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STABILITY OF THE CLASS OF DIVISIBLE S -ACTS

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ABSTRACT. We describe monoids S such that the theory of the class of all divisible S -acts is stable, superstable or, for commutative monoid, ω -stable. More precisely, we prove that the theory of the class of all divisible S -acts is stable (superstable) iff S is a linearly ordered (well ordered) monoid. We also prove that for a commutative monoid S the theory of the class of all divisible S -acts is ω -stable iff S is either an abelian group with at most countable number of subgroups or is finite and has only one proper ideal. Classes of regular, projective and strongly flat S -acts were considered in [1, 2]. Using results from [3] we obtain necessary and sufficient conditions for stability, superstability and ω -stability of theories of classes of all divisible S -acts.

Keywords: monoid, divisible S -act, stability, superstability, ω -stability.

1. PRELIMINARIES

Let us recall some definitions and facts from the theory of S -acts [4]. Let S be a monoid, i.e. a semigroup with the unit element 1. A monoid S is called *linearly (well) ordered*, if the set $\{Sa \mid a \in S\}$ is linearly ordered (well ordered) by $\supseteq$. An element $c \in S$ is said to be *right cancellable* if for all $a, b \in S$ the equality $ac = bc$ implies that $a = b$. An element $s \in S$ is called *right invertible* if there exists an element $t \in S$ such that $st = 1$.

A (*left*) S -act ${}_S A$ over a monoid S is a set A , on which an action of S is defined and the unit element of S acts on A as identity. By S -Act we denote the class of all S -acts.

Remark 1. If t is a right invertible element of S and ${}_S A$ is an S -act, then $tA = A$.

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A *congruence* on an S -act ${}_S A$ is an equivalence relation ρ on A such that $(a, a') \in \rho$ implies $(sa, sa') \in \rho$ for all $a, a' \in A, s \in S$. The smallest congruence on the S -act ${}_S A$ with respect to $\subseteq$ which contains a set X is called a *congruence of the S -act ${}_S A$, generated by X* , and is denoted by $\rho(X)$. Denote by a/ρ the class of a congruence ρ with a representative $a \in A$.

We say that an element $a \in A$ is *divisible* by $s \in S$ in ${}_S A$, if there exists $b \in A$ such that $sb = a$. A *divisible S -act* is an S -act ${}_S A$, such that $cA = A$ for every right cancellable element $c \in S$. We use $S\text{-Div}$ to denote the class of all divisible S -acts. It is clear that $S\text{-Div}$ is an elementary class.

By the *coproduct* of S -acts ${}_S A_i, i \in I$, we mean their disjoint union. The coproduct of S -acts ${}_S A_i, i \in I$, is denoted by $\coprod_{i \in I} {}_S A_i$. Note that a coproduct of

divisible S -acts is also a divisible S -act. A map $\theta : A \rightarrow B$ such that $\theta(sa) = s\theta(a)$ for all $a \in A, s \in S$, is called an *S -morphism* from the S -act A to the S -act B . An S -act F is said to be *free* in $S\text{-Act}$ (with a set of free generators X) if for every S -act A and every map $\theta : X \rightarrow A$ there exists a unique S -morphism $\bar{\theta} : F \rightarrow A$ such that $\iota\bar{\theta} = \theta$, where $\iota : X \rightarrow F$ is an embedding.

Let T be the set of all right cancellable but not right invertible elements of S . Let ${}_S A$ be an S -act, $X = \{(t, a) \in T \times A \mid a \text{ is not divisible by } t\}$, ${}_S F(X)$ be a free S -act with a set of free generators X , $H = \{(t(t, a), a) \mid (t, a) \in X, a \in A\} \subseteq (F(X) \amalg A) \times (F(X) \amalg A)$, $\rho(H)$ be a congruence of the S -act ${}_S (F(X) \amalg A)$, generated by H , ${}_S U(T, A) = {}_S (F(X) \amalg A) / \rho(H)$. Note that there exists a natural embedding $\pi : A \rightarrow U(T, A)$. Therefore we can identify elements $a \in A$ with $\pi(a)$.

We introduce the following notations: ${}_S A_0 = {}_S A$, ${}_S A_i = {}_S U(T, A_{i-1})$ for $i \in \mathbb{N}$, $D(A) = \bigcup_{i \in \omega} A_i$.

Fact 1. [4] *The S -act ${}_S D(A)$ is divisible.*

The S -act ${}_S D(A)$ is called the *divisible extension* of the S -act ${}_S A$.

