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REMOVABLE SETS FOR SOBOLEV SPACES WITH MUCKENHOUT A_1 -WEIGHT

V.A. SHLYK

ABSTRACT. Let Ω be an open set in R^n , $n \geq 2$, and E be a relatively closed subset of Ω . In this paper we obtain a criterion of equality $L_{1,\omega}^1(\Omega \setminus E) = L_{1,\omega}^1(\Omega)$ in terms of E as an $NC_{1,\omega}$ -set in Ω with A_1 -weight ω . In addition, we establish exact characterizations of $NC_{1,\omega}$ -sets in terms of $NED_{1,\omega}$ -sets and of the $(1,\omega)$ -girth condition. In the case $\omega \equiv 1$, these results complete the studies of Vodop'yanov and Gol'dstein on removable sets for $L_p^1(\Omega)$, $p \in (1, +\infty)$.

Keywords: Sobolev space, capacity and modulus of condenser, Muckenhoupt weight, removable set.

1. INTRODUCTION

In [18] Vodop'yanov and Gol'dstein gave a criterion of removable singularities for $L_p^1(\Omega)$, $W_p^1(\Omega)$ in terms of NC_p -sets, $1 < p < \infty$. An NC_p -set can be considered as a p -analog of an NED -set, earlier introduced by Väisälä [16] as result of generalizing the concept of NED -set in R^2 [1] to R^n , $n \geq 2$. Also note the definitions of an NC_p -set in Ω or an NED -set in R^n are based on condensers whose plates are the pair of arbitrary disjoint continuums located outside this set in Ω or in R^n , respectively.

Latter [6] Dymchenko and Shlyk obtained similar assertions about removable singularities for the space $L_{p,\omega}^1(\Omega)$ in terms of $NC_{p,\omega}$ -sets in Ω , where ω is a Muckenhoupt A_p -weight, $1 < p < \infty$. Their definition of $NC_{p,\omega}$ -set in Ω (as well as the initial definition of NED -set by Ahlfors–Beurling [1] in R^2) is based on condensers formed by an arbitrary coordinate rectangles Π , $\bar{\Pi} \subset \Omega$, and by any pair of its opposite facets.

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The main aim of this paper is to define $NC_{1,\omega}$ -sets, $NED_{1,\omega}$ -sets in Ω with Muckenhoupt A_1 -weight, using proper coordinate rectangular condensers, and to give criteria of equality $L_{1,\omega}^1(\Omega \setminus E) = L_{1,\omega}^1(\Omega)$ in terms of E as an $NC_{1,\omega}$ -set, $NED_{1,\omega}$ -set or with the $(1, \omega)$ -girth condition (see Theorem 3).

Here we note the difficulties in proving the main results of the paper, which are not present at $p > 1$. Namely, the equality of $(1, \omega)$ -modulus and $(1, \omega)$ -capacity of the condenser is not known in general, even for $\omega \equiv 1$. Similarly, the equality $M_{1,\omega}(\bigcup_j \Gamma_j) = \lim_{j \rightarrow \infty} M_{1,\omega}(\Gamma_j)$ is unknown if $\Gamma_j \subset \Gamma_{j+1}$, $j \geq 1$, and $M_{1,\omega}(\cdot)$ is a $(1, \omega)$ -modulus of a curve family in R^n .

2. PRELIMINARIES

2.1. Some definitions and notations. Throughout the text the symbol Ω denotes a non-empty open set in Euclidean space $R^n = \{x = (x_1, \dots, x_n)\}$, where $n \geq 2$. Respectively, E denotes a relatively closed subset of Ω , the norm of a point $x = (x_1, \dots, x_n)$ is given by $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. We put $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, +\infty)$.

If $F \subset R^n$ then ∂F , $\bar{F}$ stand for the boundary and the closure of F in R^n , respectively. The distance between two sets $A, B \subset R^n$ is denoted by $\text{dist}(A, B)$. For an open set $U \subset R^n$, we use the notation $U \Subset \Omega$ in order to indicate that U is bounded and $\bar{U} \subset \Omega$. The restriction of a function f to a set F is denoted by $f|_F$. Given $x \in R^n$ and $r > 0$, let $B(x, r) = B_r(x) = \{y \in R^n : |y - x| < r\}$. If $a > 0$ then $aB_r(x) = B_{ar}(x)$. The symbol $\mathcal{H}^s$ stands for the usual s -dimensional Hausdorff measure in R^n ; m_n is a Lebesgue measure in R^n and put $|F| = m_n(F)$ for m_n -measurable set $F \subset R^n$.

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions in Ω ; the space of functions in $C^\infty(R^n)$ with a compact support in Ω is denoted by $C_0^\infty(\Omega)$.

We will use the abbreviation "a.e." for "almost everywhere" with respect to m_n -measure. Similarly, "measurable" and "locally integrable" always mean Lebesgue measurable and locally integrable with respect to m_n -measure.

Let F be a measurable subset of R^n , and u is a measurable real-valued function on F . For $1 \leq p < \infty$ let

$$\|u\|_{L_p(F)} = \left(\int_F |u(x)|^p dx \right)^{1/p}.$$

Assume that $u(x)$ is a measurable function defined on Ω . We say that u is locally integrable to the power $p \in [1, +\infty)$ on Ω (and write $u \in L_p(\Omega, \text{loc})$) if $\|u\|_{L_p(F)} < \infty$ for every compact set $F \subset \Omega$. The class of all functions u such that $\|u\|_{L_p(\Omega)} < \infty$ is denoted by $L_p(\Omega)$.

If $\Omega = R^n$ we shall often omit Ω in notations of spaces and norm. Integration without indication of limits extends over R^n .

Let $C, C_1, C_2, \dots$ denote positive constants that depend on "dimensionless" parameters n, p, m and the like.

We call the quantities a and b equivalent and write $a \sim b$ if $C_1 a \leq b \leq C_2 a$.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of nonnegative integers α_i , we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ which has degree $|\alpha| = \sum_{i=1}^n \alpha_i$.

Similarly, if $D_j = \frac{\partial}{\partial x_j}$ then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. Note that $D^{(0,\dots,0)}u = u$ for real-valued functions u . We use the notations $\nabla_m = \{D^\alpha : |\alpha| = m\}$, $\nabla = \nabla_1$.

By a weight we shall mean a locally integrable function ω on R^n such that $\omega > 0$ for a.e. $x \in R^n$.

Then for $1 \leq p < \infty$ we define $L_{p,\omega}(\Omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L_{p,\omega}(\Omega)} = \left(\int_{\Omega} |f|^p \omega dx \right)^{1/p} < \infty.$$

As usual, any two functions f and g in $L_{p,\omega}(\Omega)$ that are equal a.e. on Ω will be identified.

Let $\mathcal{F}_1$ be a class of functions given on Ω , and $\mathcal{F}_2$ be another class of functions given on Ω' , where $\Omega' \subset \Omega$. Below if $f \in \mathcal{F}_1$ then $f \in \mathcal{F}_2$ means $f|_{\Omega'} \in \mathcal{F}_2$.

Denote by $L_{p,\omega}(\Omega, loc)$ the class of all measurable functions f on Ω such that $f \in L_{p,\omega}(\Omega')$ for all open sets $\Omega' \Subset \Omega$.

2.2. A_1 -weights. Following B. Muckenhoupt [12] a weight ω is called an A_1 -weight, if there exists a positive constant A such that for every ball $B \subset R^n$,

$$(1) \quad \left(\frac{1}{|B|} \int_B \omega dx \right) \operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \leq A,$$

The infimum over all such constants A is called the A_1 -constant of ω . Denote by A_1 the class of A_1 -weights. Throughout the text let $m \in \mathbb{N}$, $\omega \in A_1$ unless otherwise stated.

Set

$$M\omega(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y) dy.$$

We mention two assertions concerning A_1 -weight.

Proposition 1 ([15, Remark 1.2.4], [10, Theorem 2.7]). *If $\omega \in A_1$ then $L_{1,\omega}(\Omega)$ is complete space in norm $\|\cdot\|_{L_{1,\omega}(\Omega)}$ and $L_{1,\omega}(\Omega) \subset L_1(\Omega, loc)$. In addition, $L_{1,\omega}(\Omega) \subset L_1(\Omega)$ in the case of a bounded $\Omega \subset R^n$.*

Proposition 2 ([15, Remark 1.2.4, Properties 7 and 8]). *If $\omega \in A_1$ then there exist constants C, C_1 such that $\omega(x) \geq \frac{C}{(1+|x|)^n}$ and $M\omega(x) \leq C_1\omega(x)$ for a.e. $x \in R^n$.*

2.3. Weighted Sobolev spaces. Suppose that $u : \Omega \rightarrow R$ is a function in $L_1(\Omega, loc)$. This function u on Ω has a weak derivative of order $|\alpha|$ if there is a locally integrable function (denoted by $D^\alpha u$) such that

$$\int_{\Omega} u \cdot D^\alpha \varphi dx = (-1)^{|\alpha|} \int_{\Omega} D^\alpha u \cdot \varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$. For $m \in \mathbb{N}$ and any $\omega \in A_1$, $L_{1+,\omega}^m(\Omega)$ is the space of functions u having on Ω weak derivatives $D^\alpha u$ for all orders $|\alpha|$, $|\alpha| \leq m$, and satisfying

$$\|u\|_{L_{1+,\omega}^m(\Omega)} = \int_{\Omega} |\nabla_m u| \omega dx < \infty,$$

where $|\nabla_m u| = \left(\sum_{|\alpha|=m} (D^\alpha u)^2 \right)^{1/2}$. For $m = 0$ set $L_{1,\omega}^0(\Omega) = L_{1,\omega}(\Omega)$, $\nabla_0 u = u$.

Introduce the spaces

$$W_{1,\omega}^m(\Omega) = L_{1,\omega}^m(\Omega) \cap L_{1,\omega}(\Omega), \quad W_{\omega}^{m,1}(\Omega) = \bigcap_{k=0}^m L_{1,\omega}^k(\Omega),$$

equipped with the norms

$$\|u\|_{W_{1,\omega}^m(\Omega)} = \|u\|_{L_{1,\omega}^m(\Omega)} + \|u\|_{L_{1,\omega}(\Omega)}, \quad \|u\|_{W_{\omega}^{m,1}(\Omega)} = \sum_{k=0}^m \|\nabla_k u\|_{L_{1,\omega}(\Omega)}.$$

Next we let $\mathcal{P}_{m-1}$ be the collection of all polynomials of degree $\leq m-1$. Let us consider the factor-space $\check{L}_{1,\omega}^m(\Omega) = L_{1,\omega}^m(\Omega)/\mathcal{P}_{m-1}$ (with norm $\|\cdot\|_{L_{1,\omega}^m(\Omega)}$). Elements of the space $\check{L}_{1,\omega}^m(\Omega)$ on each connected component D of the set Ω are classes $\{u + P_D\}$ where $u \in L_{1,\omega}^m(\Omega)$ and $P_D \in \mathcal{P}_{m-1}$.

