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EXPONENTIAL CONVEXITY AND TOTAL POSITIVITY

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ABSTRACT. Class of exponentially convex functions is a sub-class of convex functions on a given interval (a, b) . For exponentially convex function $f(x)$ S. N. Bernstein's integral representation holds. A condition for $f(x)$, providing the kernel $K(x, y) = f(x+y)$ to be totally positive is given. New examples of totally positive kernels are obtained. For example the kernel $(x+y)^{-\alpha}$ is totally positive for any $\alpha > 0$.

Keywords: exponential convexity, total positivity, kernel.

1. INTRODUCTION

Function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called exponentially convex (e. c.) on $\mathbb{R}$ if it is continuous and

$$(1) \quad \sum_{i,j=1}^n f(x_i + x_j) \xi_i \xi_j \geq 0$$

for any $x_i, \xi_i \in \mathbb{R}$ and any n .

The classic examples include the functions $f(x) = e^{ax}$ for any $a \in \mathbb{R}$, for them condition (1) can be checked directly.

E. c. functions were introduced by S. N. Bernstein [1]. An overview of properties of such functions is given in [2] and [4]. Inequality (1) means that the matrix

$$A = [f(x_i + x_j)]_{i,j=1}^n$$

is non-negative definite for any values $x_1, \dots, x_n \in \mathbb{R}$. Therefore all principal minors of matrix A are non-negative. Hence simplest properties follow. For $n = 1$ we get $f(2x_1)\xi_1^2 \geq 0$. So $f(x) \geq 0$ for all x .

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Proposition. *If f is exponentially convex and does not vanish identically then $f(x) > 0$ for all $x \in \mathbb{R}$.*

This statement can be found in [2] (Remark 2.7.).

The most important property of e. c. functions is their integral representation ([2], p. 184).

Theorem 1. *Function f is exponentially convex iff*

$$(2) \quad f(x) = \int_{-\infty}^{\infty} e^{xt} d\mu(t), \quad x \in \mathbb{R},$$

where μ is a non-negative Borel measure on $\mathbb{R}$.

A class of e. c. functions is rather wide. It contains functions e^x , $\frac{e^x-1}{x}$, $\frac{\sinh x}{x}$, e^{x^2} . However function e^{-x^2} is not e. c. because it's not convex.

2. E. C. FUNCTIONS AND TOTALLY POSITIVE KERNELS

2.1. Totally non-negative matrices. Consider the matrix $[f(x_i+x_j)]_{i,j=1}^n$ where f is e. c. function from perspective of the theory of totally non-negative matrices [5].

Definition. *A matrix $A = [a_{ij}]_{i,j=1}^n$ is called totally non-negative if all its minors (of arbitrary order) are non-negative. If all minors are positive then A is called totally positive.*

Let A, B be totally non-negative matrices (TNM). Then, their product is also a TNM ([5]) and their sum may not be a TNM. Let's give an example (we haven't found such examples in literature):

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 \\ 4 & 1 \end{bmatrix}, \quad C = A + B = \begin{bmatrix} 10 & 5 \\ 5 & 2 \end{bmatrix}.$$

Here A, B are total positive matrices and C is not even a TNM since $\det(C) < 0$. Now let f be an e. c. function. Then the matrix

$$A = [f(x_i + x_j)]_{i,j=1}^n$$

is non-negative definite. That's why principal minors are non-negative. But will all its minors be non-negative? Let's check it with the help of an example of "typical" e. c. function $f(x) = e^{ax}$, $a \in \mathbb{R}$. Calculate in the matrix $A = [e^{a(x_i+x_j)}]_{i,j=1}^n$ one minor of order 2

$$\begin{vmatrix} e^{ax_1}e^{ax_3} & e^{ax_1}e^{ax_4} \\ e^{ax_2}e^{ax_3} & e^{ax_2}e^{ax_4} \end{vmatrix} = e^{ax_1}e^{ax_2}e^{ax_3}e^{ax_4} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Such method of taking out a factor from the row or from the column can be applied to any minor. Finally we obtain that all minors of the matrix A , starting from 2nd order, are equal to zero. So, A is a TNM.

2.2. Totally positive kernels and e. c. functions. A kernel $K(x, y)$ is called totally positive on $(a, b) \times (a, b)$, if the determinant

$$K \begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} = \det [K(x_i, y_j)]_{i,j=1}^n$$

is positive for any n and for any points $x_1 < \dots < x_n, y_1 < \dots < y_n$ from (a, b) .

In this section we examine the kernels $K(x, y) = f(x+y)$ where f is e. c. function on (a, b) . The example of the function $f(x) = \exp ax$ shows that the kernel $f(x+y)$ will not always be totally positive.

