I. M. Tarasova, V. A. Shlyk, Weighted Sobolev spaces, capacities and exceptional sets,
Sib. Èlektron. Mat. Izv., 2020, Volume 17, 1552–1570

DOI: <https://doi.org/10.33048/semi.2020.17.108>

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April 26, 2023, 13:26:18

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 17, вып. 1552–1570 (2020)

УДК 517.51

DOI 10.33048/semi.2020.17.108

MSC 46E35, 31C45

WEIGHTED SOBOLEV SPACES, CAPACITIES AND EXCEPTIONAL SETS

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ABSTRACT. We consider the weighted Sobolev space $W_{\omega}^{m,p}(\Omega)$, where Ω is an open subset of R^n , $n \geq 2$, and ω is a Muckenhoupt A_p -weight on R^n , $1 \leq p < \infty$, $m \in \mathbb{N}$. For the equalities $W_{\omega}^{m,p}(\Omega \setminus E) = W_{\omega}^{m,p}(\Omega)$, $\dot{W}_{\omega}^{m,p}(\Omega \setminus E) = \dot{W}_{\omega}^{m,p}(\Omega)$ to hold, conditions are obtained in terms of E as a set of zero (p, m, ω) -capacity, or an $NC_{p,\omega}$ -set for the first equality. For the equality $W_{p,\omega}^{m,p}(\Omega) = \dot{W}_{p,\omega}^{m,p}(\Omega)$, the conditions are established for $R^n \setminus \Omega$ as a set of zero (p, m, ω) -capacity. Similar results are partially true for $W_{p,\omega}^m(\Omega)$, $L_{p,\omega}^m(\Omega)$.

Keywords: Sobolev space, capacity, Muckenhoupt weight, exceptional set.

1. INTRODUCTION

Suppose that Ω is an open set on the Euclidean space R^n , $n \geq 2$, and E is a relatively closed subset on Ω . Let $W(\Omega)$ be a Sobolev space with a norm (with a semi-norm) $\|\cdot\|_{W(\Omega)}$, whose elements are functions (classes of equivalent functions) defined on Ω , and whose partial derivatives satisfy certain integrability conditions. We denote the closure of $C_0^\infty(G)$ on $W(\Omega)$ by $\dot{W}(\Omega)$.

The following problems are well-known in the theory of Sobolev spaces: find the conditions for E which need to be satisfied for the equalities $W(\Omega \setminus E) = W(\Omega)$ (problem (i)); $\dot{W}(\Omega) = \dot{W}(\Omega \setminus E)$ (problem (ii)); $W(\Omega) = \dot{W}(R^n \setminus E)$, where $E = R^n \setminus \Omega$ (problem (iii)), to hold respectively. More information about the equality of spaces can be found in Remark 1 below.

In problems (i)–(iii), the set E , for which the equalities are realized, is called exceptional. In particular, with regard to the equation $W(\Omega) = L_p^1(\Omega)$, the criterion for the

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Received August, 9, 2019, published September, 28, 2020.

set E to be exceptional in (i) was obtained by S. Vodop'yanov and V. Gol'dstein [15] in terms of E as an NC_p -set, $1 < p < \infty$. L. Hedberg obtained a criterion for the exceptional set E in problem (ii) as a set of zero p -capacity, where in our notation $W(\Omega) = L_p^1(\Omega)$, $1 < p < \infty$, and Ω is a bounded set in R^n [7, Theorems 1,2].

For a weighted space $H^{1,p}(\Omega, \mu)$ with a p -admissible measure μ , $1 < p < \infty$, all three problems (i)–(iii) were solved in [8, Theorems 2.43–2.45] in terms of E as a set of zero Sobolev (p, μ) -capacity. The necessary and sufficient conditions for an exceptional set E in (ii)–(iii) for $W(\Omega) = W^{m,p}(\Omega)$ are provided in [1, Theorems 3.28, 3.33] in terms of E as a (m, p) -polar set. The criteria for the exceptional set E in (iii) for $W(\Omega) = W_p^m(\Omega)$ are obtained in [10, §9.2, Theorems 1,2] in terms of E as a (m, p) -polar set and a set of zero (p, m) -capacity.

In this paper, the criteria for the exceptional set E in (i) for $W(\Omega) = W_\omega^{m,p}(\Omega)$, $W_{p,\omega}^m(\Omega)$, $L_{p,\omega}^m(\Omega)$, $1 < p < \infty$, and $m \in \mathbb{N}$, are established in terms of E as an $NC_{p,\omega}$ -set, see Theorems 6,7. The criteria for the exceptional set E in (ii) for $W(\Omega) = W_\omega^{m,p}(\Omega)$, $W_{p,\omega}^m(\Omega)$ and (iii) for $W_\omega^{m,p}(\Omega)$ are established in terms of E as a set of zero (p, m, ω) -capacity, where $1 \leq p < \infty$, $m \in \mathbb{N}$, see Theorems 9,10.

In addition, sufficient condition for the exceptional set E in (i) for

$$W(\Omega) = W_\omega^{m,p}(\Omega), W_{p,\omega}^m(\Omega), L_{p,\omega}^m(\Omega)$$

are given in terms of E as a zero (p, m, ω) -capacity set, where $1 \leq p < \infty$, $m \in \mathbb{N}$, see Theorem 8, Corollary 6.

2. PRELIMINARIES

2.1. Some definitions and notations. Throughout the text, Ω is used to denote an open set on $R^n = \{x = (x_1, \dots, x_n)\}$, while E denotes a relatively closed subset on Ω .

The norm of a point $x = (x_1, \dots, x_n) \in R^n$ has the form $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. We put $\mathbb{N} = \{1, 2, \dots\}$, $R = (-\infty, +\infty)$. If $F \subset R^n$, then $\partial F, \bar{F}$ denote the boundary and the closure of F on R^n , respectively. The distance between two sets $A, B \subset R^n$ is denoted by $\text{dist}(A, B)$.

For an open set $U \subset R^n$, we use the notation $U \Subset \Omega$ in order to indicate that U is bounded and $\bar{U} \subset \Omega$. The restriction of the function f to the set F is denoted by $f|_F$. Let χ_F be a characteristic function of the set F .

Given $x \in R^n$ and $r > 0$, suppose that $B(x, r)$ or $B_r(x) = \{y \in R^n : |y - x| < r\}$. If $a > 0$, then we have that $aB_r(x) = B_{ar}(x)$. We use the symbol $\mathcal{H}^s$ to denote an ordinary s -dimensional Hausdorff measure on R^n ; m_n is Lebesgue measure on R^n , and we put $m_n(F) = |F|$.

Let $C^\infty(\Omega)$ be a space of infinitely differentiable functions on Ω ; the space of functions in $C^\infty(R^n)$ with a compact support on Ω is denoted by $C_0^\infty(\Omega)$.

The support of a function u will be denoted by $\text{supp } u$.

For $1 \leq p < \infty$, we define $L_p(\Omega)$ as a set of m_n -measurable functions f on Ω , such that

$$\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f|^p dx \right)^{1/p} < \infty,$$

and suppose that $L_p(\Omega, \text{loc})$ is a space of m_n -measurable functions f on Ω , such that $|f|^p$ is a locally integrable function on Ω .

We will use the abbreviation "a.e." for the phrase "almost everywhere" with respect to m_n -measure. Similarly, when we use the words "measurable" and "locally integrable", we always mean "Lebesgue measurable" and "locally integrable with respect to m_n -measure".

For the case $\Omega = R^n$, we normally drop the reference to Ω in the notation of spaces and norms. Integration without specifying integration limits is extended to R^n by agreement.

Within proofs of, say, theorems, the letter C will be used to denote a generic positive constant which depends only on the parameters in the statement of the theorem. The quantities A and B are said to be "equivalent", if there exist two positive constants, C_1 and C_2 , such that $C_1 A \leq B \leq C_2 A$.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of non-negative integers α_i , we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$, which has a degree $|\alpha| = \sum_{i=1}^n \alpha_i$.

Similarly, if $D_j = \frac{\partial}{\partial x_j}$, then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. Note that $D^{(0,0,\dots,0)}u = u$.

If α and β are multi-indices, we say that $\beta \leq \alpha$ provided that $\beta_i \leq \alpha_i$ for $1 \leq i \leq n$. In this case, $\alpha - \beta$ is also a multi-index and $|\alpha - \beta| + |\beta| = |\alpha|$. Put $\alpha! = \alpha_1! \dots \alpha_n!$, then for $\beta \leq \alpha$, we have that

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!} = \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_n}{\beta_n}.$$

This allows us to write the Leibnitz formula in the form

$$(1) \quad D^\alpha(uv)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha - \beta} v(x),$$

which holds for functions u and v , that are $|\alpha|$ times continuously differentiable near x .

We use the notations $\nabla_m = \{D^\alpha : |\alpha| = m\}$, $\nabla = \nabla_1$.

By a weight we mean a locally integrable function ω on R^n , such that $\omega > 0$ for a.e. $x \in R^n$.

Then for $1 \leq p < \infty$, we define $L_{p,\omega}(\Omega)$ as a set of measurable functions f on Ω , such that

$$\|f\|_{L_{p,\omega}(\Omega)} = \left(\int_{\Omega} |f|^p \omega dx \right)^{1/p} < \infty.$$

As usual, any two functions f and g from $L_{p,\omega}(\Omega)$ that are equal a.e. on Ω will be identified. It is well-known (see [9, Theorem 2.7]) that $L_{p,\omega}(\Omega)$ is complete with respect to the norm $\|\cdot\|_{L_{p,\omega}(\Omega)}$.

Let $\mathcal{F}_1$ be a space of functions given on Ω , and $\mathcal{F}_2$ be another space of functions given on Ω' , where $\Omega' \subset \Omega$. Below, if $f \in \mathcal{F}_1$, then $f \in \mathcal{F}_2$ implies that $f|_{\Omega'} \in \mathcal{F}_2$.

We denote by $L_{p,\omega}(\Omega, loc)$ a set of all m_n -measurable functions f on Ω , such that $f \in L_{p,\omega}(\Omega')$ for all open sets $\Omega' \Subset \Omega$.

2.2. A_p -weights. Suppose that $1 \leq p < \infty$. According to B. Muckenhoupt [11], a weight ω is called an A_p -weight, if there exists a positive constant A , such that for every ball $B \subset R^n$, the inequality

$$(2) \quad \left(\frac{1}{|B|} \int_B \omega dx \right) \left(\frac{1}{|B|} \int_B \omega^{-\frac{1}{p-1}} dx \right)^{p-1} \leq A,$$

holds, if $p > 1$, and

$$(3) \quad \left(\frac{1}{|B|} \int_B \omega dx \right) \operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \leq A,$$

holds, if $p = 1$. The infimum of all such constants A is called the A_p -constant of ω . We denote by A_p , $1 \leq p < \infty$, a set of A_p -weights. Throughout the text, suppose that $1 \leq p < \infty$, $m \in \mathbb{N}$, $\omega \in A_p$, unless otherwise stated.

We should mention one result concerning A_p -weight [14, Remark 1.2.4].