In the case $\omega \equiv 1$, the weight spaces considered above with the weight ω will be written below without the symbol ω .

Note that a number of important properties of spaces $W_{\omega}^{m,1}(\Omega)$, $L_{1,\omega}^m(\Omega)$ (in other notations and with equivalent norms) were obtained in [5, 15]. We use the following ones below.

Proposition 3 ([5, Theorem 4.9]). *If Ω is an open connected set and $\omega \in A_1$ then $\check{L}_{1,\omega}^m(\Omega)$ is a Banach space. In particular, if $\{u_j\}$ is a Cauchy sequence in $L_{1,\omega}^m(\Omega)$ then there exists $u_0 \in L_{1,\omega}^m(\Omega)$ such that $\nabla_m u_j \rightarrow \nabla_m u_0$ in $L_{1,\omega}(\Omega)$ as $j \rightarrow \infty$.*

Proposition 4 ([5, Corollary 4.10]). *Let Ω be an open connected set, let $\{u_j\}$ be a Cauchy sequence in $L_{1,\omega}^m(\Omega)$, and let u be a function in $L_{1,\omega}^m(\Omega)$ such that $\|\nabla_m(u_j - u)\|_{L_{1,\omega}(\Omega)} \rightarrow 0$. Then there exists a sequence of polynomials $\{P_j\} \subset \mathcal{P}_{m-1}$ with $u_j - P_j \rightarrow u$ in $L_{1,\omega}(K)$ for all compact sets $K \subset \Omega$.*

Proposition 5 ([5, Theorem 4.2]). *Let $\omega \in A_1$. If $u \in L_{1,\omega}^m(\Omega)$ then*

$$(2) \quad \int_K |D^\alpha u| \omega \, dx < \infty$$

for all compact $K \subset \Omega$, $0 \leq |\alpha| \leq m$.

Definition 1. *If the restriction operator $\theta : L_{1,\omega}^m(\Omega) \rightarrow L_{1,\omega}^m(\Omega \setminus E)$ ($\theta u = u|_{\Omega \setminus E}$) induces the isometric isomorphism of the normed spaces $\check{L}_{1,\omega}^m(\Omega)$ and $\check{L}_{1,\omega}^m(\Omega \setminus E)$, then we write $L_{1,\omega}^m(\Omega) = L_{1,\omega}^m(\Omega \setminus E)$. In other words, this means that $|E| = 0$ and for every function $u \in L_{1,\omega}^m(\Omega \setminus E)$ there is a function $v \in L_{1,\omega}^m(\Omega)$ for which $v|_{\Omega \setminus E} = u$. In this case, the function v is called an extension of the function u in $L_{1,\omega}^m(\Omega)$ and E is called a removable set for $L_{1,\omega}^m(\Omega)$.*

Similarly we define removable sets for $W_{\omega}^{m,1}(\Omega)$ and $W_{1,\omega}^m(\Omega)$.

2.4. Mollifications. Let $\psi \in C_0^\infty(R^n)$ be nonnegative function such that $\text{supp } \psi \subset B_1(0)$ and $\int \psi(x) dx = 1$. For any function $u \in L_1(\Omega)$, extended by zero on $R^n \setminus \Omega$, we define the family of its mollifications

$$(M_\varepsilon u)(x) = \varepsilon^{-n} \int u(y) \psi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{|\xi| < 1} u(x + \varepsilon \xi) \psi(\xi) d\xi, \quad 0 < \varepsilon \leq 1.$$

The number ε shall be called a radius of mollification.

The following result is known.

Proposition 6 ([15, Theorem 2.1.4, Corollary 2.1.5]). *Suppose that $u \in W_{\omega}^{m,1}(\Omega)$ and let Ω' be an open set, $\Omega' \Subset \Omega$. Then $(M_{\varepsilon}u)(x) \in C^{\infty}(\Omega) \cap L_{1,\omega}(\Omega)$ and for $0 < \varepsilon < \min(\text{dist}(\Omega', \partial\Omega), 1)$ the equality $D^{\alpha}M_{\varepsilon}u = M_{\varepsilon}D^{\alpha}u$ is true on Ω' , $1 \leq |\alpha| \leq m$; $M_{\varepsilon}u \rightarrow u$ in $W_{\omega}^{m,1}(\Omega')$ as $\varepsilon \rightarrow 0$. In the case $\Omega = R^n$ we have convergence $M_{\varepsilon}u \rightarrow u$ in $W_{\omega}^{m,1}(R^n)$ as $\varepsilon \rightarrow 0$.*

Using Proposition 6 and an approach due to Maz'ya [11, Sec. 1.1.5, Theorem 1] (see proof of Theorem 2, Sec. 4), we obtain another assertion.

Proposition 7. *Let $u \in L_{1,\omega}^1(\Omega)$ and $\{\Omega_j\}$ be some sequence of open sets Ω_j such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\bigcup_j \Omega_j = \Omega$. Then there exists a sequence of bounded functions $u_j \in L_{1,\omega}^1(\Omega) \cap C^{\infty}(\Omega)$, $j \geq 1$, such that*

$$(3) \quad \int_{\Omega_j} |u - u_j| \omega dx < \frac{1}{j}, \quad \lim_{j \rightarrow \infty} \|u - u_j\|_{L_{1,\omega}^1(\Omega)} = 0.$$

2.5. $(1, \omega)$ -modulus and $(1, \omega)$ -capacity. Let Γ be a family of locally rectifiable curves in R^n . We denote by $\text{adm } \Gamma$ the set of Borel functions $\rho : R^n \rightarrow [0; +\infty]$ satisfying the condition: for every $\gamma \in \Gamma$ we have $\int_{\gamma} \rho ds \geq 1$. In the case $\Gamma = \emptyset$ we assume that $\text{adm } \Gamma$ contains the function $\rho \equiv 0$. The $(1, \omega)$ -modulus of Γ , denoted by $M_{1,\omega}(\Gamma)$, is defined as

$$M_{1,\omega}(\Gamma) = \inf \int_{R^n} \rho \omega dx,$$

where the infimum is taken over all $\rho \in \text{adm } \Gamma$. For the basic facts about the (p, ω) -modulus, $1 \leq p < \infty$, see [13]. Now let F_0, F_1 be compact disjoint sets in R^n . Then a triple of sets (F_0, F_1, Ω) is called a condenser in Ω . Let $\Gamma(F_0, F_1, \Omega)$ be the family of all locally rectifiable curves connecting $F_0 \cap \bar{\Omega}$ and $F_1 \cap \bar{\Omega}$ in Ω . More precisely, if $\gamma \in \Gamma(F_0, F_1, \Omega)$ then there exists a representation $x(s) : I \rightarrow \Omega$ of curve γ in terms of arc length (see [13, Sec. 2.1]), where I is an open interval, $\overline{x(I)} \cap F_0$ and $\overline{x(I)} \cap F_1$ are both non-empty.

We write $M_{1,\omega}(F_0, F_1, \Omega)$ for the $(1, \omega)$ -modulus of $\Gamma(F_0, F_1, \Omega)$. By definition, $M_{1,\omega}(F_0, F_1, \Omega) = M_{1,\omega}(F_0 \cap \bar{\Omega}, F_1 \cap \bar{\Omega}, \Omega)$ and $M_{1,\omega}(F_0, F_1, \Omega) = 0$ if at least $F_0 \cap \bar{\Omega} = \emptyset$ or $F_1 \cap \bar{\Omega} = \emptyset$. The number $M_{1,\omega}(F_0, F_1, \Omega)$ will also be called the $(1, \omega)$ -modulus of condenser (F_0, F_1, Ω) .

Now let's define $(1, \omega)$ -capacity $C_{1,\omega}(F_0, F_1, \Omega)$ of the condenser (F_0, F_1, Ω) . Suppose that $F_0 \cup F_1 \subset \bar{\Omega}$. Then we set $C_{1,\omega}(F_0, F_1, \Omega) = 0$ if, at least, $F_0 = \emptyset$ or $F_1 = \emptyset$. If F_0 and F_1 are non-empty sets then

$$C_{1,\omega}(F_0, F_1, \Omega) = \inf \int_{\Omega} |\nabla u| \omega dx,$$

where the infimum is taken over all real-valued functions u such that $u|_{\Omega}$ satisfies locally the Lipschitz condition and $u = j$ in some neighborhood of F_j , $j = 0, 1$.

Denote the set of all admissible functions of this kind by $\text{Adm}(F_0, F_1, \Omega)$. In general, we define $(1, \omega)$ -capacity of a condenser (F_0, F_1, Ω) as $C_{1,\omega}(F_0, F_1, \Omega) = C_{1,\omega}(F_0 \cap \bar{\Omega}, F_1 \cap \bar{\Omega}, \Omega)$.

By Rademacher's theorem, the function $u \in \text{Adm}(F_0, F_1, \Omega)$ is differentiable a.e. on Ω . Set for $x \in \Omega$

$$L(x, u) = \limsup_{h \rightarrow 0} \frac{|u(x+h) - u(x)|}{|h|}.$$

Then $L(x, u)$ is a Borel function on Ω and $|\nabla u(x)| = L(x, u)$ at differentiability points of u . If u is not differentiable at $x \in \Omega$, we set $|\nabla u(x)| = L(x, u)$ (see [17, Theorem 5.1]).

It follows from the Vitali-Carathéodory theorem (see [14, p.37, Theorem 2.24]) that given $f : R^n \rightarrow [0, +\infty]$, $f \in L_{1,\omega}$, there exists a lower semi-continuous function $g \geq f$ with $\|g\|_{L_{1,\omega}}$ arbitrarily closed to $\|f\|_{L_{1,\omega}}$. We shall apply below the following assertion.

Proposition 8 ([13, Sec. 2.2, p.19]).

$$M_{1,\omega}(\Gamma) = \inf_{\rho} \left\{ \int \rho \omega dx : \rho \text{ is lower semi-continuous and } \rho \in \text{adm } \Gamma \right\}.$$

2.6. Removable sets. Here we define three types of sets E in Ω (recall that E is a relatively closed subset of Ω) which will be removable singularities for $L_{1,\omega}^1(\Omega)$. Let here and further Π be any coordinate rectangle

$$\{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\},$$

where $a_i, b_i \in R$. Denote the facets of this rectangle, parallel to the hyperplane $x_i = 0$, by $\sigma_{0i} \subset \{x : x_i = a_i\}$ and $\sigma_{1i} \subset \{x : x_i = b_i\}$, $i = 1, \dots, n$. If

$$(4) \quad C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) = C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi), \quad i = 1, \dots, n,$$

for every coordinate rectangle Π with $\bar{\Pi} \subset \Omega$, then E is called $NC_{1,\omega}$ -set in Ω .