Fact 2. [5] *Let ${}_S A$ be an S -act, $a, b \in A$, $d \in D(A) \setminus A$, $a, b \in Sd$. Then $a, b \in Sc$ for some $c \in A$.*

The following facts from model theory can be found in [6, 7]. Let T be a consistent theory in a language L , $X = \{x_i \mid 1 \leq i \leq n\}$, $L_n = L_X$. Every set p of sentences of a language L_n is called an *n -type* of L . If a theory $p \cup T$ is consistent, then p is said to be an *n -type over T* . If p is a complete theory, **then** p is called a *complete n -type* of L . If also $T \subseteq p$, we say that p is a *full n -type* over T . The set of all complete n -types over T is denoted by $S_n(T)$.

Let $\mathcal{A}$ be an algebraic system of a language L , $X \subseteq A$, and $a \in A$. The *type* of an element a over the set X is the set $tp(a, X) = \{\Phi(x) \mid \mathcal{A}_X \models \Phi(a)\}$. It is easy to see that $tp(a, X)$ is a complete 1-type over $Th(\mathcal{A}_X)$. Denote $S_1(Th(\mathcal{A}_X))$ by $S(X)$.

A theory T is called *stable for the cardinality κ* or κ -stable, if $|S(X)| \leq \kappa$ for every model $\mathcal{A}$ of a theory T and every $X \subseteq A$ of cardinality κ . If a theory T is κ -stable for some infinite κ then T is called *stable*. If a theory T is κ -stable for every $\kappa \geq 2^{|T|}$, then T is said to be *superstable*. If a theory T is not stable, then T is called *unstable*.

Fact 3. [1] *A complete theory is unstable iff there exists a formula $\Phi(\bar{x}, \bar{y})$ with $2n$ variables, a model $\mathcal{A}$ of a theory T and $\bar{a}_i \in A^n$, $i \in \omega$, such that for every i, j , $i \neq j$,*

$$i < j \iff \mathcal{A} \models \Phi(\bar{a}_i, \bar{a}_j).$$

Let K be a class of S -acts. A monoid S is called a K -stabiliser (K -superstabiliser, K - ω -stabiliser), if $Th(_S A)$ is stable (superstable, ω -stable) for every S -act $_S A \in K$. If $K = S\text{-Act}$, then K -stabiliser (K -superstabiliser, K - ω -stabiliser) is called the stabiliser (superstabiliser, ω -superstabiliser).

Let us introduce the following notation:

$$\begin{aligned} \exists^n y \phi(x, y) \Leftrightarrow \exists y_1 \dots \exists y_n \left(\bigwedge_{1 \leq i < j \leq n} \neg(y_i = y_j) \wedge \bigwedge_{1 \leq i \leq n} \phi(x, y_i) \wedge \right. \\ \left. \wedge \forall y (\phi(x, y) \rightarrow \bigvee_{1 \leq i \leq n} y = y_i) \right). \end{aligned}$$

Fact 4. [3] *A monoid S is a stabiliser iff S is a linearly ordered monoid.*

Fact 5. [3] *A monoid S is a superstabiliser iff S is a well ordered monoid.*

Fact 6. [7] *If a theory T is stable for a countable cardinality (ω -stable), then it is stable for all infinite cardinalities.*

Fact 7. [8] *Let S be an arbitrary countable commutative monoid. Then the following statements are equivalent:*

- 1) S is an ω -stabiliser;
- 2) Either S is an abelian group with at most countable number of subgroups, or S is finite and has a unique proper ideal.

2. STABILITY OF THE S -Div CLASS

Theorem 1. *Given a monoid S , the following statements are equivalent:*

- 1) S is a S -Div-stabiliser;
- 2) S is a stabiliser;
- 3) S is a linearly ordered monoid.

Proof. The implication $2 \Rightarrow 1$ is trivial.

The implication $2 \Leftrightarrow 3$ follows from Fact 4.

Let us prove $1 \Rightarrow 3$.