Proposition 1. *If $\omega \in A_p$, then $L_{p,\omega}(\Omega)$ is a complete space with respect to the norm $\|\cdot\|_{L_{p,\omega}(\Omega)}$, and $L_{p,\omega}(\Omega) \subset L_1(\Omega, \text{loc})$.*

2.3. Weighted Sobolev spaces. Suppose that $u : \Omega \rightarrow \mathbb{R}$ is a function of class $L_1(\Omega, \text{loc})$. The function u on Ω has a weak derivative of order $|\alpha|$, if there is a locally integrable function (denoted by $D^\alpha u$), such that

$$\int_{\Omega} u \cdot D^\alpha \varphi dx = (-1)^{|\alpha|} \int_{\Omega} D^\alpha u \cdot \varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$. For $1 \leq p < \infty$, $m \in \mathbb{N}$ and every $\omega \in A_p$, $L_{p,\omega}^m(\Omega)$ is a space of functions which have weak derivatives $D^\alpha u$ of all orders $|\alpha|$, $|\alpha| \leq m$, and that satisfy the condition

$$\|f\|_{L_{p,\omega}^m(\Omega)} = \left(\int_{\Omega} |\nabla_m u|^p \omega dx \right)^{1/p} < \infty,$$

where $|\nabla_m u| = \left(\sum_{|\alpha|=m} (D^\alpha u)^2 \right)^{1/2}$. For $m = 0$, set $L_{p,\omega}^m(\Omega) = L_{p,\omega}(\Omega)$, $\nabla_0 u = u$.

We introduce the spaces

$$W_{p,\omega}^m(\Omega) = L_{p,\omega}^m(\Omega) \cap L_{p,\omega}(\Omega), \quad W_{\omega}^{m,p}(\Omega) = \bigcap_{k=0}^m L_{p,\omega}^k(\Omega),$$

equipped with the norms

$$\|u\|_{W_{p,\omega}^m(\Omega)} = \|u\|_{L_{p,\omega}^m(\Omega)} + \|u\|_{L_{p,\omega}(\Omega)}, \quad \|u\|_{W_{\omega}^{m,p}(\Omega)} = \sum_{k=0}^m \|\nabla_k u\|_{L_{p,\omega}(\Omega)}.$$

We denote by $\overset{\circ}{L}_{p,\omega}^m(\Omega)$, $\overset{\circ}{W}_{p,\omega}^m(\omega)$, $\overset{\circ}{W}_{\omega}^{m,p}(\Omega)$ the closures of $C_0^\infty(\Omega)$ in $L_{p,\omega}^m(\Omega)$, $W_{p,\omega}^m(\omega)$, $W_{\omega}^{m,p}(\Omega)$, respectively. In addition, we set $W_{\omega}^{m,p}(\Omega, \text{loc}) = \bigcap_{\Omega'} W_{\omega}^{m,p}(\Omega')$, where the intersection is taken over all open sets $\Omega' \Subset \Omega$. Below, $W_{\omega}^{m,p}(\Omega, \text{loc})$ will be considered a countably normed space with a system of semi-norms $\|u\|_{W_{\omega}^{m,p}(\Omega_k)}$. Here, $\{\Omega_k\}_{k \geq 1}$ is a sequence of open sets $\Omega_k \Subset \Omega_{k+1} \subset \Omega$, $\bigcup_k \Omega_k = \Omega$.

For the case when $\omega \equiv 1$, the weighted spaces considered above with the weight ω will be written below without the symbol ω .

Next, let $\mathcal{P}_{m-1}$ be a collection of all polynomials of degree $\leq m-1$. Consider the factor space $\check{L}_{p,\omega}^m(\Omega) = L_{p,\omega}^m(\Omega)/\mathcal{P}_{m-1}$ (with the norm $\|\cdot\|_{L_{p,\omega}^m(\Omega)}$). Elements of the space $\check{L}_{p,\omega}^m(\Omega)$ are classes $\tilde{u} = \{u + P\}$, where $u \in L_{p,\omega}^m(\Omega)$ and $P \in \mathcal{P}_{m-1}$.

Note that a number of important properties of spaces $W_{\omega}^{m,p}(\Omega)$, $L_{p,\omega}^m(\Omega)$ (in other notations and with equivalent norms) were obtained in [3, 14]. Below, we use the following properties.

Proposition 2 ([3, Theorem 4.9]). *If Ω is an open connected set and $\omega \in A_p$, $1 \leq p < \infty$, then $\check{L}_{p,\omega}^m(\Omega)$ is a Banach space. In particular, if $\{u_j\}$ is a Cauchy sequence in $L_{p,\omega}^m(\Omega)$, then there exists $u_0 \in L_{p,\omega}^m(\Omega)$, such that $\nabla_m u_j \rightarrow \nabla_m u_0$ in $L_{p,\omega}(\Omega)$ as $j \rightarrow \infty$.*

Proposition 3 ([3, Corollary 4.10]). *Suppose that Ω is an open connected set, $\{u_j\}$ is a Cauchy sequence in $L_{p,\omega}^m(\Omega)$, and u is a function in $L_{p,\omega}^m(\Omega)$, such that $\|\nabla_m(u_j - u)\|_{L_{p,\omega}(\Omega)} \rightarrow 0$. Then there exists a sequence of polynomials $\{P_j\} \subset \mathcal{P}_{m-1}$ with $u_j - P_j \rightarrow u$ in $L_{p,\omega}(K)$ for all compact sets $K \subset \Omega$.*

Proposition 4 ([3, Theorem 4.2]). *Suppose that $1 \leq p < \infty$, $\omega \in A_p$. If $u \in L_{p,\omega}^m(\Omega)$, then*

$$(4) \quad \int_K |D^\alpha u|^p \omega dx < \infty$$

for all compact $K \subset \Omega$, $0 \leq |\alpha| \leq m$.

Proposition 5 ([14, Theorem 2.1.14]). *Suppose that $\omega \in A_p$, $1 \leq p < \infty$, and $k, m \in \mathbb{N}$, $1 \leq k < m$. Let $B \subset R^n$ be a ball. Then there is a positive constant C depending only on k, m, p, n and the A_p -constant of ω , such that*

$$(5) \quad \int_B |\nabla_k u|^p \omega \, dx \leq C \left(|B|^{-\frac{kp}{n}} \int_B |u|^p \omega \, dx + |B|^{\frac{(m-k)p}{n}} \int_B |\nabla_m u|^p \omega \, dx \right)$$

for all $u \in W_{\omega}^{m,p}(B)$.

Remark 1. *If there is an isometric isomorphism between two normed or countably normed spaces X and Y , then we have that $X = Y$. In particular, $W_{\omega}^{m,p}(\Omega, \text{loc}) = W_{\omega}^{m,p}(\Omega \setminus E, \text{loc})$ implies that $|E| = 0$, and for every function $u \in W_{\omega}^{m,p}(\Omega \setminus E, \text{loc})$ there is a function $v \in W_{\omega}^{m,p}(\Omega, \text{loc})$, for which $v|_{\Omega \setminus E} = u$. Therefore, similar conditions can be written for $W_{\omega}^{m,p}(\Omega)$, $W_{p,\omega}^m(\Omega)$, and, by Proposition 4, for $L_{p,\omega}^m(\Omega)$ as subspaces of $W_{\omega}^{m,p}(\Omega, \text{loc})$. For example, $L_{p,\omega}^m(\Omega \setminus E) = L_{p,\omega}^m(\Omega)$ implies that $|E| = 0$, and for every $u \in L_{p,\omega}^m(\Omega \setminus E)$ there is a function $v \in L_{p,\omega}^m(\Omega)$, for which $v|_{\Omega \setminus E} = u$. Similarly, for example, by $W_{p,\omega}^m(\Omega \setminus E) = W_{p,\omega}^m(\Omega)$, we mean that every function $u \in W_{p,\omega}^m(\Omega)$ can be approximated in $\|\cdot\|_{W_{p,\omega}^m(\Omega)}$ by functions from $C_0^\infty(\Omega \setminus E)$. Finally, for example, by $W_{p,\omega}^m(\Omega) = W_{p,\omega}^m(\Omega)$, we imply that every function $u \in W_{p,\omega}^m(\Omega)$ can be approximated in $\|\cdot\|_{W_{p,\omega}^m(\Omega)}$ by functions from $C_0^\infty(\Omega)$.*

Remark 2. *Suppose that $u \in L_{p,\omega}^m(\Omega)$. Then by virtue of Propositions 1,4, partial derivatives $D^\alpha u$ belong to the space $W^{1,1}(\Omega, \text{loc})$ for all $0 \leq |\alpha| \leq m-1$. In addition, $D^\alpha u$ belongs to the space $L_{p,\omega}^1(\Omega)$ for every multi-index α of order $m-1$. Hence (see [10, Sec. 1.1.3, Theorem 1], [13, Theorem 2.5]), every partial derivative $D^\alpha u$ (perhaps, modified on a set of zero m_n -measure) is absolutely continuous in Ω on almost all straight lines (see [13, p.19] for a detailed discussion on "almost all straight lines") parallel to any coordinate axis, $0 \leq |\alpha| \leq m-1$. The weak gradient of $D^\alpha u$ coincides a.e. with the ordinary gradient. Conversely (see [10, Sec. 1.1.3, Theorem 2]), if every partial derivative $D^\alpha u$ is absolutely continuous on Ω on almost all lines which are parallel to the coordinate axes, and its first-order derivatives belong to $L_{p,\omega}(\Omega, \text{loc})$ for $0 \leq |\alpha| < m-1$ and to $L_{p,\omega}(\Omega)$ for $|\alpha| = m-1$, then $u \in L_{p,\omega}^m(\Omega)$.*

2.4. Mollifications. Let $\psi \in C_0^\infty(R^n)$ be a non-negative function, such that $\text{supp } \psi \subset B_1(0)$ and $\int \psi(x) dx = 1$. For any function $u \in L_1(\Omega)$ extended by zero on $R^n \setminus \Omega$, we define the family of its mollifications by the equalities

$$(M_\varepsilon u)(x) = \varepsilon^{-n} \int u(y) \psi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{|\xi| < 1} u(x + \varepsilon \xi) \psi(\xi) d\xi, \quad 0 < \varepsilon \leq 1.$$

The number ε is called a radius of mollification.

The following result is well-known.

Proposition 6 ([14, Theorem 2.1.4, Corollary 2.1.5]). *Suppose that $u \in W_{\omega}^{m,p}(\Omega)$, and let Ω' be an open set, $\Omega' \Subset \Omega$. Then $(M_\varepsilon u)(x) \in C^\infty(\Omega) \cap L_{p,\omega}(\Omega)$, and for $0 < \varepsilon < \min(\text{dist}(\Omega', \partial\Omega), 1)$ the equality $D^\alpha M_\varepsilon u = M_\varepsilon D^\alpha u$ is true on Ω' , $1 \leq |\alpha| \leq m$; and $M_\varepsilon u \rightarrow u$ holds on $W_{\omega}^{m,p}(\Omega')$ as $\varepsilon \rightarrow 0$. For the case when $\Omega = R^n$, we have a convergence $M_\varepsilon u \rightarrow u$ on $W_{\omega}^{m,p}(R^n)$.*

2.5. Capacity and $NC_{p,\omega}$ -sets. A triple of sets (F_0, F_1, Ω) , where F_0 and F_1 are disjoint compact subsets of R^n , is called a condenser. Suppose that $F_0 \cup F_1 \subset \bar{\Omega}$. Then we define (see [2, Proposition 5]) (p, ω) -capacity of a condenser (F_0, F_1, Ω) by $C_{p,\omega}(F_0, F_1, \Omega) = 0$,

if at least one of the following is true: $F_0 = \emptyset$ or $F_1 = \emptyset$. If F_0 and F_1 are nonempty sets, then the definition has the form

$$C_{p,\omega}(F_0, F_1, \Omega) = \inf_u \int_{\Omega} |\nabla u|^p \omega \, dx,$$

where the infimum is taken for all real-valued bounded functions u , such that $u|_{\Omega} \in C^\infty(\Omega) \cap L^1_{p,\omega}(\Omega)$ and $u = j$ in some neighborhood of F_j , $j = 0, 1$.