Now let $m_n(E) = 0$, and let $e \subset E$ be an arbitrary compact. Set $\mathcal{K}_j(e, \Omega) = \{\Pi : \text{dist}(\sigma_{0j} \cup \sigma_{1j}, e) > 0, \bar{\Pi} \subset \Omega\}$, $j = 1, \dots, n$. In addition, for $\Pi \in \mathcal{K}_j(e, \Omega)$ put $\Pi_{j,\delta} = \{x = (x_1, \dots, x_n) \in R^n : a_j < x_j < b_j, a_i + \delta < x_i < b_i - \delta, i \neq j\}$, where $0 < \delta < \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}$. If, regardless of choice of compact set $e \subset E$ the estimate

$$(5) \quad C_1 M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) \geq \lim_{\delta \rightarrow 0} M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi_{j,\delta})$$

is valid for every coordinate rectangle $\Pi \in \mathcal{K}_j(e, \Omega)$ and all $j = 1, \dots, n$, then E is called a $NED_{1,\omega}$ -set in Ω . In the definition C_1 is a constant from Proposition 2. Observe that in the case $\omega \equiv 1$, inequality (5) is equivalent to equality (see Corollary 1 from Sec. 4)

$$(6) \quad M_1(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) = M_1(\sigma_{0j}, \sigma_{1j}, \Pi)$$

for every compact $e \subset E$ and coordinate rectangle $\Pi \in \mathcal{K}_j(e, \Omega)$, $j = 1, \dots, n$.

In order to define another type of removable sets for $L_{1,\omega}^1(\Omega)$, we introduce the following concepts.

Let X_i be the family of straight lines in R^n , parallel to the coordinate x_i -axis, $i = 1, \dots, n$. Index every line $l \in X_i$ by the point $a \in l \cap H_i$, where $H_i = \{x = (x_1, \dots, x_n) \in R^n : x_i = 0\}$. Then say that some property holds for m_{n-1} -almost all lines in X_i (or segments on these lines), whenever the corresponding set of points a on H_i for lines (or their segments) in X_i violating this property, is of m_{n-1} -measure zero.

We shall say that Ω does not partitioned locally by E if $B \setminus E$ is a domain for all balls $B \Subset \Omega$. It is easily to see that in this definition the balls $B \Subset \Omega$ can be replaced with coordinate rectangles $\Pi \Subset \Omega$.

Definition 2. Take a compact set $e \subset R^n$ with $m_n(e) = 0$ so that R^n does not partitioned locally by e . We say that e satisfies the $(1, \omega)$ -girth condition respect to X_i if every Borel function $\rho : R^n \setminus e \rightarrow [0, +\infty)$, $\rho \in L_{1,\omega}(R^n \setminus e)$, locally bounded on $R^n \setminus e$, satisfies the following ε -girth condition for given $\varepsilon > 0$:

Let Π be some coordinate rectangle in R^n and $e \subset \Pi$. Let $\Gamma_i(\Pi) = \{l \cap \bar{\Pi} : l \in X_i, l \cap e \neq \emptyset\}$. Then for m_{n-1} -almost all segments $\tau \in \Gamma_i(\Pi)$ we can indicate a finite sequence of mutually disjoint intervals $(c_k, d_k) \subset \tau$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$, joining c_k with d_k , $k = 1, \dots, k_1$, such that

$$\sum_{k=1}^{k_1} \int_{\gamma_k} \rho ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{\gamma_k} ds < \varepsilon, \quad \bigcup_{k=1}^{k_1} (c_k, d_k) \supset \tau \cap e.$$

Refer to the last requirement as the ε -girth condition on τ for the ρ .

Remark. Let's denote the family of all such segments τ for the function ρ by $\Gamma_i(e, \Pi, \varepsilon)$. Take $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$ and set $\Gamma_{0i}(e, \Pi) = \bigcap_k \Gamma_i(e, \Pi, \frac{1}{k})$, $F_k = \{l \cap H_i : l \in X_i \text{ and there exists a segment } \tau \in \Gamma_i(e, \Pi, \frac{1}{k}), \tau \subset l\}$, $F_0 = \{l \cap H_i : l \in X_i \text{ and there exists a segment } \tau \in \Gamma_{0i}(e, \Pi), \tau \subset l\}$. It is easily to see that $F_0 \subset F_{k+1} \subset F_k$ and $m_{n-1}(F_k) = m_{n-1}(F_0)$ for all $k \geq 1$ and ρ satisfies the ε -girth condition on all $\tau \in \Gamma_{0i}(e, \Pi)$ for any $\varepsilon > 0$.

Moreover, the function ρ satisfies the ε -girth condition on any segment $[a, b] \subset \tau$, $\tau \in \Gamma_{0i}(e, \Pi)$, if a and $b \notin e$, for arbitrary $\varepsilon > 0$.

Definition 3. If e satisfies the $(1, \omega)$ -girth condition with respect to X_i for all $i = 1, \dots, n$, then we say that e satisfies the $(1, \omega)$ -girth condition in R^n .

Definition 4. We say that E , $m_n(E) = 0$ and Ω does not partitioned locally by E , satisfies the $(1, \omega)$ -girth condition in Ω , if any compact set $e \subset E$ satisfies the $(1, \omega)$ -girth condition in R^n .

The family of all such sets $E \subset \Omega$ is denoted by $\mathcal{G}_{1,\omega}(\Omega)$. Further $\omega \equiv 1$ we shall often omit in notations of spaces, norms, families. For example, in the case $\omega \equiv 1$ we write $\mathcal{G}_{1,\omega}$ as $\mathcal{G}_1$.

3. ABOUT ONE CLASS OF ADMISSIBLE METRICS

In this section F_0, F_1 will be compact non-empty sets in $\bar{\Omega}$. Any function $\rho \in \text{adm } \Gamma(F_0, F_1, \Omega)$ will also be called an admissible metric for $\Gamma(F_0, F_1, \Omega)$. If $\Gamma(F_0, F_1, \Omega) = \emptyset$ then, by definition, the admissible metric for $\Gamma(F_0, F_1, \Omega)$ is an arbitrary Borel function $\rho : R^n \rightarrow [0, +\infty]$. We let $d : R^n \rightarrow [0, +\infty]$ be the function defined by $d(x) = \text{dist}(x, (R^n \setminus \Omega) \cup F_0 \cup F_1)$. It is well-known that $d(x)$ satisfies the Lipschitz condition with the Lipschitz constant $\text{Lip}(d) \leq 1$. In the case where $d(x)$ is differentiable at $x \in \Omega \setminus (F_0 \cup F_1)$, it follows that $|\nabla d| = 1$ (see [7, Sec. 3.2.34]). Further we use the following technical results.

Lemma 1. $M_{1,\omega}(F_0, F_1, \Omega)$ and $C_{1,\omega}(F_0, F_1, \Omega)$ are finite.

Proof. Let U_0 and U_1 be open bounded sets in R^n such that $F_0 \subset U_0$, $F_1 \subset U_1$, $\overline{U_0} \cap \overline{U_1} = \emptyset$. In addition, take the ball $B_0 = B(0, r)$ so that $\overline{U_0} \cup \overline{U_1} \subset B_0$.

Set $\varphi \in C_0^\infty(B_0)$, $0 \leq \varphi \leq 1$, $\varphi = 0$ on U_0 and $\varphi = 1$ on $\overline{U_1}$ (see [8, Chapter 1, Theorem 2.6]). Because of choice of φ and Proposition 1 it follows that $\varphi \in \text{Adm}(F_0, F_1, R^n) \cap \text{Adm}(F_0, F_1, \Omega)$, $|\nabla \varphi| \in \text{adm}(F_0, F_1, R^n) \cap \text{adm}(F_0, F_1, \Omega)$ and $\int_{R^n} |\nabla \varphi| \omega dx < \infty$. This implies $M_{1,\omega}(F_0, F_1, \Omega) < \infty$, $C_{1,\omega}(F_0, F_1, \Omega) < \infty$. The Lemma is proved. $\square$

Lemma 2. *For every $\varepsilon > 0$, there is a function $\rho \in \text{adm}(F_0, F_1, \Omega) \cap L_{1,\omega}(\Omega)$, for which the following conditions are realized:*

- (1) ρ is lower semi-continuous on R^n ;
- (2) ρ is continuous on $\Omega \setminus (F_0 \cup F_1)$;
- (3) ρ is a positive function on R^n such that for any compact set $K \subset R^n$ $\inf_K \rho > 0$ and

$$(7) \quad M_{1,\omega}(F_0, F_1, \Omega) \leq \int_{R^n} \rho \omega dx \leq C_1 M_{1,\omega}(F_0, F_1, \Omega) + \varepsilon,$$

where the constant C_1 does not depend on ε . In the case $\omega \equiv 1$, put in (7) $C_1 = 1$.

Proof. For $\varepsilon > 0$, by Proposition 8 and Lemma 1, let ρ_1 be some admissible metric for $\Gamma(F_0, F_1, \Omega)$, $\rho_1 \in L_{1,\omega}(R^n)$ and ρ_1 be a lower semi-continuous on R^n with

$$M_{1,\omega}(F_0, F_1, \Omega) \leq \int_{R^n} \rho_1 \omega dx < M_{1,\omega}(F_0, F_1, \Omega) + \frac{\varepsilon}{3C_1}.$$

Here C_1 is the constant from Proposition 2.

Define ρ_2 as

$$(8) \quad \rho_2(x) = T_k \rho_1(x) = \frac{1}{|B(0, 1)|} \int_{B(0,1)} \rho_1 \left(x + \frac{d(x)}{2k} y \right) dy,$$

where T_k is the averaging operator used in [3] and studied in details in [9, Lemma 4.3]. Due to the known properties of the operator the function ρ_2 is lower semi-continuous on R^n and continuous on $\Omega \setminus (F_0 \cup F_1)$. Here note that to prove these properties, only local integrability of ρ_1 is required in addition.