Assume that S is a S -Div-stabiliser but not a linearly ordered monoid, i.e. there exist $t, s \in S$ such that $St \not\subseteq Ss$ and $Ss \not\subseteq St$. Let $K = \{\langle i, j \rangle \mid j \leq i < \omega\}$; $_S S_{ij}$ – a copy of an S -act $_S S$ ($\langle i, j \rangle \in K$) and $S_{ij} \cap S_{kl} = \emptyset$, if $\langle i, j \rangle \neq \langle k, l \rangle$; d_{ij} is a copy of $d \in S$ in S_{ij} . Let $_S A$ be an S -act $\bigcup_{\langle i, j \rangle \in K} _S S_{ij} / \theta$, where θ is a congruence of the S -act $\bigcup_{\langle i, j \rangle \in K} _S S_{ij}$, generated by the set $\{\langle t_{ij}, t_{il} \rangle \mid \langle i, j \rangle \in K, \langle i, l \rangle \in K\} \cup \{\langle s_{ij}, s_{lj} \rangle \mid \langle i, j \rangle, \langle l, j \rangle \in K\}$; let t_i be an equivalence class of θ with a representative t_{ij} ; let s_j be an equivalence class of θ with a representative s_{ij} ; let $\varphi(x, y)$ be a formula $\exists z(x = tz \wedge y = sz)$. It is clear that the restriction of θ on the S -act $_S A_{ij}$ ($\langle i, j \rangle \in K$) is a zero-congruence.

Let us prove that

$$(1) \quad _S D(A) \models \varphi(t_i, s_j) \iff i \geq j.$$

If $i \geq j$, then $t_i = t1_{ij}/\theta$ and $s_j = s1_{ij}/\theta$, i.e. $_S D(A) \models \varphi(t_i, s_j)$. Let $i < j$. Assume that $t_i = tu$ and $s_j = su$ for some $u \in D(A)$. Then Fact 2 implies that $t_i, s_j \in Sc/\theta$ for some $c/\theta \in A$, which is not true.

From Fact 3 it follows that (1) contradicts the stability of $Th(_S D(A))$. Therefore S is a linearly ordered monoid. $\square$

3. SUPERSTABILITY OF THE S -Div CLASS

Theorem 2. *Given a monoid S , the following statements are equivalent:*

- (1) S is a S -Div-superstabiliser;
- (2) S is a superstabiliser;
- (3) S is a well ordered monoid.

Proof. The implication $2 \Rightarrow 1$ is trivial.

The implication $2 \Leftrightarrow 3$ follows from Fact 5.

Let us prove $1 \Rightarrow 3$.

Let S be a S -Div-superstabiliser. Then S is a S -Div-stabiliser and from Theorem 1 it is linearly ordered. We want to show that S is a well ordered monoid. Assume the contrary, i.e. there exist $a_i \in S$ such that $Sa_i \subset Sa_{i+1}$ ($i \in \omega$). Let T be a theory S -Div, κ a cardinal number, such that $\kappa \geq 2^{|T|}$. For $\eta \in \kappa^\omega$, we denote copies of an S -act by ${}_S S_\eta$ and copies of elements $c \in S$ by $c_\eta \in S_\eta$. Let $\eta|0 = \emptyset$, $\eta|i = (\eta(0), \eta(1), \dots, \eta(i-1))$, where $i \in \omega \setminus \{0\}$.

Let ${}_S A = \bigsqcup_{\eta \in \kappa^\omega} {}_S S_\eta / \Theta$, where Θ is a congruence of the S -act $\bigsqcup_{\eta \in \kappa^\omega} {}_S S_\eta$ generated by the set $\{((a_i)_\eta, (a_i)_\varepsilon) \mid \eta, \varepsilon \in \kappa^\omega, \eta|i = \varepsilon|i\}$, $b_{\eta|i} = (a_i)_\eta / \Theta$, $b_\eta = 1_\eta / \Theta$, where $\eta \in \kappa^\omega$, $B = \{b_{\eta|i} \mid \eta \in \kappa^\omega, i \in \omega\}$. It is clear that $|b_\eta| = 1$ for every $\eta \in \kappa^\omega$ and $|B| = \kappa^\omega$. Let $\eta, \varepsilon \in \kappa^\omega$, $\eta \neq \varepsilon$. We show that $tp(b_\eta, B)$ and $tp(b_\varepsilon, B)$ are distinct 1-types over a theory $Th(D({}_S A))$. Since $\eta \neq \varepsilon$, it follows that $\eta|i \neq \varepsilon|i$ for some $i \in \omega$ and $b_{\eta|i} \neq b_{\varepsilon|i}$. Furthermore, $b_{\eta|i} = a_i b_\eta$ and $b_{\varepsilon|i} = a_i b_\varepsilon$. Then $b_{\eta|i} = a_i x \in tp(b_\eta, B) \setminus tp(b_\varepsilon, B)$. Consequently, $|S(B)| \geq |\{b_\eta \mid \eta \in \kappa^\omega\}| = \kappa^\omega > \kappa$ and it follows that the theory $Th(D({}_S A))$ is not superstable. This implies that monoid S is not a S -Div-superstabiliser and with that we obtain a contradiction. Therefore, S is a well ordered monoid. $\square$

4. ω -STABILITY OF THE S -Div CLASS

Theorem 3. *Given a commutative countable monoid S , the following statements are equivalent:*

- (1) S is a S -Div- ω -superstabiliser;
- (2) S is an ω -stabiliser;
- (3) Either S is an abelian group with at most countable number of subgroups or S is finite and has a unique proper ideal.