We denote the set of all admissible functions of this kind by $\text{Adm}_{p,\omega}(F_0, F_1, \Omega)$.

In general, we define a (p, ω) -capacity condenser (F_0, F_1, Ω) by

$$C_{p,\omega}(F_0, F_1, \Omega) = C_{p,\omega}(F_0 \cap \bar{\Omega}, F_1 \cap \bar{\Omega}, \Omega).$$

Consider a relatively closed subset $E \subset \Omega$, and let Π be a coordinate rectangle

$$\{x = (x_1, \dots, x_n) \in R^n : a_i < x_i < b_i, i = 1, \dots, n\}.$$

We denote the facets of this rectangle parallel to the hyperplane $x_i = 0$ by $\sigma_{0i} \subset \{x : x_i = a_i\}$ and $\sigma_{1i} \subset \{x : x_i = b_i\}$. If

$$(6) \quad C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) = C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi), \quad i = 1, 2, \dots, n.$$

for every coordinate rectangle Π with $\bar{\Pi} \subset \Omega$, then E is called an $NC_{p,\omega}$ -set in Ω .

Similarly to the case $\omega = 1$ [15], the $NC_{p,\omega}$ -set has zero m_n -measure (see [5, Lemma 5], [4, Theorem 1]) and $\tau \setminus E$ is an open connected set for every connected component τ of Ω [4, Theorem 8].

Remark. We have provided a capacity definition of an $NC_{p,\omega}$ -set. Since the capacity of condenser is equal to the modulus of this condenser [4, Theorem 1], this definition is equivalent to the modulus definition of an $NC_{p,\omega}$ -set in [5, Sec. 3].

We now define another kind of capacity. For a compact set $e \subset \Omega$, we put $\mathfrak{M}(e, \Omega) = \{u \in C_0^\infty(\Omega) : u = 1 \text{ in some neighborhood of } e\}$ and suppose that $S_{p,\omega}^m(\Omega)$ is one of the spaces $W_{\omega}^{m,p}(\Omega)$, $W_{p,\omega}^m(\Omega)$, $L_{p,\omega}^m(\Omega)$. Following [10, §9.1], we define the capacity $\text{Cap}(e, S_{p,\omega}^m(\Omega))$ of e by $\inf \|u\|_{S_{p,\omega}^m(\Omega)}^p$, where the infimum is taken for all $u \in \mathfrak{M}(e, \Omega)$. The definition is extended to an arbitrary Borel set $F \subset \Omega$ by setting $\text{Cap}(F, S_{p,\omega}^m(\Omega)) = \sup\{\text{Cap}(e, S_{p,\omega}^m(\Omega)) : e \subset F, e \text{ compact}\}$. The number $\text{Cap}(F, W_{\omega}^{m,p}(R^n))$ will be called a (p, m, ω) -capacity of the Borel set $F \subset R^n$. As usual, for the case when $\Omega = R^n$, we will drop the reference to Ω , as follows: $\text{Cap}(F, S_{p,\omega}^m)$.

Remark 3. Using the truncation [12, Theorem 4.14] $v = \min(\max(0, u), 1) \in W_{\omega}^{1,p}(\Omega)$ for $u \in \mathfrak{M}(e, \Omega)$ and its subsequent mollification in R^n (see Proposition 6), the $\mathfrak{M}(e, \Omega)$ class in the definition of $\text{Cap}(e, L_{p,\omega}^1(\Omega))$ can be replaced by the class

$$\tilde{\mathfrak{M}}(e, \Omega) = \{u \in C_0^\infty(\Omega) : 0 \leq u \leq 1 \text{ in } \Omega, u = 1 \text{ in some neighborhood of } e\}.$$

Remark 4. From the definition of $\text{Cap}(F, W_{\omega}^{m,p}(\Omega))$, it immediately follows that

$$(7) \quad \text{Cap}(F, L_{p,\omega}^1(\Omega)) \leq \text{Cap}(F, W_{\omega}^{1,p}(\Omega)) \leq \text{Cap}(F, W_{\omega}^{m,p}(\Omega))$$

for $1 \leq p < \infty$ and $m \in \mathbb{N}$.

2.6. Coverings. First, we will present the following version of Besicovitch theorem [6, Sec. 1, p. 5].

Proposition 7 ([10, Sec. 1.2.1, Theorem 1]). *Let S be a bounded set in R^n . For each $x \in S$, a ball $B_{r(x)}(x)$ is given, $r(x) > 0$. Then, one can choose among the given balls $\{B_{r(x)}(x)\}_{x \in S}$ a sequence $\{B_k\}$ (possibly finite), such that 1) $S \subset \bigcup_k B_k$; 2) no point of R^n belongs to more than θ_n (a number which only depends on n) balls of the sequence $\{B_k\}$, i.e. for every $z \in R^n$, we have that $\sum_k \chi_{B_k}(z) \leq \theta_n$.*

Now let Ω_j be a sequence of open sets, such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\Omega = \bigcup_j \Omega_j$. Set $F_1 = \overline{\Omega_1}$, $F_2 = \overline{\Omega_2 \setminus \Omega_1}$, $F_3 = \overline{\Omega_3 \setminus \Omega_2}$, $\dots$. Suppose that D_j is another sequence of open sets, such that $D_j \Subset \Omega_j \Subset D_{j+1}$.

For each $x \in F_1$, we define a ball $B_{r_1(x)}(x)$, where $0 < r_1(x) < \min(1, \text{dist}(F_1, \partial D_2))$. For $j \geq 2$, we define an inequality $0 < r_j(x) < \min\left(\frac{1}{j}, \text{dist}(F_j, \overline{D_{j-1}}), \text{dist}(F_j, \partial D_{j+1})\right)$ and a ball $B_{r_j(x)}(x)$ if $x \in F_j$.

Assuming that in Proposition 7 we have that $S = F_j$, $\{B_{r_j(x)}(x)\}_{x \in F_j}$, we get a finite sequence of balls $B_{jk} \in \{B_{r_j(x)}(x)\}_{x \in F_j}$, $1 \leq k \leq k_j$, for which

- 1) $F_j \subset \bigcup_k B_{jk}$;
- 2) $\sum_k \chi_{B_{jk}}(x) \leq \theta_n, \forall x \in R^n$.

Next, choosing the value of j from 1 to ∞ , we come to another covering result.

Corollary 1. *Given Ω , there exists a sequence of balls $B_j = B_{r_j}(x_j)$, $r_j > 0$, such that*

- 1) $\bigcup_j B_j = \Omega$;
- 2) $\lim_{j \rightarrow \infty} r_j = 0$;
- 3) *for each $x \in R^n$, we have that $\sum_j \chi_{B_j}(x) \leq 2\theta_n$, i.e. $\{B_j\}$ is a covering of the set Ω of bounded multiplicity;*
- 4) $\{B_j\}$ is a locally finite covering of Ω .

3. APPROXIMATION BY SMOOTH FUNCTIONS ON Ω , COMPLETENESS OF $W_{p,\omega}^m(\Omega)$

3.1. Approximation on $S_{p,\omega}^m(\Omega)$. The following two theorems state that functions from $S_{p,\omega}^m(\Omega)$ can be approximated by smooth functions on $S_{p,\omega}^m(\Omega)$.

Theorem 1. *If $u \in L_{p,\omega}^m(\Omega)$, then there exists a sequence of functions $u_k \in L_{p,\omega}^m(\Omega) \cap C^\infty(\Omega)$, such that*

- 1) $u_k \rightarrow u$ in $L_{p,\omega}^m(\Omega)$;
- 2) $u_k \rightarrow u$ in $W_{\omega}^{m,p}(\Omega')$ for every open set $\Omega' \Subset \Omega$;
- 3) *from the estimate $|u| \leq C$ on Ω it follows that $|u_k| \leq 2C\theta_n$ on Ω for all $k \geq 1$, where θ_n is the constant from Proposition 7.*

Proof. We will prove assertions 1) and 2), using an approach by Maz'ya [10, Sec. 1.1.5, Theorem 1]. Suppose that $\{B_j\}$ is a covering of the bounded multiplicity on Ω from Corollary 1, and $\{\varphi_j\}$ is a partition of a unity subordinated to this covering. Consider a function $u \in L_{p,\omega}^m(\Omega)$. Then, $u \in W_{\omega}^{m,p}(B_j)$ by Proposition 5, and $u \in W^{m,1}(B_j)$ by Proposition 1 for every ball B_j .

Since $\varphi_j \in C_0^\infty(B_j)$, we have that $u\varphi_j \in W^{m,1}(B_j)$, and that every partial derivative $D^\alpha u$ is written, as usual, using the Leibnitz formula (1) [9, Sec. 6.12]. This implies that $u\varphi_j \in W_{\omega}^{m,p}(B_j)$. By virtue of $\text{supp } \varphi_j \subset B_j$, we can assume that $u\varphi_j \in W_{\omega}^{m,p}(R^n)$, if we put $u\varphi_j = 0$ on $R^n \setminus B_j$, $j \geq 1$.

Here, note that $D^\alpha(u\varphi_j) = 0$ on $\Omega \setminus \text{supp } \varphi_j$ for all $0 \leq |\alpha| \leq m$, $j \geq 1$. We take $0 < \varepsilon < \frac{1}{2}$ and denote by z_j the mollification of the function $v_j = u\varphi_j$, where the radius of mollification is ρ_j , $0 < \rho_j < \min(1, \text{dist}(\text{supp } \varphi_j, \partial B_j))$. Moreover, according to Proposition 6, the choice of ρ_j is made in a way that

$$(8) \quad \|v_j - z_j\|_{W_{\omega}^{m,p}(R^n)} < \varepsilon^j.$$

By the choice of $\{B_j\}$, on every open set $\Omega' \Subset \Omega$, the equality $u = \sum_j v_j$ is valid. Here, the sum contains a finite number of non-zero terms v_j . The same property holds for the

sum $g = \sum_j z_j$ on Ω' , and therefore $g \in C^\infty(\Omega)$. Thus, by (8), we have that

$$(9) \quad \|u - g\|_{L_{p,\omega}^m(\Omega')} \leq \|u - g\|_{W_{\omega}^{m,p}(\Omega')} \leq \sum_j \|v_j - z_j\|_{W_{\omega}^{m,p}(R^n)} < \frac{\varepsilon}{1-\varepsilon} \leq 2\varepsilon.$$

This inequality along with arbitrary choice of Ω' imply that $g \in L_{p,\omega}^m(\Omega)$ and $\|u - g\|_{L_{p,\omega}^m(\Omega)} \leq 2\varepsilon$.

In (9), we set $\varepsilon = \frac{1}{4k}$, $k \in \mathbb{N}$ and denote the corresponding function g by u_k . Then we have that

$$\|u - u_k\|_{L_{p,\omega}^m(\Omega)} \rightarrow 0, \quad \|u - u_k\|_{W_{\omega}^{m,p}(\Omega')} \rightarrow 0$$

for every open set $\Omega' \Subset \Omega$ as $k \rightarrow \infty$. This implies assertions 1) and 2) of the Theorem.