By integration (8), we get

$$\int_{R^n} \rho_2(x) \omega dx = \int_{R^n} \left[\frac{1}{|B(0, 1)|} \int_{B(0,1)} \rho_1 \left(x + \frac{d(x)}{2k} y \right) dy \right] \omega(x) dx.$$

Interchanging the order of integration gives

$$(9) \quad \|\rho_2\|_{L_{1,\omega}} = \frac{1}{|B(0, 1)|} \int_{B(0,1)} \int_{R^n} \rho_1 \left(x + \frac{d(x)}{2k} y \right) \omega(x) dy.$$

Define for $y \in B(0, 1)$, $\theta_{y,k} : R^n \rightarrow R^n$ by $z = \theta_{y,k}(x) = x + \frac{d(x)}{2k} y$. By construction, for all $x, x' \in R^n$ (see [9, Theorem 2.1])

$$\left(1 - \frac{1}{2k}\right) |x - x'| \leq |\theta_{y,k}(x) - \theta_{y,k}(x')| \leq \left(1 + \frac{1}{2k}\right) |x - x'|.$$

Thus, the mapping $\theta_{y,k}$ is a Lipschitz homeomorphism and $\theta_{y,k}(\Omega \setminus (F_0 \cup F_1)) = \Omega \setminus (F_0 \cup F_1)$. In addition, Jacobian of the mapping $\theta_{y,k}$ is equal $1 + \frac{y}{2k} \nabla d(x)$ a.e. on R^n . Hence [9, Lemma 4.3], by the changing of variables formula with $z = \theta_{y,k}$ as the mapping function, we obtain in (9)

$$\|\rho_2\|_{L_{1,\omega}} \leq \frac{1}{|B(0,1)|(1 - \frac{1}{2k})} \int_{B(0,1)} \int_{R^n} \rho_1(z) \omega(x(z)) dz dy.$$

Repeated interchanging of order of integration gives

$$\|\rho_2\|_{L_{1,\omega}} \leq \frac{1}{(1 - \frac{1}{2k})} \int_{R^n} \left(\frac{1}{|B(0,1)|} \int_{B(0,1)} w(x(z)) dy \right) \rho_1(z) dz.$$

By Proposition 2 and $x = z - \frac{1}{2k} d(x(z))y$, we deduce

$$\frac{1}{|B(0,1)|} \int_{B(0,1)} \omega \left(z - \frac{1}{2k} d(x(z))y \right) dy = \frac{1}{|B(z, \frac{d(x(z))}{2k})|} \int_{B(z, \frac{d(x(z))}{2k})} \omega(y) dy \leq$$

$$(10) \quad M\omega(z) \leq C_1 \omega(z)$$

a.e. on R^n . Hence

$$\|\rho_2\|_{L_{1,\omega}} \leq \frac{C_1}{(1 - \frac{1}{2k})} \int_{R^n} \rho_1(z) \omega(z) dz = \frac{C_1}{(1 - \frac{1}{2k})} \|\rho_1\|_{L_{1,\omega}}.$$

Here note that for $\omega \equiv 1$ we have the equality $M\omega(x) = 1$ on R^n . In other words, we can assume $C_1 = 1$ for $\omega \equiv 1$ in (10).

Moreover, using standard arguments (see proofs [9, Lemma 4.3], [4, Theorem 2.1]), we have $g_k = (1 + \frac{1}{2k}) \rho_2 \in \text{adm}(F_0, F_1, \Omega)$. This implies

$$\begin{aligned} M_{1,\omega}(F_0, F_1, \Omega) &\leq \int_{R^n} g_k \omega dx \leq \\ &\frac{C_1 (1 + \frac{1}{2k})}{(1 - \frac{1}{2k})} \int_{R^n} \rho_1 \omega dx \leq C_1 \frac{(1 + \frac{1}{2k})}{(1 - \frac{1}{2k})} M_{1,\omega}(F_0, F_1, \Omega) + \frac{\varepsilon}{3}. \end{aligned}$$

Let $g(x)$ be a positive continuous function on R^n with $\int_{R^n} g(x) \omega(x) dx < \frac{\varepsilon}{3}$. Its construction is similar to the construction of a positive function $\alpha(x) \in C^\infty$ in [13, Lemma 2.4.1]. Then it is clear that for large $k \in \mathbb{N}$ the function $\rho = g_k + g$ satisfies the conditions of the Lemma. Thus, the Lemma is proved. $\square$

4. SOME PROPERTIES OF NED_{1,ω^-} , $NC_{1,\omega}$ -SETS

In this section, we will establish a number of properties of sets that will be removable for $L_{1,\omega}^1(\Omega)$ and use these properties in proving the main results of our paper.

Property 1. *For any coordinate rectangle $\Pi = \{x \in R^n : a_i < x_i < b_i, i = 1, \dots, n\}$ we have (see (4))*

$$(11) \quad M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \text{ess sup}_{\Pi} \frac{1}{\omega} \geq m_{n-1}(\sigma_{0j}), j = 1, \dots, n.$$

Proof. By Lemma 1, it follows that $M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) < \infty$ for all $j = 1, \dots, n$. Fix $j \in \{1, \dots, n\}$. Then for each $\varepsilon > 0$ we can find $\rho \in \text{adm}(\sigma_{0j}, \sigma_{1j}, \Pi)$ such that

$$M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \leq \int_{R^n} \rho \omega dx < M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) + \varepsilon.$$

In addition, $\int_{a_j}^{b_j} \rho(x) dx_j \geq 1$ for all x' , where

$x' \in \Pi' = \{x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in R^{n-1} : a_i < x_i < b_i, i \neq j\}$. This implies

$$\int_{\Pi'} \left(\int_{a_j}^{b_j} \rho \omega \frac{1}{\omega} dx_j \right) dx' \geq m_{n-1}(\sigma_{0j}).$$

Hence, $\int_{\Pi} \rho \omega dx \cdot \text{ess sup}_{\Pi} \frac{1}{\omega} \geq m_{n-1}(\sigma_{0j})$. Thus, the Property is proved. $\square$

Set $\omega \equiv 1$ in (11) and let $\rho_0 = \frac{1}{b_j - a_j}$ on Π and $\rho_0 = 0$ on $R^n \setminus \Pi$. By choice, $\text{ess sup}_{\Pi} \frac{1}{\omega} = 1$, $\rho_0 \in \text{adm}(\sigma_{0j}, \sigma_{1j}, \Pi)$ and $\int_{R^n} \rho_0 dx = m_{n-1}(\sigma_{0j})$. It follows from (11) another assertion.

Corollary 1. *In the case of $\omega \equiv 1$ we see*

$$M_1(\sigma_{0j}, \sigma_{1j}, \Pi) = m_{n-1}(\sigma_{0j}), \quad j = 1, \dots, n,$$

and, by (5),

$$\begin{aligned} \lim_{\delta \rightarrow +0} M_1(\sigma_{0j}, \sigma_{1j}, \Pi_{j,\delta}) &= \lim_{\delta \rightarrow +0} M_1(\sigma_{0j} \cap \overline{\Pi_{j,\delta}}, \sigma_{1j} \cap \overline{\Pi_{j,\delta}}, \Pi_{j,\delta}) = \\ \lim_{\delta \rightarrow +0} m_{n-1}(\sigma_{0j} \cap \overline{\Pi_{j,\delta}}) &= m_{n-1}(\sigma_{0j}) = M_1(\sigma_{0j}, \sigma_{1j}, \Pi). \end{aligned}$$

Moreover, in view of Proposition 2, we have also the estimate

$$(12) \quad M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \geq C_1 \cdot \frac{1}{\sup_{\Pi} (1 + |x|)^n} m_{n-1}(\sigma_{0j})$$

for all $j = 1, \dots, n$.

Property 2.

$$C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \geq M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi).$$

Proof. For given $\varepsilon > 0$ we can find $u_\varepsilon \in \text{Adm}(\sigma_{0j}, \sigma_{1j}, \Pi)$ such that

$$C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \leq \int_{\Pi} |\nabla u_\varepsilon| \omega dx < C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) + \varepsilon.$$

In addition, we see

$$1 \leq \left| \int_{\gamma} \frac{\partial u_\varepsilon}{\partial s} ds \right| \leq \int_{\gamma} |\nabla u_\varepsilon| ds$$

for all $\gamma \in \Gamma(\sigma_{0j}, \sigma_{1j}, \Pi)$. Hence $\rho \in \text{adm}(\sigma_{0j}, \sigma_{1j}, \Pi)$ if $\rho = |\nabla u_\varepsilon|$ on Π and $\rho = 0$ on $R^n \setminus \Pi$. This implies

$$M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \leq \int_{\Pi} \rho \omega dx \leq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) + \varepsilon.$$

The arbitrariness of ε yields the inequality required in the Property. $\square$

Property 3. *If E is an $NC_{1,\omega}$ -set in Ω then $\sigma_{0j} \cap \overline{\Pi \setminus E} = \sigma_{0j}$, $\sigma_{1j} \cap \overline{\Pi \setminus E} = \sigma_{1j}$ for all $j = 1, \dots, n$.*

Proof. Let $j = 1$ and let, for example, some $x_0 \in \sigma_{01}$ not to be a boundary point of $\Pi \setminus E$. Hence there exists a ball $B(x_0, r)$ such that $(B(x_0, r) \cap \sigma_{01}) \cap \Pi \setminus E = \emptyset$. Then we will define another coordinate rectangle $\Pi_1 \subset \Pi$, whose opposite facets $\sigma_{01}^1, \sigma_{11}^1$, orthogonal to the x_1 -axis, lie in $B(x_0, r) \cap \sigma_{01}, \sigma_{11}$, respectively.

By definition, $\sigma_{01}^1 \cap \overline{\Pi_1 \setminus E} = \emptyset$ and therefore $C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus E) = C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1) = 0$. This, by Property 2, contradicts the inequality (12). Thus, $\sigma_{01} \cap \overline{\Pi \setminus E} = \sigma_{01}$, $\sigma_{11} \cap \overline{\Pi \setminus E} = \sigma_{11}$.

Similarly, we establish the required equalities for $j = 2, \dots, n$. This completes the proof of the Property. $\square$

Property 4. *If E is an $NC_{1,\omega}$ -set in Ω and $u_j = \frac{x_j - a_j}{b_j - a_j}$ on $\Pi \Subset \Omega$, then*

$$(13) \quad \int_{\Pi \setminus E} |\nabla u_j| \omega dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus E), \quad \int_{\Pi} |\nabla u_j| \omega dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi)$$

for all $j = 1, \dots, n$.

Proof. As in the proof of Property 3, we derive the inequalities (13) only for $j = 1$.