Proof. Let S be a commutative countable monoid.

The implication $2 \Rightarrow 1$ is trivial.

The implication $2 \Leftrightarrow 3$ follows from Fact 7.

We shall prove that $1 \Rightarrow 3$.

For a proof assume that S is a S -Div- ω -stabiliser. Then Fact 6. implies that S is a S -Div-superstabiliser and from theorem 2 we obtain that S is a well ordered monoid.

If S contains no cancellable and non-invertible elements then by definition of divisible S -acts it follows that S -Div = S -Act. Then S is an ω -stabiliser.

Assume that such element exists in S . Since S is a well ordered monoid, there exists a cancellable and non-invertible element $g \in S$ such that

$$(2) \quad \forall a \in S (Sg \subset Sa \Rightarrow a \text{ is non-cancellable or invertible}).$$

We denote $1 \in S$ by g^0 .

Consider an element $i \in \omega$. We will show that

$$(3) \quad \forall a \in S(Sg^{i+1} \subset Sa \subseteq Sg^i \Rightarrow Sa = Sg^i).$$

Assume that $Sg^{i+1} \subset Sa \subseteq Sg^i$. Then $a = sg^i$ and $s \notin Sg$. Since S is linearly ordered, we have that $Sg \subset Ss$ and $g = ts$ for some $t \in S$. From (2) it follows that s is either non-cancellable or invertible. Let us check that s is a cancellable element of S . Indeed, if $xs = ys$ for some $x, y \in S$, then $xts = yts$, i.e. $xg = yg$ and $x = y$. Therefore, s is invertible and $Ss = S$. Hence $Sa = Ssg^i = Sg^i$ and (3) is proved.

We will now show that $Sg^{i+1} \subset Sg^i$. Assume that $Sg^i = Sg^{i+1}$. Then $g^i = sg^{i+1}$ for some $s \in S$. Since g is cancellable, it follows that $1 = sg$. This contradicts the fact that g is not invertible.

Consider elements $s, t \in S$. We shall prove that

$$(4) \quad s \in Sg^i \setminus Sg^{i+1} \wedge t \in Sg^j \setminus Sg^{j+1} \Rightarrow st \in Sg^{i+j} \setminus Sg^{i+j+1}.$$

Let $s \in Sg^i \setminus Sg^{i+1}$, $t \in Sg^j \setminus Sg^{j+1}$. As $s \in Sg^i$, then $s = s_1g^i$ for some $s_1 \in S$. Since $t \in Sg^j$, it follows that $t = t_1g^j$ for some $t_1 \in S$. Then $st = s_1t_1g^{i+j} \in Sg^{i+j}$. Because $s \notin Sg^{i+1}$, we obtain that $s_1 \notin Sg$; also $t \notin Sg^{j+1}$, implies that $t_1 \notin Sg$. Furthermore since S is linearly ordered we have that $Sg \subset Ss_1$, $Sg \subset St_1$ and from (3), we derive that $Ss_1 = S$ and $St_1 = S$. Assume $st \in Sg^{i+j+1}$, i.e. $st = rg^{i+j+1}$ for some $r \in S$. Then $rg^{i+j+1} = s_1t_1g^{i+j}$. From the fact that g is cancellable we see that $rg = s_1t_1$. Thus, $s_1t_1 \in Sg$, i.e. $S = Ss_1t_1 \subseteq Sg$, which is a contradiction. Hence $st \in Sg^{i+j} \setminus Sg^{i+j+1}$.

Let $\sim$ be the following relation on the set S :

$$a \sim b \iff \exists i \in \omega : a, b \in Sg^i \setminus Sg^{i+1}.$$

It is easy to show that $\sim$ is an equivalence relation. Let us prove that $\sim$ is also a congruence of the S -act ${}_S S$. Consider elements $a, b, s \in S$, such that $s \in Sg^i \setminus Sg^{i+1}$, where $i \in \omega$, and $a \sim b$. We want to show that $sa \sim sb$. Since $a \sim b$, it follows that $a, b \in Sg^j \setminus Sg^{j+1}$ for some $j \in \omega$. From (4), we obtain that $sa, sb \in Sg^{i+j} \setminus Sg^{i+j+1}$, i.e. $sa \sim sb$.