Now, suppose that $|u(x)| \leq C$ on Ω , then obviously the estimates $|v_j(x)| \leq C$, $|z_j(x)| \leq C$ are valid at every point $x \in \Omega$ for all $j \geq 1$. In addition, in the equality $z(x) = \sum_j z_j(x)$

mentioned above, depending on the choice of $\{B_j\}$, the sum contains no more than $2\theta_n$ non-zero terms $z_j(x)$, $x \in \Omega$. Hence, we have that $|z(x)| \leq 2C\theta_n$, and therefore $|u_k(x)| \leq 2C\theta_n$ for all $x \in \Omega$ and $k \geq 1$. This completes the proof of the theorem. $\square$

Using the same reasoning as in Theorem 1, we get another result.

Theorem 2. *If $u \in W_{p,\omega}^m(\Omega)$, then there exists a sequence of functions $u_k \in W_{p,\omega}^m(\Omega) \cap C^\infty(\Omega)$, such that 1) $u_k \rightarrow u$ in $W_{p,\omega}^m(\Omega)$; 2) $u_k \rightarrow u$ in $W_{\omega}^{m,p}(\Omega')$ for every open set, $\Omega' \Subset \Omega$; 3) from the estimate $|u| \leq C$ on Ω it follows that $|u_k| \leq 2C\theta_n$ on Ω for all $k \geq 1$, where θ_n is the constant from Proposition 7.*

Similarly, if $u \in W_{\omega}^{m,p}(\Omega)$, then there exists a sequence of functions $u_k \in W_{p,\omega}^m(\Omega) \cap C^\infty(\Omega)$, such that 1) $u_k \rightarrow u$ in $W_{\omega}^{m,p}(\Omega)$; 2) from the estimate $|u| \leq C$ on Ω it follows that $|u_k| \leq 2C\theta_n$ on Ω for all $k \geq 1$.

Theorem 3. $W_{\omega}^{m,p}(R^n) = \overset{\circ}{W}_{\omega}^{m,p}(R^n)$.

Proof. Here we use the same approach as in [1, Theorem 3.22]. According to Theorem 2, it is sufficient to show that every function $u \in C^\infty(R^n) \cap W_{\omega}^{m,p}(R^n)$ can be approximated in $W_{\omega}^{m,p}(R^n)$ by functions from $C_0^\infty(R^n)$. Let f be a fixed function in $C_0^\infty(R^n)$, satisfying the following conditions: 1) $f(x) = 1$, if $|x| < 1$; 2) $f(x) = 0$, if $|x| \geq 2$; 3) $|D^\alpha f(x)| \leq C_1$ (constant) for all $x \in R^n$, and $0 \leq |\alpha| \leq m$.

For $\varepsilon > 0$, suppose that $f_\varepsilon(x) = f(\varepsilon x)$. Then $f_\varepsilon(x) = 1$ for $|x| \leq \frac{1}{\varepsilon}$, and also $|D^\alpha f_\varepsilon(x)| \leq C_1 \varepsilon^{|\alpha|} \leq C_1$, if $\varepsilon \leq 1$.

If $u \in W_{\omega}^{m,p}(R^n) \cap C^\infty(R^n)$, then $u_\varepsilon = f_\varepsilon u$ belongs to $W_{\omega}^{m,p}(R^n)$ and has a bounded support. For $0 < \varepsilon \leq 1$ and $|\alpha| \leq m$ by (1), we have that

$$|D^\alpha u_\varepsilon(x)| = \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(x) \cdot D^{\alpha-\beta} f_\varepsilon(x) \right| \leq C_1 \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta u(x)|.$$

Therefore, setting $\Omega_\varepsilon = \{x \in \Omega : |x| > \frac{1}{\varepsilon}\}$, we obtain

$\|u - u_\varepsilon\|_{W_{\omega}^{m,p}(\Omega)} = \|u - u_\varepsilon\|_{W_{\omega}^{m,p}(\Omega_\varepsilon)} \leq \|u\|_{W_{\omega}^{m,p}(\Omega_\varepsilon)} + \|u_\varepsilon\|_{W_{\omega}^{m,p}(\Omega_\varepsilon)} \leq C_2 \|u\|_{W_{\omega}^{m,p}(\Omega_\varepsilon)}$, where the constant C_2 does not depend on the choice of u . The right side approaches to zero as $\varepsilon \rightarrow 0$. The proof is complete. $\square$

3.2. Completeness of $W_{p,\omega}^m(\Omega)$.

Theorem 4. *If $1 \leq p < \infty$, $m \in \mathbb{N}$, then $W_{p,\omega}^m(\Omega)$ is a Banach space.*

Proof. Let $\{u_j\}$ be a Cauchy sequence in $W_{p,\omega}^m(\Omega)$, i.e. $\|u_i - u_j\|_{W_{p,\omega}^m(\Omega)} \rightarrow 0$ as $i, j \rightarrow \infty$. Because of completeness of $L_{p,\omega}(\Omega)$ (see Proposition 1), there exist functions u , T^α , $|\alpha| = m$, such that $u_j \rightarrow u$, $D^\alpha u_j \rightarrow T^\alpha$ in $L_{p,\omega}(\Omega)$ for all α , $|\alpha| = m$, as $j \rightarrow \infty$. Then by Proposition 4, we have that $u_j \in W_{\omega}^{m,p}(\Omega, \text{loc})$ for all $j \geq 1$. This, with Proposition

5, allows us to conclude that $\{u_j\}$ is a Cauchy sequence in $W_{\omega}^{m,p}(B)$ for every $B \Subset \Omega$. Hence, $D^\beta u_j \rightarrow D^\beta u$ in $L_{p,\omega}(B)$ for these B , and $D^\beta u \in L_{p,\omega}(B) \cap L_1(\Omega, \text{loc})$ for all multi-indices β , $1 \leq |\beta| \leq m-1$. In addition, we have that

$$\int_{\Omega} u D^\alpha \varphi dx = \lim_{j \rightarrow \infty} \int_{\Omega} u_j D^\alpha \varphi dx = (-1)^m \lim_{j \rightarrow \infty} \int_{\Omega} \varphi D^\alpha u_j dx = (-1)^m \int_{\Omega} \varphi T^\alpha dx$$

for any function $\varphi \in C_0^\infty(\Omega)$ and for every multi-index α of order m . Consequently, $D^\alpha u = T^\alpha$, and the sequence $\{u_j\}$ converges to the function u in $W_{p,\omega}^m(\Omega)$. The theorem is proved. $\square$

Remark 5. The assertion about completeness of the space $W_{\omega}^{m,p}(\Omega)$ can be found in [14, Proposition 2.1.2].

4. SOME PROPERTIES OF CAPACITY $\text{Cap}(F, S_{p,\omega}^m(\Omega))$

Here we consider the question of equivalence of equalities $\text{Cap}(F, W_{\omega}^{m,p}(\Omega)) = 0$, $\text{Cap}(F, W_{p,\omega}^m(\Omega)) = 0$, $\text{Cap}(F, L_{p,\omega}^m(\Omega)) = 0$ on a Borel set $F \subset \Omega$.

Lemma 1. *If e is a compact subset of Ω , then the equalities $\text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0$ and $\text{Cap}(e, W_{\omega}^{m,p}(\Omega)) = 0$ are equivalent.*

Proof. If $\text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0$, then the inclusion $\mathfrak{M}(e, \Omega) \subset \mathfrak{M}(e, R^n)$ immediately implies that $\text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0$. Conversely, suppose that $\text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0$, and that Ω_1 and Ω_2 are open sets, such that $e \subset \Omega_1 \Subset \Omega_2 \Subset \Omega$. Consider the function $\varphi \in C_0^\infty(\Omega_2)$, satisfying the following conditions: 1) $0 \leq \varphi(x) \leq 1$, if $x \in R^n$; 2) $\varphi(x) = 1$, if $x \in \Omega_1$. By construction, $\max_{|\alpha| \leq m} |D^\alpha u| \leq C_1$, if $x \in R^n$. Moreover, suppose that $u_j \in \mathfrak{M}(e, R^n)$, $j \geq 1$, and that $\|u_j\|_{W_{p,\omega}^m}^p \rightarrow \text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0$ as $j \rightarrow \infty$. Then from (5), we have that $\|u_j\|_{W_{\omega}^{m,p}(B)} \rightarrow 0$ for every ball $B \subset R^n$ as $j \rightarrow \infty$. Hence, $\lim_{j \rightarrow \infty} \|u_j\|_{W_{\omega}^{m,p}(\Omega_2)} \rightarrow 0$.

Since $\varphi u_j \in \mathfrak{M}(e, \Omega)$, from (1) it follows that

$$\lim_{j \rightarrow \infty} \|\varphi u_j\|_{W_{\omega}^{m,p}(\Omega_2)} = \lim_{j \rightarrow \infty} \|\varphi u_j\|_{W_{\omega}^{m,p}(\Omega)} = 0 = \text{Cap}(e, W_{\omega}^{m,p}(\Omega)).$$

It is obvious that $\text{Cap}(e, W_{p,\omega}^m(\Omega)) \leq \text{Cap}(e, W_{\omega}^{m,p}(\Omega))$. Therefore,

$$0 = \text{Cap}(e, W_{\omega}^{m,p}(\Omega)) = \text{Cap}(e, W_{p,\omega}^m(\Omega)).$$

Thus, the required assertion of the Lemma is proved along with the equivalence of the following equalities:

$$(10) \quad \text{Cap}(e, W_{p,\omega}^m(\Omega)) = 0, \quad \text{Cap}(e, W_{\omega}^{m,p}(\Omega)) = 0, \quad \text{Cap}(e, W_{\omega}^{m,p}) = 0.$$

$\square$

Taking into account (10), Lemma 1, and the definition of $\text{Cap}(F, S_{p,\omega}^m(\Omega))$, we also obtain

Corollary 2. *Let F be a Borel subset of Ω . Then the equalities*

$$\text{Cap}(F, W_{p,\omega}^m(\Omega)) = 0, \quad \text{Cap}(F, W_{\omega}^{m,p}(\Omega)) = 0, \quad \text{Cap}(F, W_{\omega}^{m,p}) = 0$$

are equivalent.

Consider some compact $e \subset \Omega$. For all $u \in \mathfrak{M}(e, \Omega)$, it is easily seen that

$$\|u\|_{L_{p,\omega}^m(\Omega)} \leq \|u\|_{W_{\omega}^{m,p}(\Omega)}.$$

In addition, for the case when $p = 1$, since we know the estimate $\omega(x) \geq \frac{C}{(1+|x|)^n}$ for a.e. $x \in R^n$ [14, Remark 1.2.4, Property 8], we obtain

$$(11) \quad m_n(e) = \int_e |u| dx \leq C_1 \int_{\Omega_1} |u| \omega dx \leq C_1 \|u\|_{W_{\omega}^{m,1}(\Omega)}.$$

Similarly, for the case when $p > 1$, due to Hölder's inequality, we get that

$$(12) \quad m_n(e) = \int_e |u| dx \leq \left(\int_e |u|^p \omega dx \right)^{1/p} \left(\int_e \omega^{-\frac{1}{p-1}} dx \right)^{\frac{p-1}{p}} \leq C_2 \|u\|_{W_{\omega}^{m,p}(\Omega)}.$$

Using these estimates for $u = \varphi u_j$ and taking into account the proof of Lemma 1, we easily deduce the following

Lemma 2. *If e is a compact set of zero (p, m, ω) -capacity on Ω , then $m_n(e) = 0$, and $\text{Cap}(e, L_{p,\omega}^m(\Omega)) = 0$.*

Replacing e in Lemma 2 by a Borel set $F \subset \Omega$, we get a more general result.