Let $\varepsilon \in \left(0, \frac{b_1 - a_1}{2}\right)$ and set

$$u_{1,\varepsilon}(x) = \begin{cases} \frac{x_1 - a_1 - \varepsilon}{b_1 - a_1 - 2\varepsilon}, & x_1 \in [a_1 + \varepsilon, b_1 - \varepsilon], \\ 0, & x_1 < a_1 + \varepsilon, \\ 1, & x_1 > b_1 - \varepsilon \end{cases}$$

for $x = (x_1, \dots, x_n) \in R^n$. Obviously, $u_{1,\varepsilon} \in \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi) \cap \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi \setminus E)$. This implies

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) \leq \int_{\Pi} |\nabla u_{1,\varepsilon}| \omega dx = \int_{\Pi} |\nabla u_1| \omega dx + o(1),$$

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \leq \int_{\Pi \setminus E} |\nabla u_{1,\varepsilon}| \omega dx = \int_{\Pi \setminus E} |\nabla u_1| \omega dx + o(1),$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, taking $\varepsilon \rightarrow 0$, we derive the required inequalities in the Property for $j = 1$. $\square$

Corollary 2. *It is clear that for u_1 from the proof of Property 4 we have $|\nabla u_1| = \frac{\rho_1}{b_1 - a_1}$, where $\rho_1 = 1$ on $\Pi \setminus E$ and $\rho_1 = 0$ on $\Pi \cap E$. Therefore, if E is an*

$NC_{1,\omega}$ -set in Ω then

$$(14) \quad \int_{\Pi \setminus E} \frac{\rho_1}{b_1 - a_1} \omega \, dx \geq C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E)$$

for all $\Pi \in \Omega$.

Property 5. *If E is an $NC_{1,\omega}$ -set in Ω then $m_n(E) = 0$.*

Proof. Suppose that $m_n(E) > 0$. Consider the coordinate rectangle $\Pi = \{x = (x_1, \dots, x_n) \in R^n : a_i < x_i < b_i, i = 1, \dots, n\}$ and $m_n(E \cap \Pi) > 0$. Let ρ_1 be a function from Corollary 2. Given two rationals $r_1 < r_2$ in (a_1, b_1) , put $\Pi(r_1, r_2) = \Pi \cap \{x \in R^n : r_1 < x_1 < r_2\}$, $\sigma_0 = \sigma_0(r_1, r_2) = \bar{\Pi} \cap \{x : x_1 = r_1\}$ and $\sigma_1 = \sigma_1(r_1, r_2) = \bar{\Pi} \cap \{x : x_1 = r_2\}$. It is clear that $m_{n-1}(\Pi') = m_{n-1}(\sigma_0) = m_{n-1}(\sigma_1)$, where $\Pi' = \{x' = (x_2, \dots, x_n) : a_i < x_i < b_i, i = 2, \dots, n\}$. Then from Property 2, (12) and (14) we deduce easily

$$(15) \quad \int_{\Pi'} dx' \int_{r_1}^{r_2} \frac{\rho_1 \omega}{r_2 - r_1} dx_1 \geq C_1 \frac{1}{\sup_{\Pi} (1 + |x|)^n} \cdot m_{n-1}(\Pi') \geq C_2 m_{n-1}(\Pi'),$$

where $C_2 = C_1 \frac{1}{\sup_{\Pi} (1 + |x|)^n}$.

Put $\Phi(x', r_1, r_2) = \int_{r_1}^{r_2} \frac{\rho_1 \omega \, dx_1}{r_2 - r_1}$. By Fubini's theorem (see [2]), there exists a set $\tilde{\Pi}'(r_1, r_2)$ in Π' with $m_{n-1}(\tilde{\Pi}'(r_1, r_2)) = m_{n-1}(\Pi')$, all of whose points are Lebesgue points for $\Phi(x', r_1, r_2)$ (differentiability points of the integral of this function over Π'). Put $\tilde{\Pi}' = \bigcap_{r_1 < r_2} \tilde{\Pi}'(r_1, r_2)$. It is obvious that $m_{n-1}(\tilde{\Pi}') = m_{n-1}(\Pi')$. Choose $x' \in \tilde{\Pi}'$ so that $\mathcal{H}^1(l(x') \cap E) > 0$ and $\mathcal{H}^1$ -almost all points on the segment $l(x') = \{(x_1, x') : a_1 \leq x_1 \leq b_1\}$ are Lebesgue points for the function $\rho_1 \omega$ and $\omega < \infty$.

From (15) we obtain

$$(16) \quad \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \rho_1 \omega \, dx_1 \geq C_2$$

for m_{n-1} -almost all $x' \in \Pi'$.

Since the choice of r_1 and r_2 in (a_1, b_1) is arbitrary, the absolute continuity of the integral enables us to consider r_1 and r_2 in (16) as real from (a_1, b_1) . By construction, there exists $x_1^0 \in (a_1, b_1)$ and $(x_1^0, x') \in E \cap l(x')$ such that (x_1^0, x') is a Lebesgue point for $\rho_1 \omega$ with fixed x' .

Putting in (16) $r_1 = x_1^0 - \delta$ and $r_2 = x_1^0 + \delta$ with $\delta > 0$, and letting $\delta \rightarrow 0$, we get a contradiction to $0 = \rho(x_1^0, x') \omega(x_1^0, x') \geq C_2$. Consequently, $m_n(E) = 0$. Thus the Property is proved. $\square$

Property 6. *If E is an $NC_{1,\omega}$ -set in Ω then $\Pi \setminus E$ is a domain for arbitrary coordinate rectangle $\Pi \in \Omega$.*

Proof. Suppose that there exists a coordinate rectangle $\Pi \in \Omega$, which is partitioned by E into two non-empty open sets D_0 and D_1 ($\Pi \setminus E = D_0 \cup D_1$, $D_0 \cap D_1 = \emptyset$). Take two points $d_0 \in D_0$, $d_1 \in D_1$ and let $B_0 = B(d_0, r_0)$, $B_1 = B(d_1, r_1)$ be such that $\bar{B}_0 \subset D_0$, $\bar{B}_1 \subset D_1$.

Let $L_1 = L_1(a_1, \dots, a_k)$ be a simple polyline in Π composed of straight segments $[a_i, a_{i+1}]$, $i = 1, \dots, k-1$, where each segment $[a_i, a_{i+1}]$ is parallel to some coordinate axis and $a_1 = d_0$, $a_k = d_1$. Below, any polyline whose links are parallel to the coordinate axes, will be called a coordinate polyline. Then it follows that there exists some segment $[a, b] \subset \Pi$, parallel to one of the links of L_1 and $a \in D_0$, $b \in D_1$. Indeed, in the ball $B(a_2, r_2)$ with $0 < r_2 < \min(r_0, r_1, \text{dist}(L_1, \partial\Pi))$, by virtue of $m_n(E) = 0$, there is a point $a_2^1 \notin E$. Consider a parallel translation $T_{h_1} : x \rightarrow x + h_1$ on R^n , where $h_1 = a_2^1 - a_2$. By construction, $T_{h_1}(L_1)$ is a coordinate polyline $L_2 = L_2(a_1^1, \dots, a_k^1)$ with $a_1^1 = T_{h_1}(a_1) \in D_0$, $a_2^1 = T_{h_1}(a_2) \notin E$, $\dots$, $a_k^1 = T_{h_1}(a_k) \in B(a_k, r_1) \subset D_1$. If $a_2^1 \in D_1$ then we choose $[a, b] = [a_1^1, a_2^1]$.

Suppose that $a_2^1 \in D_0$. Then in $B(a_3^1, r_3)$ with $0 < r_3 < \min(\text{dist}(a_2^1, \partial B(a_2, r_2) \cup E), \text{dist}(L_2, \Pi), \text{dist}(a_k^1, \partial B(a_k, r_1)))$, as above, there is a point $a_3^2 \notin E$. Put $L_3 = L_3(a_2^1, \dots, a_k^1)$ and let $T_{h_2} : x \rightarrow x + h_2$, where $h_2 = a_3^2 - a_3^1$. Hence $T_{h_2}(L_3)$ is a coordinate polyline $L_4 = L_4(a_2^2, \dots, a_k^2)$ with $a_2^2 \in D_0$, $a_3^2 \notin E$, $\dots$, $a_k^2 = T_{h_2}(a_k^1) \in B(a_k, r_1) \subset D_1$.

If $a_3^2 \in D_1$ then we choose $[a, b] = [a_2^2, a_3^2]$. If $a_3^2 \in D_0$ then we continue this process. Taking into account that, by the choice, $T_{h_1}(a_k^1) \in D_1$, $T_{h_2}(a_k^2) \in D_1$ and so far, at most after k steps we get the required segment $[a, b]$. Without loss of generality, we assume that the segment $[a, b]$ is parallel to the x_1 -axis.

Let's choose a coordinate rectangle $\Pi_1 \subset \Pi$ so that its opposite facets $\sigma_{01}^1, \sigma_{11}^1$ lie in D_0 and D_1 , respectively, and are orthogonal to the x_1 -axis. In addition $a \in \sigma_{01}^1$ and $b \in \sigma_{11}^1$.

Set $u = 0$ on D_0 and $u = 1$ on D_1 . Then $u \in \text{Adm}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E)$ and $\int_{\Pi \setminus E} |\nabla u| \omega dx = 0$. This implies $C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E) = 0$. On the other hand, by (4), (12) and Property 2, we see that $C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E) > 0$. The resulting contradiction completes the proof of this Property. $\square$

Note that in the proof of the Property 6, only the condition $m_n(E) = 0$ was used to construct the rectangle Π_1 . And $\Pi_1 \in \mathcal{K}_1(e, \Omega)$, where $e = \overline{\Pi_1} \cap E$ and $\mathcal{K}_1(e, \Omega)$ is from definition of $NED_{1,\omega}$ -sets in Ω .

It is clear that $\Gamma(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = \emptyset$ and hence $M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = 0$. On the other hand, write $\Pi_1 = \{x = (x_1, \dots, x_n) : a_i^1 < x_i < b_i^1, i = 1, \dots, n\}$ and set $\Pi_{1,\delta} = \{x = (x_1, \dots, x_n) : a_1^1 < x_1 < b_1^1, a_i^1 + \delta < x_i < b_i^1 - \delta, i = 2, \dots, n\}$, where $0 < \delta < \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}$. Then, applying (5) and (12), we have

$$C_1 M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) \geq \lim_{\delta \rightarrow 0} M_{1,\omega}(\sigma_{01}^1 \cap \overline{\Pi_{1,\delta}}, \sigma_{11}^1 \cap \overline{\Pi_{1,\delta}}, \Pi_{1,\delta}) \geq \frac{C_1}{\max_{\Pi_1} (1 + |x|)^n} m_{n-1}(\sigma_{01}^1) > 0,$$

that contradicts the condition $M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = 0$. Thus, we come to another assertion

Property 7. *If E is an $NED_{1,\omega}$ -set in Ω then $\Pi \setminus E$ is a domain for all $\Pi \in \Omega$.*

Property 8. *If E is an $NC_{1,\omega}$ -set in Ω then E is an $NED_{1,\omega}$ -set in Ω .*

Proof. Take some compact $e \subset E$, coordinate rectangle $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ and make sure that inequalities (5) are true if $j = 1$

and $\Pi \in \mathcal{K}_1(e, \Omega)$. For the remaining j and other coordinate rectangles $\in \mathcal{K}_j(e, \Omega)$, $j = 2, \dots, n$ and e the proof of this fact is performed in the same way.

