u_+) of (21) has additional regularity, by applying the generalized Green formula, we are about to deduce the following differential equations and boundary conditions for the functions u_- and u_+ :

$$(23) \quad \begin{aligned} \mu_{\pm} \Delta^2 u_{\pm} &= f_{\pm} \quad \text{in } \Omega_{\pm}, \\ u_{\pm} &= \frac{\partial u_{\pm}}{\partial n} = 0 \quad \text{on } \Gamma_{D\pm}, \\ m(u_{\pm}) &= t(u_{\pm}) = 0 \quad \text{on } \Gamma_{N\pm}, \\ m(u_{\pm}) &= 0, \quad t(u_{\pm}) + \frac{3\mu_m}{2d^3} (u_+ - u_-) = 0 \quad \text{on } S, \end{aligned}$$

where

$$m(u_{\pm}) = \mu_{\pm} \left(k_{\pm} \Delta u_{\pm} + (1 - k_{\pm}) \frac{\partial^2 u_{\pm}}{\partial \nu^2} \right) \quad \text{and} \quad t(u_{\pm}) = \mu_{\pm} \frac{\partial}{\partial \nu} \left(\Delta u_{\pm} + (1 - k_{\pm}) \frac{\partial^2 u_{\pm}}{\partial \tau^2} \right).$$

are bending moment and transverse force, respectively.

Note that the last condition in (23) is an analog of spring type interface condition which is widely used in elasticity and means that the transverse forces on the common interface of adherents are proportional to the jump of deflections. This explains the choice of the dependence of the elastic properties of the adhesive layer on ε as ε^3 .

REFERENCES

- [1] M. Goland, E. Reissner, *The stresses in cemented joints*, J. Appl. Mech., ASME, **11** (1944), A17–A27.
- [2] A. Klabring, *Derivation of a model of adhesively bonded joints by the asymptotic expansion method*, Int. J. Eng. Sci., **29**:4 (1991), 493–512. Zbl 0762.73069
- [3] H. Brezis, L.A. Caffarelli, A. Friedman, *Reinforcement problem for elliptic equations and variational inequalities*, Ann. Mat. Pura Appl., IV. Ser., **123** (1980), 219–246. Zbl 0434.35079

- [4] G. Geymonat, F. Krasucki, S. Lenci, *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math. Mech. Solids, **4**:2 (1999), 201–225. Zbl 1001.74591
- [5] Y. Benveniste, T. Miloh, *Imperfect soft and stiff interfaces in two-dimensional elasticity*, Mech Mater., **33** (2001), 309–323.
- [6] S. Dumont, F. Lebon, R. Rizzoni, *Imperfect interfaces with graded materials and unilateral conditions: theoretical and numerical study*, Math. Mech. Solids, **23**:3 (2018), 445–460. Zbl 1404.74123
- [7] E. Bonetti, G. Bonfanti, F. Lebon, *Derivation of imperfect interface models coupling damage and temperature*, Comput. Math. Appl., **77** (2009), 2906–2916. MR3958639
- [8] A.M. Khludnev, *On modeling elastic bodies with defects*, Sib. Electron. Mat. Izv., **15** (2018), 153–166. Zbl 1390.35354
- [9] M. Serpilli, *Asymptotic interface models in magneto-electro-thermoelastic composites*, Meccanica, **52**:6 (2017), 1407–1424. Zbl 1380.74076
- [10] A. Furtsev, H. Itou, E. Rudoy, *Modeling of bonded elastic structures by a variational method: Theoretical analysis and numerical simulation*, International Journal of Solids and Structures, **182–183** (2020), 100–111.
- [11] N.A. Kazarinov, E.M. Rudoy, V.Yu. Slesarenko, V.V. Shcherbakov, *Mathematical and numerical simulation of equilibrium of an elastic body reinforced by a thin elastic inclusion*, Comput. Math. Math. Phys., **58**:5 (2018), 761–774. Zbl 06920540
- [12] G. Sweers, *A survey on boundary conditions for the biharmonic*, Complex Var. Elliptic Equ., **54**:2 (2009), 79–93. Zbl 1171.35034
- [13] A.M. Khludnev A.M., J. Sokolowski, *Modelling and Control in Solid Mechanics*, Birkhauser Verlag, Basel-Boston-Berlin, 1997. Zbl 0865.73003
- [14] M. Serpilli, R. Rizzoni, F. Lebon, S. Dumont, *An asymptotic derivation of a general imperfect interface law for linear multiphysics composites*, International Journal of Solids and Structures, **180–181** (2019), 97–107.

EVGENY MIKHAILOVICH RUDOY
 LAVRENTYEV INSTITUTE OF HYDRODYNAMICS OF SB RAS,
 15, LAVRENYEVA AVE.,
 NOVOSIBIRSK, 630090, RUSSIA
 NOVOSIBIRSK STATE UNIVERSITY,
 1, PIROGOV STR.,
 NOVOSIBIRSK, 630090, RUSSIA
 E-mail address: `rem@hydro.nsc.ru`