Corollary 3. *If F is a Borel set of zero (p, m, ω) -capacity on Ω , then $m_n(F) = 0$, and $\text{Cap}(F, L_{p,\omega}^m(\Omega)) = 0$.*

Lemma 3. *Suppose that e is a compact set of zero (p, m, ω) -capacity and $e \subset B_1 = B(a, r_1)$, then $B_1 \setminus e$ is a connected open set.*

Proof. First, assume that $m = 1$. Suppose, on the contrary, that the set $B_1 \setminus e$ has a nonempty connected component G , where $G \Subset B_1$ and $\partial G \subset e$. By Lemma 2, $\text{Cap}(e, L_{p,\omega}^1(B_1)) = 0$ and $m_n(e) = 0$.

According to Remark 3, there exists a sequence of functions $u_j \in \tilde{\mathfrak{M}}(e, B_1)$, $j \geq 1$, such that

$$(13) \quad \lim_{j \rightarrow \infty} \|u_j\|_{L_{p,\omega}^1(B_1)} = 0.$$

Since $u_j \in C_0^\infty(B_1)$, we have that $u_j = 0$ on $R^n \setminus B_1$. Therefore, in (13), the ball B_1 can be replaced with the ball $B_2 = B(a, r_2)$ for every $r_2 > r_1$. In other words,

$$(14) \quad \lim_{j \rightarrow \infty} \|u_j\|_{L_{p,\omega}^1(B_2)} = 0.$$

Set $v_j = 1$, if $x \in G$, and $v_j = u_j$, if $x \in R^n \setminus G$. It is obvious that $0 \leq v_j \leq 1$, $\text{supp } v_j \subset B_1$, $v_j = 1$ in some neighborhood O_j of a compact set $e \cup G$, where $O_j \Subset B_1$.

Moreover, v_j satisfies the Lipschitz condition on R^n , and so $v_j \in W_{\omega}^{1,p}(B_2)$. By construction, $\int_{B_2} |\nabla v_j|^p \omega dx \leq \int_{B_2} |\nabla u_j|^p \omega dx$ for all $j \geq 1$. From this relation and equality (14), we derive that

$$(15) \quad \lim_{j \rightarrow \infty} \|\nabla v_j\|_{L_{p,\omega}(B_2)} = 0.$$

Equality (15), Propositions 3,4, and the arbitrariness of $r_2 > r_1$ imply the existence of a sequence $\{c_j\}$ of constants, such that

$$(16) \quad \lim_{j \rightarrow \infty} \|v_j - c_j\|_{W_{\omega}^{1,p}(B_2)} = 0.$$

On the other hand, we know that $v_j = 0$ on $B_2 \setminus B_1$ for all $j \geq 1$. It follows that $\lim_{j \rightarrow \infty} c_j = 0$. Now, from (16) we deduce that $\lim_{j \rightarrow \infty} \|v_j\|_{W_{\omega}^{1,p}(B_2)} = 0$.

Using estimates (11), (12), we see that

$$(17) \quad m_n(G) \leq C \|v_j\|_{W_{\omega}^{1,p}(B_2)}.$$

Since $m_n(G) > 0$, setting $j \rightarrow \infty$ in (17), we arrive at a contradiction. Thus, $B_1 \setminus e$ is a connected open set.

Now suppose that $m > 1$ and $\text{Cap}(e, W_{\omega}^{m,p}) = 0$. Remark 4 implies that

$$\text{Cap}(e, L_{p,\omega}^1(B_1)) = 0.$$

According to the above arguments, $B_1 \setminus e$ is a connected open set. The lemma is proved. $\square$

Lemma 4. *Let e be a compact subset of an open bounded set Ω . Then the equalities $\text{Cap}(e, L_{p,\omega}^m(\Omega)) = 0$ and $\text{Cap}(e, W_{\omega}^{m,p}(\Omega)) = 0$ are equivalent.*

Proof. Taking into account Corollary 2 and Lemma 2, it is enough to show that the equality $\text{Cap}(e, L_{p,\omega}^m(\Omega)) = 0$ implies $\text{Cap}(e, W_{\omega}^{m,p}(\Omega)) = 0$. In fact, suppose that

$$\text{Cap}(e, L_{p,\omega}^m(\Omega)) = 0$$

and $\{u_j\}$ is a sequence of functions $u_j \in \mathfrak{M}(e, \Omega)$, such that

$$(18) \quad \|\nabla_m u_j\|_{L_{p,\omega}(\Omega)}^p \rightarrow \text{Cap}(e, L_{p,\omega}^m(\Omega)) = 0 \text{ as } j \rightarrow \infty.$$

By construction, $u_j \in C_0^\infty(\Omega)$, and therefore, $u_j \in C_0^\infty(B)$ for all $j \geq 1$, and for every ball, $B = B(0, r)$, where $\bar{\Omega} \subset B$. Obviously, $D^\alpha u_j \in C_0^\infty(\Omega)$, and $D^\alpha u_j = 0$ on $B \setminus \Omega$ for all multi-indices α of order at most m . Moreover, from (18), it follows that $\lim_{j \rightarrow \infty} \|D^\alpha u_j\|_{L_{p,\omega}^1(B)} = 0$ for all multi-indices α of order $m-1$. By Propositions 3, 4 there exists a sequence $\{c_j\}$ of constants, such that

$$(19) \quad \lim_{j \rightarrow \infty} \|D^\alpha u_j - c_j\|_{W_{\omega}^{1,p}(B)} = 0$$

for all multi-indices α of order $m-1$ and every ball $B(0, r)$, $\bar{\Omega} \subset B(0, r)$.

Here, the equality $\lim_{j \rightarrow \infty} c_j = 0$ is obtained in a similar way as in the proof of Lemma 3.

Therefore, (19) implies the relation

$$(20) \quad \lim_{j \rightarrow \infty} \|D^\alpha u_j\|_{W_{\omega}^{1,p}(B)} = 0$$

with the same α and B .

Replacing the multi-index α , $|\alpha| = m-1$, in the above argument sequentially by a multi-index of order less than $m-1$, if necessary, we conclude that $\lim_{j \rightarrow \infty} \|D^\alpha u_j\|_{W_{\omega}^{1,p}(B)} = 0$ with $|\alpha| \leq m-1$, and $B \supset \bar{\Omega}$. Thus, $\lim_{j \rightarrow \infty} \|u_j\|_{W_{\omega}^{m,p}(B)} = 0$.

Since $u_j \in \mathfrak{M}(e, B)$, we have that $\text{Cap}(e, W_{\omega}^{m,p}(B)) = 0$, and therefore, by Corollary 2, we obtain $\text{Cap}(e, W_{\omega}^{m,p}) = 0 = \text{Cap}(e, W_{\omega}^{m,p}(\Omega))$. The Lemma is proved. $\square$

For the case when $m = 1$, by Remark 3, the condition $u_j \in \mathfrak{M}(e, \Omega)$ in the proof of Lemma 4 can be replaced by the condition $u_j \in \tilde{\mathfrak{M}}(e, \Omega)$, which gives rise to the following result.

Corollary 4. *Suppose that e is a compact subset of open bounded set Ω , and $\text{Cap}(e, W_{\omega}^{1,p}) = 0$. Then the class $\mathfrak{M}(e, \Omega)$ in the definition of $\text{Cap}(e, W_{\omega}^{1,p})$ can be replaced by the class $\tilde{\mathfrak{M}}(e, \Omega)$. Moreover, if F is a Borel subset of open bounded set Ω , then the equalities $\text{Cap}(F, L_{p,\omega}^m(\Omega)) = 0$ and $\text{Cap}(F, W_{\omega}^{m,p}(\Omega)) = 0$ are equivalent.*

Remark 6. *For the case when $\omega = 1$, the Lemma 4 was proved by Maz'ya [10, Sec. 9.1.4].*

5. EXCEPTIONAL SETS IN PROBLEM (I) FOR $S_{p,\omega}^m(\Omega)$

By Remark 1, the equality $S_{p,\omega}^m(\Omega) = S_{p,\omega}^m(\Omega \setminus E)$ implies that $m_n(E) = 0$, and for each $u \in S_{p,\omega}^m(\Omega \setminus E)$ there exists $v \in S_{p,\omega}^m(\Omega)$, for which $v|_{\Omega \setminus E} = u$.

Recall that by definition, E is a relatively closed subset on Ω . In this case, the function v will be called the extension of u in $S_{p,\omega}^m(\Omega)$.

First, we refine the statement of Theorem 1 for $L_{p,\omega}^1(\Omega)$.

Theorem 5. *Let $u \in L_{p,\omega}^1(\Omega)$ and $\{\Omega_j\}$ be some sequence of open sets Ω_j , such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$, and $\bigcup_j \Omega_j = \Omega$. Then there exists a sequence of bounded functions*

$u_j \in L_{p,\omega}^1(\Omega) \cap C^\infty(\Omega)$, $j \geq 1$, such that

$$(21) \quad \left(\int_{\Omega_j} |u - u_j|^p \omega \, dx \right)^{1/p} < \frac{1}{j}, \quad \lim_{j \rightarrow \infty} \|u - u_j\|_{L_{p,\omega}^1(\Omega)} = 0.$$

Proof. By Theorem 1, there exists a function $v_j \in C^\infty(\Omega) \cap L^1_{p,\omega}(\Omega)$, for which the estimates

$$(22) \quad \int_{\Omega_j} |u - v_j|^p \omega \, dx < \frac{1}{(3j)^p},$$

$$(23) \quad \int_{\Omega} |\nabla u - \nabla v_j|^p \omega \, dx < \frac{1}{(3j)^p}$$

are valid for all $j \geq 1$.

Now we choose an open set $G_j \Subset \Omega$, such that $\Omega_j \Subset G_j$ and

$$(24) \quad \int_{\Omega \setminus G_j} |\nabla v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p}.$$

For $l \in \mathbb{N}$, we set $\Omega_{j,l} = \{x \in \Omega : -l < v_j < l\}$ and choose $l = l_j$, such that

$$(25) \quad \int_{\Omega_j \setminus \tilde{\Omega}_j} |v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p},$$

$$(26) \quad \int_{G_j \setminus \tilde{\Omega}_j} |\nabla v_j|^p \omega \, dx < \frac{1}{2^{p+1}(3j)^p},$$

where $\tilde{\Omega}_j = \Omega_{j,l_j}$.

Now suppose that $h_j = \max(-l_j, v_j)$, $g_j = \min(l_j, h_j)$. From the well-known properties of truncation (see [8, Theorem 1.20], [12, Theorem 4.14] for detailed information), it follows that $h_j = g_j = v_j$ and $\nabla h_j = \nabla g_j = \nabla v_j$ on $\tilde{\Omega}_j$, $|\nabla h_j| \leq |\nabla v_j|$ and $|\nabla g_j| \leq |\nabla h_j|$ a.e. on Ω , $|g_j| \leq l_j$ on Ω . Using these relations and the inequalities (22)–(26), we have that

$$(27) \quad \int_{\Omega_j} |v_j - g_j|^p \omega \, dx \leq 2^p \int_{\Omega_j \setminus \tilde{\Omega}_j} |v_j|^p \omega \, dx < \frac{1}{(3j)^p},$$

$$(28) \quad \int_{\Omega} |\nabla v_j - \nabla g_j|^p \omega \, dx \leq 2^p \int_{\Omega \setminus G_j} |\nabla v_j|^p \omega \, dx + 2^p \int_{G_j \setminus \tilde{\Omega}_j} |\nabla v_j|^p \omega \, dx < \frac{1}{(3j)^p}.$$

For the bounded function g_j , by Theorem 1, there exists a bounded function $u_j \in L^1_{p,\omega}(\Omega) \cap C^\infty(\Omega)$, $j \geq 1$, such that

$$(29) \quad \int_{\Omega_j} |g_j - u_j|^p \omega \, dx < \frac{1}{(3j)^p}, \quad \int_{\Omega} |\nabla g_j - \nabla u_j|^p \omega \, dx < \frac{1}{(3j)^p}.$$

Applying the properties of the norm $\|\cdot\|_{L_{p,\omega}(\Omega)}$ and combining (22)–(23) with (27)–(29), we obtain the relations in (21), which completes the proof of the Theorem. $\square$

It is known that an $NC_{p,\omega}$ -set is exceptional for $L^1_{p,\omega}(\Omega)$, $1 < p < \infty$ [5, Corollary 2]. Below, we will provide an addition to this result and extend it to $L^m_{p,\omega}(\Omega)$.