Let's introduce a few notations. Let $\sigma_0 = \sigma_{01}$, $\sigma_1 = \sigma_{11}$ be opposite facets of Π such as in (4). Let $\Pi_{1,\delta}$ be such as in (5) and set $\sigma_0 \cap \overline{\Pi_{1,\delta}} = \sigma_0(\delta)$, $\sigma_1 \cap \overline{\Pi_{1,\delta}} = \sigma_1(\delta)$.

Fix $0 < \delta < \min_i \frac{b_i - a_i}{2}$.

Since $\Pi \in \mathcal{K}_1(e, \Omega)$, then $\eta = \text{dist}(\sigma_0 \cup \sigma_1, e) > 0$. Hence there exist two sequences $\{\Omega_{0,k}\}$, $\{\Omega_{1,k}\}$ of coordinate rectangles $\Omega_{0,k}, \Omega_{1,k} \Subset \Omega$ such that $\text{dist}(\Omega_{0,k} \cup \Omega_{1,k}, e) > 0$ for all $k = 1, \dots$; $\sigma_0 \subset \Omega_{0,k+1} \Subset \Omega_{0,k}$, $\bigcap_k \Omega_{0,k} = \sigma_0$; $\sigma_1 \subset \Omega_{1,k+1} \Subset \Omega_{1,k}$,

$\bigcap_k \Omega_{1,k} = \sigma_1$ and $\overline{\Omega_{0,1}} \cap \overline{\Omega_{1,1}} = \emptyset$.

Set $\sigma_0^k = \partial\Omega_{0,k} \cap \overline{\Pi_{1,\delta}}$, $\sigma_1^k = \partial\Omega_{1,k} \cap \overline{\Pi_{1,\delta}}$.

Put in Lemma 2 $\Omega = \Pi \setminus e$, $F_0 = \sigma_0$, $F_1 = \sigma_1$. Then for a given $\varepsilon > 0$ there is a metric $\rho \in \text{adm } \Gamma(\sigma_0, \sigma_1, \Pi \setminus e) \cap L_{1,\omega}$ that satisfies the conditions of Lemma 2. In particular, ρ is a continuous function on $\Pi \setminus e$,

$$(17) \quad \inf_K \rho > 0 \text{ for all compact set } K \subset R^n$$

and

$$(18) \quad M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) \leq \int_{R^n} \rho \omega dx < C_1 M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) + \varepsilon.$$

In addition, the metric ρ is continuous on $\partial\Pi_{1,\delta} \setminus (e \cup \sigma_0^k \cup \sigma_1^k)$ for all $k = 1, 2, \dots$ and $\int_\gamma \rho ds \geq 1$ for any locally rectifiable curve γ , connecting $\sigma_0(\delta)$ and $\sigma_1(\delta)$ in $\overline{\Pi_{1,\delta}} \setminus (\sigma_0(\delta) \cup \sigma_1(\delta) \cup e)$. Finally, let $\Pi_{1,\delta}^k = \Pi_{1,\delta} \setminus (\overline{\Omega_{0,k}} \cup \overline{\Omega_{1,k}})$, $\Gamma_k = \Gamma(\sigma_0^k, \sigma_1^k, \Pi_{1,\delta}^k \setminus e)$.

Take $\beta \in (0, 1)$. We shall show that there is $k_0 = k_0(\beta) \in \mathbb{N}$ for which

$$(19) \quad \int_\gamma \rho ds \geq 1 - \beta$$

for all $\gamma \in \Gamma_k$ if $k \geq k_0$. Indeed, admitting the opposite, we obtain a sequence $\{\gamma_k\}_{k=1}^\infty$ such that $\int_{\gamma_k} \rho ds < 1 - \beta$ for all $k = 1, 2, \dots$.

By (17), it is inferred that γ_k is a rectifiable curve and for any k and j , $1 \leq j < k$, γ_k contains points in σ_0^j and σ_1^j .

Let $x_0(j, k)$ (resp. $x_1(j, k)$) be the first (resp. last) point in σ_0^j (resp. σ_1^j) coming along γ_k from σ_0^k . Choose a subsequence $\{\gamma_{k_j}\}$ of $\{\gamma_k\}$ so that $x_s(j, k_j)$ converges to a point $x_{js} \in \sigma_s$, $s = 0, 1$. Denote this subsequence by $\{\gamma_{1k}\}$.

Since ρ is continuous in some neighborhood of σ_s^j , we can find a closed ball $B_{1s} = \overline{B(x_{1s}, r_{1s})}$ in $\Pi \setminus e$ so that $\int_l \rho ds < \frac{\beta}{2^s}$ for any segment l in B_{1s} . We may assume that all γ_{1k} meet B_{1s} , $s = 0, 1$.

Similarly we find a subsequence $\{\gamma_{2k}\}$ of $\{\gamma_{1k}\}$ and a closed balls B_{20}, B_{21} in $\Pi \setminus e$ such that $\int_l \rho ds < \frac{\beta}{2^7}$ for any segment $l \subset B_{20} \cup B_{21}$. We continue this process.

Denote the diagonal subsequence $\{\gamma_{kk}\}$ by $\{\gamma_k\}$.

For each $k \geq 1$ we modify each γ_k in every B_{js} so that all γ_k , $\gamma_k \cap B_{js} \neq \emptyset$, pass through x_{js} , as follows.

We replace $\gamma_k \cap B_{js}$ by two radii, terminating at $\gamma_k \cap \partial B_{js}$ and denote the resulting curve still by γ_k . Using further standard construction from the proof

of Lemma 4.10 in [13, p. 138–139], we obtain a curve $\tilde{\gamma}$, satisfying the following conditions:

- (1) $\int_{\tilde{\gamma}} \rho ds < 1 - \frac{\beta}{8}$ and therefore, by (17), $\tilde{\gamma}$ is a rectifiable curve,
- (2) $\tilde{\gamma} \in \Gamma(\sigma_0, \sigma_1, \Pi \setminus e)$ and $\tilde{\gamma} \subset \overline{\Pi_{1,\delta}} \setminus (\sigma_0(\delta) \cup \sigma_1(\delta) \cup e)$,
- (3) x_{j0} and $x_{j1} \in \tilde{\gamma}$ for all $j = 1, 2, \dots$

Hence, by condition 2), we have $\int_{\tilde{\gamma}} \rho ds \geq 1$, that contradicts condition 1). Thus,

the inequality (19) is true.

Take $k_1 > k_0(\beta)$ and let $\sigma_0^{k_1} \subset \{x = (x_1, \dots, x_n) : x_1 = a_{1,k_1}\}$, $\sigma_1^{k_1} \subset \{x = (x_1, \dots, x_n) : x_1 = b_{1,k_1}\}$.

Set

$$\rho_1 = \begin{cases} 0, & x_1 \leq a_{1,k_1}, \\ \frac{\rho}{1-\beta}, & x_1 > a_{1,k_1}, x \in \Pi_{1,\delta} \setminus e, \end{cases}$$

and let $G_1 = \{x = (x_1, \dots, x_n) : x_1 \leq a_{1,k_1}\} \cup (\Pi_{1,\delta} \setminus e)$, $u_1(x) = 0$, if $x_1 \leq a_{1,k_1}$, $u_1(x) = \inf_{\gamma_x} \int \rho ds$, where γ_x is arbitrary rectifiable curve in $\Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e)$, connecting $\sigma_0^{k_1+1}$ and $x \in \Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e)$.

In view of local boundedness ρ_1 on G_1 and known properties of functions such as u_1 [13, Lemma 5.2], it follows that u_1 satisfies the Lipschitz condition in G_1 , $|\nabla u_1| \leq \rho_1$ a.e. on G_1 , $u_1 = 0$ on $\{x : x_1 \leq a_{1,k_1}\}$.

Taking the cutting $u_2 = \min(1, u_1)$ on G_1 , we get

$$(20) \quad \int_{\Pi_{1,\delta} \setminus e} |\nabla u_2| \omega dx \leq \int_{\Pi_{1,\delta} \setminus e} |\nabla u_1| \omega dx,$$

where u_2 , by properties of the cutting, satisfies the Lipschitz condition on G_1 and $u_2 = 1$ on $\Pi_{1,\delta} \cap \Omega_{1,k_1}$, $u_2 = 0$ on Ω_{0,k_1} . Set

$$u_3 = \begin{cases} u_2, & x \in G_1 \setminus \Omega_{1,k_1+1}, \\ 1, & x \in \Omega_{1,k_1+1}. \end{cases}$$

It is easily to see that $u_3 \in \text{Adm}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e)$. Then, by (18), (20), Property 2 and the arbitrariness of β , we have for $NC_{1,\omega}$ -set E

$$M_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) \leq C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) = C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e) \leq \int_{\Pi_{1,\delta} \setminus e} |\nabla u_3| \omega dx \leq \int_{\Pi_{1,\delta} \setminus e} \rho \omega dx + o(1) \leq C_1 M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) + \varepsilon + o(1),$$

where $o(1) \rightarrow 0$ as $\beta \rightarrow 0$. Hence, taking $\beta \rightarrow 0$, $\delta \rightarrow 0$, $\varepsilon \rightarrow 0$, we deduce (5). $\square$

Remark 1. In the proof of Property 8 we applied the cutting $\min(1, u_1)$. Similarly using $\max(\min(1, u), 0)$ to the function $u \in \text{Adm}(F_0, F_1, G)$, we come the following definition of the capacity of the condenser.

Definition. If F_0, F_1 are compact disjoint non-empty sets in $\bar{\Omega}$ then

$$C_{1,\omega}(F_0, F_1, \Omega) = \inf_u \int_{\Omega} |\nabla u| \omega dx,$$

where the infimum is taken over all real-valued bounded functions u such that $u|_\Omega$ satisfies locally the Lipschitz condition and $u = j$ in some neighborhood of F_j , $j = 0, 1$.

Remark 2. Let F_0, F_1 be compact non-empty disjoint sets in Ω . Let $\{D_j\}$ be a sequence (maybe finite) of all mutually disjoint connected components D_j of Ω .