Let's consider measures defined by positive continuous density $p(t)$ on (c, d) . Let integral

$$(3) \quad f(x) = \int_c^d e^{xt} p(t) dt$$

be finite for any $x \in (a, b)$. Then function $f(x)$ will be exponentially convex on (a, b) .

Now let the interval (a, b) be either $(-\infty, \infty)$, or $(0, \infty)$. Then $x + y \in (a, b)$ for all $x, y \in (a, b)$ and the kernel can be defined as $K(x, y) = f(x+y)$.

Theorem 2. *Let a function $f(x)$ be represented by formula (3). Then the kernel $K(x, y) = f(x+y)$ is totally positive on $(a, b) \times (a, b)$.*

Proof. We have

$$f(x+y) = \int_c^d e^{xt} p(t) e^{yt} dt.$$

We see that the kernel $K(x, y)$ is a composition of two kernels

$$L(x, t) = e^{xt} p(t) \text{ and } M(t, y) = e^{ty}.$$

It means that

$$K(x, y) = \int_c^d L(x, t) M(t, y) dt.$$

It is well-known that the kernel $M(t, y) = e^{ty}$ is totally positive:

$$M \begin{bmatrix} t_1, & \dots, & t_n \\ y_1, & \dots, & y_n \end{bmatrix} > 0$$

for any $t_1 < \dots < t_n$ on (c, d) and $y_1 < \dots < y_n$ on (a, b) ([6], p. 21). The kernel $L(x, t)$ is also totally positive. Take $x_1 < \dots < x_n$ and calculate the determinant

$$L \begin{bmatrix} x_1, & \dots, & x_n \\ t_1, & \dots, & t_n \end{bmatrix} = \det [e^{x_i t_j} p(t_j)]_{i,j=1}^n = \prod_{j=1}^n p(t_j) \det [e^{x_i t_j}]_{i,j=1}^n > 0.$$

Here we took into consideration that $p(t_j) > 0$. By composition formula ([6], p. 14, Eq.(3.12)),

$$\begin{aligned} K \begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} &= \\ &= \int_{c \leq t_1 < t_2 < \dots < t_n \leq d} L \begin{bmatrix} x_1, & \dots, & x_n \\ t_1, & \dots, & t_n \end{bmatrix} M \begin{bmatrix} t_1, & \dots, & t_n \\ y_1, & \dots, & y_n \end{bmatrix} dt_1 \dots dt_n. \end{aligned}$$

This formula is an integral analogue of the Binet-Cauchy formula. As determinators L and M are positive then

$$K \begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} > 0.$$

Theorem is proved. $\square$

From the theorem we can get some interesting examples of totally positive kernels. First example is well-known.

Example 1. For the function $f(x) = e^{x^2}$ the representation holds

$$e^{x^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-t^2/4} dt, \quad x \in \mathbb{R}.$$

To calculate the integral we can use the identity $xt - t^2/4 = -\frac{1}{4}(t - 2x)^2 + x^2$.

Since the density $e^{-t^2/4}$ is positive then the kernel

$$K(x, y) = \exp((x + y)^2)$$

is totally positive on $\mathbb{R} \times \mathbb{R}$.

Example 2. For any $\alpha > 0$ we have the equality

$$x^{-\alpha} = \int_0^{\infty} e^{-xt} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt, \quad x > 0.$$

This can also be written as

$$x^{-\alpha} = \int_{-\infty}^0 e^{x\tau} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha)} d\tau, \quad x > 0.$$

By Theorem 2 the kernel $K(x, y) = (x + y)^{-\alpha}$ is totally positive for $x > 0$, $y > 0$, any $\alpha > 0$. For $\alpha = 1$ we obtain the well-known Cauchy kernel.

Example 3. For $n \geq 1$ consider the function

$$F_n(x) = \frac{1}{x^n} \left[e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} \right], \quad x \neq 0; \quad F_n(0) = \frac{1}{n!}.$$

This function is exponentially convex on $\mathbb{R}$. Indeed, let's write the Taylor formula

$$e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} = \frac{1}{(n-1)!} \int_0^x e^{\tau} (x - \tau)^{n-1} d\tau.$$

Here in the integral we make the substitution $\tau = xt$, $t \in [0, 1]$, and then divide by x^n . Finally we get

$$F_n(x) = \frac{1}{(n-1)!} \int_0^1 e^{xt} (1-t)^{n-1} dt, \quad x \in \mathbb{R}.$$

Since $(1-t)^{n-1} > 0$ on $(0, 1)$ then F_n is e. c. function. For the function $F_n(t)$ Theorem 2 holds i. e. the kernel of $F_n(x + y)$ is totally positive on $\mathbb{R} \times \mathbb{R}$.

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