Theorem 6. Suppose that $1 < p < \infty$ and $m \in \mathbb{N}$. Then E is an exceptional set in problem (i) for $L^1_{p,\omega}(\Omega)$, if and only if E is an $NC_{p,\omega}$ -set in Ω . If E is an $NC_{p,\omega}$ -set in Ω , then $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$ for all $m \in \mathbb{N}$.

Proof. Step 1. Suppose that E is an $NC_{p,\omega}$ -set in Ω , and $u \in L^1_{p,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ is a bounded function in $\Omega \setminus E$, where E as an $NC_{p,\omega}$ -set has zero m_n -measure (see Remark in Sec. 2.4). First, we prove that u can be extended to a function in $L^1_{p,\omega}(\Omega)$. Indeed, consider a sequence $\{\Omega_j\}$ of open sets Ω_j , where $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\bigcup_j \Omega_j = \Omega$.

For a fixed coordinate x_i -axis, the function u is absolutely continuous on every segment $e \subset \Omega_j \setminus E$ parallel to the x_i -axis, $i = 1, 2, \dots, n$ and $j \geq 1$. Then (see the proof for sufficiency condition in Theorem 1 from [5]) the function u can be further defined on $\Omega_j \cap E$ by u_{ji} , so that u_{ji} is absolutely continuous in Ω_j on almost all straight lines parallel to the x_i -axis (see Remark 2).

Hence, the partial derivative $\frac{\partial u_{ji}}{\partial x_i}$ in the classical sense in Ω_j is equal to $\frac{\partial u}{\partial x_i}$ on $\Omega_j \setminus E$. In addition, $u_{ji}, \frac{\partial u_{ji}}{\partial x_i} \in L_1(\Omega_j, \text{loc})$ (by virtue of $u, \frac{\partial u}{\partial x_i} \in L_1(\Omega_j, \text{loc})$). Using integration by parts and Fubini's theorem, we obtain

$$(30) \quad \int_{\Omega_j} \varphi \frac{\partial u_{ji}}{\partial x_i} dx = - \int_{\Omega_j} u_{ji} \frac{\partial \varphi}{\partial x_i} dx$$

for all $\varphi \in C_0^\infty(\Omega_j)$.

Note now that for (30) it is possible to redefine the values of $u_{ji}, \frac{\partial u_{ji}}{\partial x_i}$ on a set of zero m_n -measure in Ω_j . Then we change the values of u_{ji} on $E \cap \Omega_j$, so that $u_{ji} = u_{j1}$ on Ω_j for all $i = 2, \dots, n$.

We set $v_j = u_{j1}$ on Ω_j and suppose that

$$v = \begin{cases} v_1, & x \in \Omega_1; \\ v_j, & x \in \Omega_j \setminus \Omega_{j-1}, \text{ if } j \geq 2. \end{cases}$$

Obviously, $v \in L_{p,\omega}^1(\Omega)$ and $v|_{\Omega \setminus E} = u$. In other words, the function u is extended to a function $v \in L_{p,\omega}^1(\Omega)$.

Step 2. Now, let u be an arbitrary function in $L_{p,\omega}^1(\Omega \setminus E)$, and $\{\tau_k\}$ be a sequence (possibly, finite) of pairwise disjoint connected components of Ω . Then, $\tau_k \setminus E$ is the connected component of $\Omega \setminus E$ (see Sec. 2.4), and $\Omega \setminus E = \bigcup_k (\tau_k \setminus E)$.

By Theorem 5, there exists a sequence of bounded functions $u_j \in L_{p,\omega}^1(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$, $j \geq 1$, such that

$$(31) \quad \lim_{j \rightarrow \infty} \|u_j - u\|_{L_{p,\omega}^1(\Omega \setminus E)} = 0,$$

$$(32) \quad \lim_{j \rightarrow \infty} \|u_j - u\|_{L_{p,\omega}(\Omega')} = 0 \text{ for all } \Omega' \Subset \Omega \setminus E.$$

According to Step 1, we assume that $u_j \in L_{p,\omega}^1(\Omega)$ for all $j \geq 1$. Taking into account (31) and $m_n(E) = 0$, we get that $\{u_j\}$ is a Cauchy sequence in $L_{p,\omega}^1(\Omega)$. Then, by Proposition 2, $\{u_j\}$ converges in $L_{p,\omega}^1(\tau_k)$ to some function v_k , $k \geq 1$, as $j \rightarrow \infty$. Moreover, from (31), $|\nabla(u - v_k)| = 0$ a.e. on $\tau_k \setminus E$, and therefore, $u = v_k + c_k$ (see [3, Sec. 1.1.5]) on $\tau_k \setminus E$. Using (32), it is easy to show that $c_k = 0$, $k \geq 1$.

For all $x \in \Omega$, set $v(x) = v_k(x)$, if $x \in \tau_k$. By construction,

$$\|\nabla v\|_{L_{p,\omega}(\Omega)} = \|\nabla u\|_{L_{p,\omega}(\Omega \setminus E)}, \quad v(x)|_{\Omega \setminus E} = u(x).$$

Hence, E is an exceptional set in problem (i) for $L_{p,\omega}^1(\Omega)$, $1 < p < \infty$.

Step 3. Let E be an exceptional set in problem (i) for $L_{p,\omega}^1(\Omega)$. This implies that E is an $NC_{p,\omega}$ -set on Ω . To establish this fact, we first prove that $\tau_k \setminus E$ is a domain for every τ_k , $k \geq 1$. Here, $\{\tau_k\}$ is a sequence from Step 2.

Suppose, on the contrary, that for some k , the set $\tau_k \setminus E$ has a nonempty connected component η_0 , for which $\eta_1 = (\tau_k \setminus E) \setminus \eta_0$ is a nonempty open set. Suppose that $u_0(x) = 0$ on $\eta_0 \cup (\Omega \setminus \tau_k)$ and $u_0(x) = 1$ on η_1 . Obviously, $u_0 \in C^\infty(\Omega \setminus E) \cap L_{p,\omega}^1(\Omega \setminus E)$.

By the choice of E , u_0 can be extended to the function $v_0 \in L_{p,\omega}^1(\Omega)$. On the other hand, we will show that such extension is impossible. In fact, since τ_k is a domain, $m_n(E) = 0$, there exists a simple broken line $\gamma \subset \tau_k$ with a finite number of links, joining two given points, $a \in \eta_0$ and $b \in \eta_1$, for which $\mathcal{H}^1(\gamma \cap E) = 0$. By construction, $\gamma \cap E$

is a compact set in τ_k . Hence, we can find a ball $B_0 = B(a^0, r_0)$ satisfying the following conditions: $a^0 \in \gamma \cap \overline{\eta_0}$ and $\overline{B_0} \subset \tau_k$, $B_0 \cap \eta_0 \neq \emptyset$ and $B_0 \cap \eta_1 \neq \emptyset$.

We consider $a^1 \in B_0 \cap \eta_0$, $b^1 \in B_0 \cap \eta_1$ and an arbitrary orthogonal transformation $\mathcal{P} : R^n \rightarrow R^n$ [5, Sec. 3.1]. Set $T = a^0 + \mathcal{P}$.

By the choice of T , $T(B_0) = B_0$, $T(B(x, r))$ is a ball $B(a^0 + P(x), r)$ for all $B(x, r) \subset R^n$, and the determinant of the Jacobian matrix is equal to 1: $\det(T'(x)) = 1$. Hence, applying the change of the variable $x = T^{-1}(y)$, in (2), (3), we deduce that $\omega \circ T^{-1}$ is also an A_p -weight for $1 \leq p < \infty$.

Then the linear operator $T_{p,\omega} : L_{p,\omega}^1(B_0) \rightarrow L_{p,\omega \circ T^{-1}}^1(B_0)$, defined by $T_{p,\omega}(u) = u \circ T^{-1}$, transforms $L_{p,\omega}^1(B_0)$ boundedly into $L_{p,\omega \circ T^{-1}}^1(B_0)$ and has a bounded inverse operator [5, Theorem 3], [12, Corollary 6.1.6]. From Remark 2, it follows that every function $u \in L_{p,\omega}^1(B_0)$ is absolutely continuous on almost all straight lines parallel to an arbitrary pre-given straight line in R^n , and, in particular, to the line $a^1 b^1$.

Suppose that P is a closed rectangle in B_0 , and that σ_0, σ_1 are its opposite facets, where $a^1 \in \sigma_0 \subset \eta_0$, $b^1 \in \sigma_1 \subset \eta_1$, and the straight line $a^1 b^1 \perp \sigma_0, \sigma_1$. We denote by Γ the family of all straight segments e joining the facets σ_0, σ_1 in P and parallel to the straight line $a^1 b^1$.

According to the mentioned above, the function v_0 is absolutely continuous on almost every segment $e \in \Gamma$ satisfying the additional condition $\mathcal{H}^1(e \cap E) = 0$. This implies the existence of a limit point $x_e \in E$ on each of such segments e simultaneously for $e \cap \eta_0$ and $e \cap \eta_1$. Consequently, $v_0(x_e) = 0$ and $v_0(x_e) = 1$, which contradicts the definition of the function v_0 . Thus, $\tau_k \setminus E$ is a domain for all $k \geq 1$.

Finally, we will prove that E is an $NC_{p,\omega}$ -set in Ω for $1 < p < \infty$. Suppose that $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, 2, \dots, n\}$ is a coordinate rectangle with the facets σ_{0i}, σ_{1i} , from the definition of an $NC_{p,\omega}$ -set (see Sec. 2.4), $\overline{\Pi} \subset \Omega$. According to $m_n(E) = 0$, we get that $\sigma_{0i} \cup \sigma_{1i} \subset \partial(\Pi \setminus E)$ for all $i = 1, \dots, n$. Now we choose the connected component τ_k of the set Ω , for which $\overline{\Pi} \subset \tau_k$.

In order to prove equality (6) for Π , given $\varepsilon > 0$ and $i = 1, \dots, n$, we find an admissible function $u \in \text{Adm}(\sigma_{0i}, \sigma_{1i}, \Pi)$, such that

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) \leq \int_{\Pi \setminus E} |\nabla u|^p \omega \, dx \leq C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) + \varepsilon.$$

Suppose that G_l is an open neighborhood of the facet σ_{li} , in which $u = l$, and that G'_l is another neighborhood of the facet σ_{li} , where $G'_l \Subset G_l \Subset \tau_k$, $l = 0, 1$. Set $G = G_0 \cup G_1 \cup \Pi$ and let the sequence $\{B_j\}$ be a locally finite covering of the set G by the balls $B_j = B(a_j, r_j) \subset G$.

By Corollary 1, we assume that the covering $\{B_j\}$ has a bounded multiplicity, and that the balls from $\{B_j\}$, which have common points with $\overline{G'_l}$, belong to G_l , $l = 0, 1$.