Since F_0, F_1 are compact sets, there exists a finite family of components $D_j \in \{D_j\}$, having common points with $F_0 \cup F_1$. Without loss of generality, suppose that only $D_1, \dots, D_{j_1}$ intersect $F_0 \cup F_1$. Let there is no component D_j , $1 \leq j \leq j_1$, such that simultaneously $D_j \cap F_0 \neq \emptyset$, $D_j \cap F_1 \neq \emptyset$. This implies $\Gamma(F_0, F_1, \Omega) = \emptyset$. In addition, assuming $u = 0$ on D_j if $D_j \cap F_0 \neq \emptyset$, $u = 1$ on D_j if $D_j \cap F_1 \neq \emptyset$, $u = 1$ on D_j if $j > j_1$, we get $u \in \text{Adm}(F_0, F_1, \Omega)$ and $\int_\Omega |\nabla u| \omega dx = 0$. it follows

that $C_{1,\omega}(F_0, F_1, \Omega) = 0$.

Otherwise, there is a component D_j , $1 \leq j \leq j_1$, for which $D_j \cap F_0 \neq \emptyset$, $D_j \cap F_1 \neq \emptyset$ and hence $\Gamma(F_0, F_1, \Omega) \neq \emptyset$.

Then using the same arguments from the proof of Property 8 (see also [9, Theorem 5.5]), we obtain the next assertion.

Theorem 1. If F_0, F_1 are compact disjoint non-empty sets in Ω then

$$M_{1,\omega}(F_0, F_1, \Omega) \leq C_{1,\omega}(F_0, F_1, \Omega) \leq C_1 M_{1,\omega}(F_0, F_1, \Omega),$$

where C_1 depends only on ω and $C_1 = 1$ for $\omega \equiv 1$.

5. REMOVABLE SETS

In this section, we will establish three criteria of removable sets E for $L^1_{1,\omega}(\Omega)$. We recall (see Definition 1) that E is a removable set for $L^1_{1,\omega}(\Omega)$ if E is a relatively closed subset of Ω , $m_n(E) = 0$ and for every function $u \in L^1_{1,\omega}(\Omega \setminus E)$ there exists a function $v \in L^1_{1,\omega}(\Omega)$ such that $v|_{\Omega \setminus E} = u$.

We will need the following two lemmas. Further let $\Omega = \bigcup_j D_j$, where $\{D_j\}$ is the sequence (maybe finite) of connected components of Ω from Remark 2.

Lemma 3. If E is a removable set for $L^1_{1,\omega}(\Omega)$ then $D_j \setminus E$ is a domain for all j .

Proof. Suppose that there exists some component D_j which is partitioned by E into non-empty sets Q_0 and Q_1 , $D_j \setminus E = Q_0 \cup Q_1$, $Q_0 \cap Q_1 = \emptyset$. take two point $q_0 \in Q_0$, $q_1 \in Q_1$ and balls $B_0 = B(q_0, r_0)$, $B_1 = B(q_1, r_1)$ be such that $\overline{B_0} \subset Q_0$, $\overline{B_1} \subset Q_1$. Let $L_1 = L_1(a_1, \dots, a_k)$ be a simple polyline in a domain D_j composed of straight segments $[a_i, a_{i+1}]$, $i = 1, \dots, k-1$, where each segment $[a_i, a_{i+1}]$ is parallel to some coordinate axis and $a_1 = q_0$, $a_k = q_1$. Applying reasoning from the proof of Property 6, we obtain a coordinate rectangle Π_1 with opposite facets $\sigma_0 \subset Q_0$ and $\sigma_1 \subset Q_1$. Suppose, for example, that σ_0, σ_1 are orthogonal to the x_1 -axis. Let $u_0 = 0$ on Q_0 and $u_0 = 1$ on $Q_1 \cap (\Omega \setminus D_j)$. Obviously, $u_0 \in L^1_{1,\omega}(\Omega \setminus E)$ and, by definition of E , there is a function $v_0 \in L^1_{1,\omega}(\Omega)$, for which $v_0|_{\Omega \setminus E} = u_0$. In view of Proposition 1, $v_0 \in W^{1,1}(D_j, \text{loc})$ and therefore $v_0 \in L^1_1(\Pi_1)$. Then (see [11, Sec. 1.1.3, Theorem 1]) function v_0 is absolutely continuous on m_{n-1} -almost all segments l , $\mathcal{H}^1(l \cap E) = 0$, joining the facets σ_0, σ_1 in Π_1 and parallel to the x_1 -axis. This implies the existence of a limit point $x_l \in E \cap l$ simultaneously for $l \cap Q_0$ and $l \cap Q_1$. Hence, $v_0(x_l) = 0$ and $v_0(x_l) = 1$, that contradicts the definition of function v_0 . Thus, $D_j \setminus E$ is a domain for all j and our Lemma is proved. $\square$

Lemma 4. *Let $B = B(a, r)$ be some ball in an open set $G \subset \Omega$ and E be a removable set for $L_{1,\omega}^1(\Omega)$. Suppose that $\varphi \in C_0^\infty(B)$ and $u \in L_{1,\omega}^1(G \setminus E)$. In addition, assume that u is a bounded and locally satisfying the Lipschitz condition on $G \setminus E$, $B_1 = B(a, r_1)$ is some ball such that $0 < r_1 < r$ and $\text{supp } \varphi \in B_1$. Then there exists a function $g \in W_{\omega}^{1,1}(R^n)$ for which $g = 0$ on $R^n \setminus \overline{B_1}$, $g|_{G \setminus E} = u\varphi$.*

Proof. Set $E_1 = \overline{B_1} \cap E$ and, by Lemma 3, note that $B \setminus E_1$ is a domain. In addition, E_1 as a closed subset of E is a removable set for $L_{1,\omega}^1(\Omega)$. Put $v = u\varphi$ on $\overline{B_1} \setminus E = \overline{B_1} \setminus E_1$ and $v = 0$ on $R^n \setminus \overline{B_1}$.

We see that v satisfies locally the Lipschitz condition on $(B_1 \setminus E_1) \cap (R^n \setminus \overline{B_1})$. If $x_0 \in \partial B_1 \setminus E_1$ then there is some ball $B(x_0, r_0)$ such that $B(x_0, r_0) \subseteq (B \setminus E) \setminus \text{supp } \varphi$. This implies $v = 0$ on $B(x_0, r_0)$ and hence v satisfies the Lipschitz condition on $B(x_0, r_0)$. On account of the arbitrariness of $x_0 \in \partial B_1 \setminus E_1$ we obtain that v satisfies locally the Lipschitz condition on $R^n \setminus E$ and $v|_{G \setminus E} = u\varphi$. Since, u , φ , $|\nabla \varphi|$ are bounded functions on $\overline{B_1} \setminus E_1$, we see that $v \in W_{\omega}^{1,1}(R^n \setminus E_1)$ and hence $v \in W_{\omega}^{1,1}(\Omega \setminus E_1)$. It follows that there exists $g \in L_{1,\omega}^1(\Omega)$ such that $g|_{\Omega \setminus E_1} = v$, $g|_{G \setminus E} = u\varphi$ and $g = 0$ on $R^n \setminus \overline{B_1}$. Assuming $g = 0$ on $R^n \setminus \Omega$ we get a function that satisfies the condition of Lemma. Thus the Lemma is proved. $\square$

Theorem 2. *If E is a removable set for $L_{1,\omega}^1(\Omega)$ then E is an $NC_{1,\omega}$ -set in Ω .*

Proof. Let $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ be a coordinate rectangle with its facets σ_{0i}, σ_{1i} from definition of $NC_{1,\omega}$ -set in (4).

In view of $m_n(E) = 0$, we see that $\sigma_{0i}, \sigma_{1i} \subset \partial(\Pi \setminus E)$ for all $i = 1, \dots, n$. Verify that the equality (4) is true if $j = 1$. For the remaining $j = 2, \dots, n$ the proof of (4) is performed in the same way.

Given $\varepsilon > 0$, by Lemma 1 and Remark 1, we find a bounded function $u \in \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \cap L_{1,\omega}^1(\Pi \setminus E)$ such that

$$(21) \quad C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \leq \int_{\Pi \setminus E} |\nabla u| \omega dx < C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) + \frac{\varepsilon}{2}.$$

Let G_s be an open neighborhood of the facets σ_{s1} , in which $u = s$, where $G_s \subseteq \Omega$, $s = 0, 1$, and $\text{dist}(G_0, G_1) > 0$. In addition, let G'_s be another open neighborhood of the facets σ_{s1} , where $G'_s \subseteq G_s$, $s = 0, 1$.

Set $G = G_0 \cup G_1 \cup \Pi$ and $E_1 = (\Pi \cap E) \setminus (G_0 \cup G_1)$. It is easily to see that $u \in W_{\omega}^{1,1}(G \setminus E_1)$ and E_1 is a relatively closed subset of G .

Let the sequence $\{B_k\}$ be a locally finite covering of G by balls $B_k = B(a_k, r_k) \subseteq G$, $k \geq 1$. Additionally we required that all balls B_k having common points with G'_s be contained in G_s , $s = 0, 1$. Let $\{\varphi_k\}$ be a C^∞ -partition of unity for G subordinating to the covering $\{B_k\}$. We show that there is a function $z \in \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi)$ with $\|z\|_{L_{1,\omega}^1(\Pi)}$ arbitrarily close to $\|u\|_{L_{1,\omega}^1(\Pi \setminus E)}$.

Indeed, let $B'_k = B(a_k, r'_k)$ be a ball such that $0 < r'_k < r_k$ and $\text{supp } \varphi_k \subset B'_k$. Then, by Lemma 4, there is a function $g_k \in W_{\omega}^{1,1}(R^n)$ for which $g_k = 0$ on $R^n \setminus \overline{B'_k}$, $g|_{G \setminus E_1} = u\varphi_k$, $k \geq 1$. Set $u_k = u\varphi_k$ on $G \setminus E_1$, $k \geq 1$, and $u = \sum_k u_k$ on $G \setminus E_1$ and $g = \sum_k g_k$ on G .