Consider ${}_S \bar{S} = {}_S S / \sim$ (${}_S \bar{S}$ is an S -act). For $a \in S$ denote by $\bar{a}$ a congruence class of $\sim$ with a representative a . Then (3) yields that $\bar{S} = \{\bar{g}^i \mid i \in \omega\}$.

Define the action of the S -act S on the set $A = \{\bar{g}^n \mid n \in \mathbb{Z}\}$. Let $s \in S$. Then $s \sim g^k$ for some $k \in \omega$. Assume that $s\bar{g}^n = \bar{g}^{n+k}$ for every $n \in \mathbb{Z}$. We will prove that $s(t\bar{g}^n) = (st)\bar{g}^n$ for every $s, t \in S$, $n \in \mathbb{Z}$. Consider elements $s \sim g^k$, $t \sim g^m$. From (4) we have $st \sim g^{m+k}$. By definition of the action of the monoid S on the set A , we have that $(st)\bar{g}^n = \bar{g}^{n+m+k}$, $t\bar{g}^n = \bar{g}^{m+n}$ and $s(t\bar{g}^n) = s\bar{g}^{m+n} = \bar{g}^{k+m+n}$. Hence ${}_S A$ is an S -act. Note that ${}_S \bar{S}$ is a sub- S -act of ${}_S A$.

We will see that ${}_S A \in S\text{-Div}$. Let t be a cancellable element of S . We want to check that $tA = A$. By Remark 1. it can be assumed that t is not invertible. Let $\bar{g}^n \in A$ and $t \sim g^i$. Then $\bar{g}^n = t\bar{g}^{n-i} \in tA$. Therefore $tA = A$.

We will now show that a theory $Th({}_S A)$ is not ω -stable. Let ${}_S A_i$ ($i \in \mathbb{N}$) be copies of the S -act ${}_S A$, and $a_i \in A_i$ be copies of an element $a \in A$. We assign to every $n \in \mathbb{N}$ an S -act ${}_S A^n = \bigsqcup_{i \leq n} {}_S A_i / \Theta^n$, where Θ^n is a congruence of an S -act

$\bigsqcup_{i \leq n} {}_S A_i$ generated by the set $\{\bar{g}_i^{-2} \mid i \leq n\}$. Note that for $m \geq -2$ elements $\bar{g}_1^m / \Theta^n$, $\dots, \bar{g}_n^m / \Theta^n$ are the same. Denote by $(g^m)_n$ an element $\bar{g}_i^m / \Theta^n$ ($m \geq -2$). To each $K \subseteq \mathbb{N}$ we assign an S -act ${}_S A^K = \bigsqcup_{n \in K} {}_S A_n / \eta^K$, where η^K is a congruence of

an S -act $\bigsqcup_{n \in K} sA_n$ generated by the set $\{(\bar{g}^{-1})_n \mid n \in K\}$. Note that for $m \geq -1$ elements $(g^m)_n/\eta^K$, $n \in K$, coincide. Denote an element $(g^m)_n/\eta^K$ by $(g^m)_K$.

Let $sB = \bigsqcup_{K \subseteq \mathbb{N}} sA^K/\xi$, where ξ is a congruence of an S -act $\bigsqcup_{K \subseteq \mathbb{N}} A^K$ generated by the set $\{(g^0)_K \mid K \subseteq \mathbb{N}\}$. Note that for $m \geq 0$ elements $(g^m)_K/\xi$, $K \subseteq \omega$, coincide. For $m \geq 0$, we will denote element $(g^m)_K/\xi$ by g^m . For every $n \in \mathbb{N}$, we define:

$$\varphi_n(y) \rightleftharpoons \exists^n z(gz = y).$$

Let $K_1 \neq K_2$. We want to check that $tp((g^{-1})_{K_1}, \emptyset) \neq tp((g^{-1})_{K_2}, \emptyset)$. Assume there exists n such that $n \in K_1 \setminus K_2$. Then

$$sD(B) \models \varphi_n((g^{-1})_{K_1}) \wedge \neg \varphi_n((g^{-1})_{K_2}).$$

Hence, $|S(\emptyset)| \geq 2^{\mathbb{N}}$ and a theory $Th(sD(B))$ is not ω -stable. Therefore, a monoid S is not a Div - ω -stabiliser. $\square$

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