Let $\{\varphi_j\}$ be a C^∞ -partition of the unity for G , subordinating to the covering $\{B_j\}$. Here, by definition, $\varphi_j \in C_0^\infty(B_j)$, and therefore there is a ball $B'_j = B(a_j, \rho_j)$, such that $0 < \rho_j < r_j$ and $\text{supp } \varphi_j \subset B'_j$. Suppose that $e_j = E \cap \overline{B'_j}$ and $u_j(x) = u(x)\varphi_j(x)$, if $x \in \overline{B'_j} \setminus e_j$; $u_j(x) = 0$ if $x \in \Omega \setminus \overline{B'_j}$.

According to the above arguments, $B_j \setminus e_j$ is a domain. Moreover, it follows from the inclusions $u \in C^\infty(G \setminus E)$, $\varphi_j \in C_0^\infty(B_j)$, that $u_j|_{B_j \setminus E} = u\varphi_j$, and u_j satisfies locally the Lipschitz condition on $\Omega \setminus e_j$. This implies $u_j \in L_{p,\omega}^1(\Omega \setminus e_j)$.

With an appropriate choice of E , the function u_j extends to the function $v_j \in L_{p,\omega}^1(\Omega)$. Moreover, $v_j \in L_{p,\omega}^1(G)$ and, by construction,

$$v_j|_{B_j \setminus E} = u\varphi_j, \quad \|v_j\|_{L_{p,\omega}^1(\Omega)} = \|u\varphi_j\|_{L_{p,\omega}^1(B_j \setminus E)}.$$

Since $u = \sum_j u\varphi_j$ on $G \setminus E$, then, setting $v = \sum_j v_j$ on G , we conclude, similarly to the proof of Theorem 1, that $v \in L^1_{p,\omega}(G)$ and that

$$v|_{G \setminus E} = u, \quad \|v\|_{L^1_{p,\omega}(G)} = \|u\|_{L^1_{p,\omega}(G \setminus E)}.$$

Below, for convenience of calculations, we denote the function v by u . In the proof of Theorem 1, we replace the set Ω , the covering $\{B_j\}$, the partition $\{\varphi_j\}$ of the unity for Ω , u , with the set G , the covering $\{B_j\}$, the partition $\{\varphi_j\}$ of the unity for G , and u , considered here, respectively. In addition, note in this case, that $z_j = u\varphi_j = 0$ on every B_j , $B_j \cap \overline{G'_0} \neq \emptyset$, and $z_j = u\varphi_j = \varphi_j$ on every B_j , $B_j \cap \overline{G'_1} \neq \emptyset$.

Using the same reasoning as in Theorem 1, we get the proper function $z = \sum_j z_j \in \text{Adm}_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi)$, such that $z = 0$ on G'_0 , $z = 1$ on G'_1 , and $\int_{\Pi} |\nabla(u - z)|^p \omega \, dx \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, we have that

$$\begin{aligned} C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi) &\leq \int_{\Pi} |\nabla z|^p \omega \, dx = \int_{\Pi} |\nabla u|^p \omega \, dx + o(1) = \\ &= \int_{\Pi \setminus E} |\nabla u|^p \omega \, dx + o(1) < C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) + \varepsilon + o(1), \end{aligned}$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Here, we suppose that $\varepsilon \rightarrow 0$ and conclude that

$$C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi) = C_{p,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E)$$

for all $i = 1, \dots, n$ and $\Pi, \bar{\Pi} \subset \Omega$. Thus, E is an $NC_{p,\omega}$ -set in Ω .

Step 4. Here we show that an $NC_{p,\omega}$ -set in Ω is exceptional in problem (i) for $L^m_{p,\omega}(\Omega)$, $m = 2, 3, \dots$

Let u be a function in $L^m_{p,\omega}(\Omega \setminus E)$, where E is an $NC_{p,\omega}$ -set in Ω , and $m = 2, 3, \dots$. Below we will keep the previous notation for the extended functions. Obviously, if $D^\alpha u$ is a weak partial derivative of order $|\alpha| = m - 1$ in $\Omega \setminus E$, then $D^\alpha u \in L^1_{p,\omega}(\Omega \setminus E)$. According to Step 1, it follows that $D^\alpha u \in L^1_{p,\omega}(\Omega)$. Hence, by Proposition 4, we have that $D^\alpha u \in L_{p,\omega}(\Omega, \text{loc})$ for all α of order $|\alpha| = m - 1$. Replacing the set Ω in above arguments with an arbitrary open set $\Omega' \Subset \Omega$, the derivative $D^\alpha u$ – by $D^\nu u$, $|\nu| = m - 2$, we deduce that $D^\nu u \in L^1_{p,\omega}(\Omega')$. By Proposition 1, it follows that $D^\nu u \in W^{1,p}_\omega(\Omega, \text{loc}) \cap W^{1,1}_\omega(\Omega, \text{loc})$. Taking into account the continuity of this process, we obtain that $D^\alpha u \in W^{1,p}_\omega(\Omega, \text{loc}) \cap W^{1,1}_\omega(\Omega, \text{loc})$ for all α , $|\alpha| \leq m - 2$. In other words, $u \in L^m_{p,\omega}(\Omega)$, which completes the proof of the Theorem. $\square$

Since $W^{m,p}_\omega(\Omega)$ and $W_{p,\omega}^m(\Omega) \subset L^m_{p,\omega}(\Omega)$, a simple modification of the arguments in the proof of Theorem 6 gives rise to another statement.

Theorem 7. *If E is an $NC_{p,\omega}$ -set in Ω , $1 < p < \infty$, then $W_{p,\omega}^m(\Omega \setminus E) = W_{p,\omega}^m(\Omega)$, $W^{m,p}_\omega(\Omega \setminus E) = W^{m,p}_\omega(\Omega)$.*

The following theorem states that the set of zero (p, m, ω) -capacity is also exceptional in problem (i) for $L^m_{p,\omega}(\Omega)$ for $1 \leq p < \infty$.

Theorem 8. *If E is a set of zero (p, m, ω) -capacity in Ω , $1 \leq p < \infty$ and $m \in \mathbb{N}$, then $L^m_{p,\omega}(\Omega \setminus E) = L^m_{p,\omega}(\Omega)$, and E is an $NC_{p,\omega}$ -set in Ω for $1 < p < \infty$.*

Proof. First, suppose that $m = 1$ and $E \subset \Omega$ is a set of zero $(p, 1, \omega)$ -capacity or, in other words, by Corollary 2, we obtain $\text{Cap}(E, W^{1,p}_\omega(\Omega)) = 0$. In addition, by Corollary 4, we have that $m_n(E) = 0$. We first prove that every bounded function $u \in C^\infty(\Omega \setminus E) \cap L^1_{p,\omega}(\Omega \setminus E)$ can be extended to a function in $L^1_{p,\omega}(\Omega)$.

Here we use the construction from Step 3 of the proof of Theorem 6. Let the sequence $\{B_j\}$ be a locally finite covering of Ω , similar to the one in Corollary 1, where, in particular, $B_j = B(a_j, r_j) \subset \Omega$. Suppose that $\{\varphi_j\}$ is a C^∞ -partition of the unity for Ω , subordinated

to the covering $\{B_j\}$. We fix j and take a ball $B'_j = B(a_j, \rho_j)$, such that $0 < \rho_j < r_j$ and $\text{supp } \varphi_j \subset B'_j$.

Suppose that $e_j = E \cap \overline{B'_j}$ and set $u_j(x) = u(x)\varphi_j(x)$, if $x \in \overline{B'_j} \setminus e_j$, and $u_j(x) = 0$, if $x \in \Omega \setminus \overline{B'_j}$. By the choice of u_j , u , and by Lemma 3, we have that u_j satisfies locally the Lipschitz condition, and that $u_j \in W_{\omega}^{1,p}(\Omega \setminus e_j)$. By construction, we see that $u_j = u\varphi_j$ on $\Omega \setminus E$.

We need to show that u_j can be extended to a function in $W_{\omega}^{1,p}(\Omega)$. By virtue of $u_j = 0$ on $\Omega \setminus \overline{B'_j}$, it is sufficient to show that u_j can be extended to a function in $W_{\omega}^{1,p}(B_j)$. Using Corollary 4, we find a sequence $\{\psi_k\}$, $k \geq 1$, such that $\psi_k \in \tilde{\mathfrak{M}}(e_j, B_j)$ and

$$(33) \quad \|\psi_k\|_{W_{\omega}^{1,p}(B_j)}^p \rightarrow \text{Cap}(e_j, W_{\omega}^{1,p}(B_j)) = 0.$$

Here, by definition, $\psi_k \in C_0^\infty(B_j)$, $0 \leq \psi_k \leq 1$ on B_j and $\psi_k = 1$ on some open neighborhood O_k of a compact set e_j . Let O'_k be another open neighborhood of the set e_j , where $O'_k \Subset O_k$ and $O'_{k+1} \Subset O'_k$, $\bigcap_k O'_k = e_j$.

We set $v_{jk} = u_j(1 - \psi_k)$ on $B_j \setminus e_j$, and for a given $\varepsilon > 0$, we choose $k_0 \in \mathbb{N}$, such that

$$(34) \quad \int_{O'_{k_0} \setminus e_j} |\nabla u_j|^p \omega \, dx < \varepsilon.$$

In addition, note that $\nabla(u_j \psi_k) = u_j \nabla \psi_k + \psi_k \nabla u_j$ a.e. on $B_j \setminus e_j$; u_j , ψ_k are bounded functions on $B_j \setminus e_j$, $|\nabla u_j|$ is a bounded function on $B_j \setminus O'_{k_0}$. Hence, (33) and (34) imply the existence of $k_1 \in \mathbb{N}$

$$(35) \quad \int_{O'_{k_0} \setminus e_j} |\nabla(u_j \psi_k)|^p \omega \, dx = o(1),$$

$$(36) \quad \int_{B_j \setminus O'_{k_0}} |\nabla(u_j \psi_k)|^p \omega \, dx = o(1)$$

for all $k \geq k_1$. Here, $o(1) \rightarrow 0$, if $\varepsilon \rightarrow 0$.

The equalities (35) and (36) imply that

$$(37) \quad v_{jk} \rightarrow u_j \text{ in } W_{\omega}^{1,p}(B_j \setminus e_j) \text{ as } k \rightarrow \infty.$$

On the other hand, set $g_{jk} = u_j(1 - \psi_k)$ on $B_j \setminus O'_k$, $g_{jk} = 0$ on O'_k . Obviously, $g_{jk} \in W_{\omega}^{1,p}(B_j)$ and $v_{jk} = g_{jk}$ on $B_j \setminus e_j$.

Hence, from (33) and $m_n(e_j) = 0$, we conclude that $\{g_{jk}\}$ is a Cauchy sequence in $W_{\omega}^{1,p}(B_j)$. Due to the completeness of the space $W_{\omega}^{1,p}(B_j)$, there exists a function v_j , for which $v_j = \lim_{k \rightarrow \infty} g_{jk}$ in $W_{\omega}^{1,p}(B_j)$, and, along with that, in $L_{p,\omega}^1(B_j)$. According to (37), $v_j = u_j$ on $B_j \setminus e_j$, $j \geq 1$. Therefore, u_j is extended to v_j in $L_{p,\omega}^1(B_j)$, $j \geq 1$.

Setting $v = \sum_j v_j$ in Ω , where $v_j = 0$ on $\Omega \setminus B_j$, $j \geq 1$, as in the proof of Theorem 6

(see Step 3 there), we get that $v \in L_{p,\omega}^1(\Omega)$ and $v|_{\Omega \setminus E} = u$.

Now let u be an arbitrary function in $L_{p,\omega}^1(\Omega \setminus E)$. Repeating verbatim the reasoning in Step 2 of the proof of Theorem 6, we obtain that u extends to a function $v \in L_{p,\omega}^1(\Omega)$, and $v|_{\Omega \setminus E} = u$.

Finally, suppose that u is a function in $L_{p,\omega}^m(\Omega \setminus E)$, where $m \geq 2$, and that E is a set of zero (p, m, ω) -capacity in Ω . By Remark 4, it follows that E is the set of zero $(p, 1, \omega)$ -capacity, and $m_n(E) = 0$. As was proved above, any function $h \in L_{p,\omega}^1(\Omega \setminus E)$ extends to the function z from $L_{p,\omega}^1(\Omega)$, and $z|_{\Omega \setminus E} = h$. Using this fact and the arguments from Step 4 of the proof of Theorem 6, we deduce that the function $u \in L_{p,\omega}^m(\Omega \setminus E)$ extends to the function $v \in L_{p,\omega}^m(\Omega)$, $v|_{\Omega \setminus E} = u$. Consequently, $L_{p,\omega}^m(\Omega \setminus E) = L_{p,\omega}^m(\Omega)$ for $1 \leq p < \infty$, and $m \in \mathbb{N}$.

Now, suppose that $1 < p < \infty$, and that E is a set of zero (p, m, ω) -capacity. Then E is a set of zero $(p, 1, \omega)$ -capacity, and $L_{p,\omega}^1(\Omega \setminus E) = L_{p,\omega}^1(\Omega)$. By Theorem 6, E is an $NC_{p,\omega}$ -set. Thus, the second part and, hence, the entire theorem is proved. $\square$

We will mention two other insertions that can be proved by a simple modification of the arguments in the proof of Theorems 6, 8.

Corollary 5. *If $\text{Cap}(E, W_{\omega}^{m,p}(\Omega)) = 0$, $1 \leq p < \infty$, and $m \in \mathbb{N}$, then $\tau \setminus E$ is a domain for every connected component τ of Ω .*

Corollary 6. *If $\text{Cap}(E, W_{\omega}^{m,p}(\Omega)) = 0$, $1 \leq p < \infty$, and $m \in \mathbb{N}$, then $W_{\omega}^{m,p}(\Omega \setminus E) = W_{\omega}^{m,p}(\Omega)$, $W_{p,\omega}^m(\Omega \setminus E) = W_{p,\omega}^m(\Omega)$.*

6. EXCEPTIONAL SETS IN PROBLEMS (II)-(III) FOR $S_{p,\omega}^m(\Omega)$

According to Remark 3, the equality $\overset{\circ}{S}_{p,\omega}^m(\Omega) = \overset{\circ}{S}_{p,\omega}^m(\Omega \setminus E)$ implies that for every function $u \in \overset{\circ}{S}_{p,\omega}^m(\Omega)$, there exists a sequence of functions $u_j \in C_0^\infty(\Omega \setminus E)$, $j \geq 1$, for which $\lim_{j \rightarrow \infty} \|u - u_j\|_{S_{p,\omega}^m(\Omega)} = 0$. Similarly, from the equality $S_{p,\omega}^m(\Omega) = \overset{\circ}{S}_{p,\omega}^m(\Omega)$ it follows that for every function $u \in S_{p,\omega}^m(\Omega)$ there exists a sequence of functions $u_j \in C_0^\infty(\Omega)$, $j \geq 1$, for which $\lim_{j \rightarrow \infty} \|u - u_j\|_{S_{p,\omega}^m(\Omega)} = 0$.

Here and below, as always, E is a relatively closed subset of the open set Ω .

First, we will give the conditions under which the set E will be exceptional in problem (ii) for $S_{p,\omega}^m(\Omega)$.

Theorem 9. *The equalities $\overset{\circ}{W}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{W}_{p,\omega}^m(\Omega)$, $\overset{\circ}{W}_{\omega}^{m,p}(\Omega \setminus E) = \overset{\circ}{W}_{\omega}^{m,p}(\Omega)$ hold if and only if $\text{Cap}(E, W_{\omega}^{m,p}) = 0$. In order for the equality $L_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{L}_{p,\omega}^m(\Omega)$ to hold, it is necessary that $\text{Cap}(E, L_{p,\omega}^m) = 0$ and it is sufficient that $\text{Cap}(E, W_{\omega}^{m,p}) = 0$. In addition, for the case of a bounded set Ω , the equality $\overset{\circ}{L}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{L}_{p,\omega}^m(\Omega)$ holds if and only if $\text{Cap}(E, W_{\omega}^{m,p}) = 0$.*

Proof. Necessity. Suppose that $\overset{\circ}{W}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{W}_{p,\omega}^m(\Omega)$. Let e be a compact set in E . We choose $u \in C_0^\infty(\Omega)$ with $u = 1$ in a neighborhood of e .

Since $\overset{\circ}{W}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{W}_{p,\omega}^m(\Omega)$, we can choose a sequence of functions $u_j \in C_0^\infty(\Omega \setminus E)$, such that $u_j \rightarrow u$ in $W_{p,\omega}^m(\Omega)$. By construction, $u - u_j \in \mathfrak{M}(e, \Omega)$ for all $j \geq 1$. This implies

$$0 \leq \text{Cap}(e, W_{p,\omega}^m(\Omega)) \leq \lim_{j \rightarrow \infty} \|u - u_j\|_{W_{p,\omega}^m(\Omega)}^p = 0.$$

Hence, by Lemma 1 and Corollary 2, we have that

$$\text{Cap}(E, W_{p,\omega}^m(\Omega)) = \text{Cap}(E, W_{\omega}^{m,p}(\Omega)) = 0.$$

Similarly, from the equalities $\overset{\circ}{W}_{\omega}^{m,p}(\Omega \setminus E) = \overset{\circ}{W}_{\omega}^{m,p}(\Omega)$, $\overset{\circ}{L}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{L}_{p,\omega}^m(\Omega)$, we obtain

$$\text{Cap}(E, W_{\omega}^{m,p}) = 0, \quad \text{Cap}(E, L_{p,\omega}^m(\Omega)) = 0,$$

respectively. Moreover, if Ω is a bounded set, from Corollaries 3 and 4, we have the equivalence of the equalities $\text{Cap}(E, L_{p,\omega}^m(\Omega)) = 0$ and $\text{Cap}(E, W_{\omega}^{m,p}) = 0$. The necessity condition of the theorem is proved.

Sufficiency. Now suppose that $\text{Cap}(E, W_{\omega}^{m,p}) = 0$. We need to prove that $\overset{\circ}{L}_{p,\omega}^m(\Omega \setminus E) = \overset{\circ}{L}_{p,\omega}^m(\Omega)$. To do this, it is sufficient to prove that every function $u \in C_0^\infty(\Omega)$ can be approximated in $L_{p,\omega}^m(\Omega)$ by functions from $C_0^\infty(\Omega \setminus E)$. Indeed, we take a function $u \in C_0^\infty(\Omega)$ and suppose that Ω' is an open set, such that $\text{supp } u \subset \Omega' \Subset \Omega$. Put $e = \overline{\Omega'} \cap E$ and note that e is a compact set of zero (p, m, ω) -capacity. Then there exists a sequence of functions $\varphi_j \in \mathfrak{M}(e, R^n)$, $j \geq 1$, such that $\|\varphi_j\|_{W_{\omega}^{m,p}} \rightarrow 0 = \text{Cap}(e, W_{\omega}^{m,p})$ as $j \rightarrow \infty$.

Since all partial derivatives $D^\alpha u$, $|\alpha| \leq m$, are uniformly bounded in Ω , we have that $\|u\varphi_j\|_{W_\omega^{m,p}(\Omega)} \rightarrow 0$. This implies $u(1 - \varphi_j) \rightarrow u$ in $W_\omega^{m,p}(\Omega)$, and therefore $u(1 - \varphi_j) \rightarrow u$ in $L_{p,\omega}^m(\Omega)$ as $j \rightarrow \infty$. Obviously, $u(1 - \varphi_j) \in C_0^\infty(\Omega \setminus E)$. Consequently, $\mathring{L}_{p,\omega}^m(\Omega \setminus E) = \mathring{L}_{p,\omega}^m(\Omega)$. Similarly, we deduce that $\mathring{W}_{p,\omega}^m(\Omega \setminus E) = \mathring{W}_{p,\omega}^m(\Omega)$, $\mathring{W}_\omega^{m,p}(\Omega \setminus E) = \mathring{W}_\omega^{m,p}(\Omega)$. Hence, the sufficiency condition of the theorem is proved. Thus, the theorem is also proved. $\square$

Next, we give the conditions under which the equality $S_{p,\omega}^m(\Omega) = \mathring{S}_{p,\omega}^m(\Omega)$ holds.

Theorem 10. *The equality $W_\omega^{m,p}(\Omega) = \mathring{W}_\omega^{m,p}(\Omega)$ holds true if and only if $\text{Cap}(R^n \setminus \Omega, W_\omega^{m,p}) = 0$. If $W_{p,\omega}^m(\Omega) = \mathring{W}_{p,\omega}^m(\Omega)$ or $L_{p,\omega}^m(\Omega) = \mathring{L}_{p,\omega}^m(\Omega)$, then $\text{Cap}(R^n \setminus \Omega, W_\omega^{m,p}) = 0$ or $\text{Cap}(R^n \setminus \Omega, L_{p,\omega}^m) = 0$, respectively.*

Proof. Necessity. Suppose that, for example, $L_{p,\omega}^m(\Omega) = \mathring{L}_{p,\omega}^m(\Omega)$. Then we have that

$$\mathring{L}_{p,\omega}^m(R^n) \subset L_{p,\omega}^m(R^n) \subset L_{p,\omega}^m(\Omega) = \mathring{L}_{p,\omega}^m(\Omega) \subset \mathring{L}_{p,\omega}^m(R^n).$$

This implies $\mathring{L}_{p,\omega}^m(R^n) = \mathring{L}_{p,\omega}^m(\Omega) = \mathring{L}_{p,\omega}^m(R^n \setminus (R^n \setminus \Omega))$, and, consequently, by Theorem 9, we get $\text{Cap}(R^n \setminus \Omega, L_{p,\omega}^m) = 0$.

Similarly, from $W_\omega^{m,p}(\Omega) = \mathring{W}_\omega^{m,p}(\Omega)$ or $W_{p,\omega}^m(\Omega) = \mathring{W}_{p,\omega}^m(\Omega)$, by Corollary 2, we deduce that $\text{Cap}(R^n \setminus \Omega, W_{p,\omega}^m) = \text{Cap}(R^n \setminus \Omega, W_\omega^{m,p}) = 0$. The necessity condition of the theorem is proved.

Sufficiency. Suppose that $\text{Cap}(R^n \setminus \Omega, W_\omega^{m,p}) = 0$. By Theorems 3, 9, and Corollary 6, we infer that

$$\begin{aligned} W_\omega^{m,p}(\Omega) &= W_\omega^{m,p}(R^n \setminus (R^n \setminus \Omega)) = W_\omega^{m,p}(R^n) = \\ &= \mathring{W}_\omega^{m,p}(R^n) = \mathring{W}_\omega^{m,p}(R^n \setminus (R^n \setminus \Omega)) = \mathring{W}_\omega^{m,p}(\Omega). \end{aligned}$$

Theorem 10 is proved. $\square$

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