On any bounded open set $Q \subseteq G$ we have

$$(22) \quad 1 = \sum_k \varphi_k, \quad g = \sum_k g_k$$

and on $Q \setminus E_1$

$$(23) \quad u = \sum_k u_k,$$

where the sums in (22), (23) contain a finite number of terms. This implies

$$(24) \quad \|g\|_{W_\omega^{1,1}(Q)} \leq \infty, \quad \|u - g\|_{W_\omega^{1,1}(Q \setminus E_1)} \leq \sum_k \|u_k - g_k\|_{W_\omega^{1,1}(Q \setminus E_1)} = 0$$

On account of the arbitrariness of Q we obtain $g \in W_\omega^{1,1}(G)$ and

$$\int_{\Pi \setminus E} |\nabla g| \omega \, dx = \int_{\Pi \setminus E} |\nabla u| \omega \, dx = \int_{\Pi} |\nabla g| \omega \, dx.$$

Moreover, if $Q = G_0$ then $u = 0$ on G_0 , $\|u - g\|_{W_\omega^{1,1}(G_0)} = 0$ and therefore $g = 0$ on G_0 . Similarly if $Q = G_1$ then $u = 1 = g$ on G_1 .

Next we approximate the function g in $W_\omega^{1,1}(G)$ by smooth functions as follows.

If $B_k \cap G'_0 \neq \emptyset$, we set $z_k = g_k = 0$ on R^n . If $B_k \cap G'_1 \neq \emptyset$, we set $z_k = \varphi_k \in C_0^\infty(R^n)$. In other cases let z_k denote the mollification of $g_k \in W_\omega^{1,1}(R^n)$ with a radius $0 < \rho_k < \text{dist}(\text{supp } \varphi_k, \partial B'_k)$.

We take $\beta \in \left(0, \frac{1}{2}\right)$ and, by Proposition 6, choose ρ_k to satisfy

$$\|g_k - z_k\|_{W_\omega^{1,1}(R^n)} < \beta^k.$$

By construction, $z_k \in W_\omega^{1,1}(R^n) \cap C_0^\infty(R^n)$. Set $z = \sum_k z_k$ and on any set $Q \Subset G$ we have

$$\|g - z\|_{W_\omega^{1,1}(\Omega)} \leq \sum_k \|g_k - z_k\|_{W_\omega^{1,1}(R^n)} \leq \frac{\beta}{1 - \beta} \leq 2\beta.$$

Therefore $z \in L_{1,\omega}^1(G) \cap C^\infty(G)$ and $z = 0$ on G'_0 and $z = 1$ on G'_1 . This implies $z \in \text{Adm}(\sigma_0, \sigma_1, \Pi)$ and

$$(25) \quad \int_{\Pi} |\nabla z| \omega \, dx = \int_{\Pi} |\nabla g| \omega \, dx + o(1),$$

where $o(1) \rightarrow 0$ as $\beta \rightarrow 0$.

Connecting (21), (24) and (25), we derive

$$\begin{aligned} C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) &\leq \int_{\Pi} |\nabla z| \omega \, dx \leq \int_{\Pi \setminus E} |\nabla g| \omega \, dx + o(1) \leq \\ &\int_{\Pi \setminus E} |\nabla u| \omega \, dx + o(1) \leq C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) + o(1) + \varepsilon. \end{aligned}$$

Sequentially taking $\beta \rightarrow 0$, $\varepsilon \rightarrow 0$, we obtain

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) = C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E).$$

This is proved our theorem. $\square$

Theorem 3. *Let E be a relatively closed subset of an open set $\Omega \subset R^n$. Then the following four conditions are equivalent:*

- (1) E is a removable set for $L_{1,\omega}^1(\Omega)$;
- (2) E is an $NC_{1,\omega}$ -set in Ω ;

- (3) E is an $NED_{1,\omega}$ -set in Ω ;
- (4) $E \in \mathcal{G}_{1,\omega}(\Omega)$.

Proof. The implication 1) $\rightarrow$ 2) is established in Theorem 2 and the implication 2) $\rightarrow$ 3) is obtained in Proposition 8. So first we get the implication 3) $\rightarrow$ 4).

Let E be an $NED_{1,\omega}$ -set in Ω and hence $m_n(E) = 0$, Ω does not locally partitioned by E . Take a compact set $e \subset E$ and verify that e satisfies the $(1, \omega)$ -girth condition with respect to X_i (see Definition 1 in Sec. 2.6) if $i = 1$. For the remaining $i = 2, \dots, n$ the proof of the $(1, \omega)$ -girth condition with respect to X_i is performed in the same way.

Let ρ be a function from Definition 1 with respect to X_1 and put $\rho = 0$ on e .

Consider also some coordinate rectangle $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ such that $e \subset \Pi$. Put $\Pi' = \{x' = (x_2, \dots, x_n) : a_i < x_i < b_i, i = 2, \dots, n\}$ and consider $x' \in \Pi'$ such that $l(x') = \{x = (x_1, x') \in R^n : a_1 \leq x_1 \leq b_1\}$ has common points with e .

Verify that ε -girth condition holds for ρ on the m_{n-1} -almost all segments $l(x')$. It suffices to prove this fact for the function $\rho_1 = \rho + \rho_0$, where $\rho_0 = 1$ on $\bar{\Pi}$ and $\rho_0 = 0$ on $R^n \setminus \bar{\Pi}$. Then $\int_{\Pi'} dx' \int_{a_1}^{b_1} \rho_1 \omega dx_1 = \int_{\Pi} \rho_1 \omega dx < \infty$.

Put $\Phi(x', r_1, r_2) = \int_{r_1}^{r_2} \rho_1 \omega dx$ for all rationals $r_1, r_2 \in [a_1, b_1]$, $r_1 < r_2$. As in the proof of Property 5, we verify that there exists a m_{n-1} -measurable set $\tilde{\Pi}' \subset \Pi'$ with $m_{n-1}(\tilde{\Pi}') = m_{n-1}(\Pi')$ such that every point $x' \in \tilde{\Pi}'$ is a Lebesgue point for the function $\Phi(x', r_1, r_2)$ (a differentiability point of the integral of these functions over Π'). Choose now $x' = (x_2, \dots, x_n) \in \tilde{\Pi}'$ so that $\mathcal{H}^1(l(x) \cap e) = 0$ (by definition $l(x') \cap e \neq \emptyset$).

Cover $l(x') \cap e$ by the mutually disjoint intervals $U_k = (c_k, d_k)$, $k = 1, \dots, k_1$, where $c_k = (r_k, x')$, $d_k = (\tilde{r}_k, x')$, while r_k and $\tilde{r}_k$ are rationals in (a_1, b_1) with $r_k < \tilde{r}_k$ and

$$(26) \quad \sum_{k=1}^{k_1} \mathcal{H}^1(U_k) < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{U_k} \rho_1 \omega dx < \frac{\varepsilon C}{8},$$

where $C = \frac{1}{\sup_{\Pi} (1 + |x|)^n}$. For small $\delta > 0$ put

$$\Pi'(\delta) = \{y' = (y_2, \dots, y_n) : |y_i - x_i| < \delta, i = 2, \dots, n\},$$

$$\Pi_k(\delta) = \{(x, y') : r_k < x_1 < \tilde{r}_k, y' \in \Pi'(\delta)\} \subset \Pi.$$

Denote by $\sigma_0^k = \sigma_0^k(\delta)$, $c_k \in \sigma_0^k$, and $\sigma_1^k = \sigma_1^k(\delta)$, $d_k \in \sigma_1^k$, the facets of $\Pi_k(\delta)$, parallel to $H_1 = \{x = (x_1, \dots, x_n) : x_1 = 0\}$.

In addition, put $\Gamma_k(\delta) = \Gamma(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e)$. It is easily to see that $\Pi_k(\delta) \in \mathcal{K}_1(e, \Pi)$. By (12) and the definition of an $NED_{1,\omega}$ -set,

$$M_{1,\omega}(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e) \geq \frac{m_{n-1}(\sigma_0^k)}{\sup_{\Pi_k(\delta)} (1 + |x|)^n} \geq C \cdot m_{n-1}(\sigma_0^k),$$

where $m_{n-1}(\sigma_0^k) = m_{n-1}(\Pi'(\delta))$.

Let $L_k(\delta) = \inf_{\gamma} \{ \int_{\gamma} \rho_1 ds : \gamma \in \Gamma_k(\delta) \}$. Obviously, $L_k(\delta) \geq \tilde{r}_k - r_k$ for all $k = 1, \dots, k_1$. Show that $L_k(\delta)$ is sufficiently small when $\delta > 0$ close to zero. Indeed, $\frac{\rho_1}{L_k(\delta)} \in \text{adm } \Gamma_k(\delta)$ and

$$\int_{\Pi'(\delta)} dx' \int_{r_k}^{\tilde{r}_k} \rho_1 \omega dx_1 \geq L_k(\delta) C m_{n-1}(\sigma_0^k).$$

For sufficiently small $\delta > 0$, using the differentiability of the integrals, under consideration,

$$L_k(\delta) \leq \frac{1}{C} \int_{r_k}^{\tilde{r}_k} \rho_1 \omega dx_1 + \frac{\varepsilon}{8k_1}$$

for all $k = 1, \dots, k_1$. By the definition of infimum, there exists a curve $\gamma_k^0 \in \Gamma_k(\delta)$ such that

$$(27) \quad \int_{\gamma_k^0} \rho_1 ds \leq L_k(\delta) + \frac{\varepsilon}{8k_1} \leq \frac{1}{C} \int_{r_k}^{\tilde{r}_k} \rho_1 \omega dx_1 + \frac{\varepsilon}{4k_1}.$$

Since $\rho_1 \geq 1$ on Π we have $\int_{\gamma_k^0} ds \leq \int_{\gamma_k^0} \rho_1 ds$. Hence γ_k^0 is a rectifiable curve. By the local boundedness of ρ_1 on $\Pi \setminus e$, for sufficiently small $\delta > 0$ there exist segments $\tau_k^0 \subset \sigma_k^0(\delta)$ and $\tau_k^1 \subset \sigma_k^1(\delta)$ joining the points c_k and d_k , respectively, with the end points of γ_k^0 in $\overline{\Pi_k(\delta)} \setminus e$ so that

$$\int_{\tau_k^0} \rho_1 ds + \int_{\tau_k^1} \rho_1 ds < \frac{\varepsilon}{8k_1}.$$

Consequently, there exist simple curves $\gamma_k \subset \tau_k^0 \cup \gamma_k^0 \cup \tau_k^1 \subset \overline{\Pi_k(\delta)} \setminus e$ joining the corresponding points c_k and d_k , $k = 1, \dots, k_1$, and by (26), (27)

$$\sum_{k=1}^{k_1} \int_{\gamma_k} \rho_1 ds < \varepsilon.$$

This implies

$$(28) \quad \sum_{k=1}^{k_1} \int_{\gamma_k} ds < \sum_{k=1}^{k_1} \int_{\gamma_k} \rho_1 ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{\gamma_k} \rho ds < \sum_{k=1}^{k_1} \int_{\gamma_k} \rho_1 ds < \varepsilon.$$

In other words, the ε -girth condition holds for ρ and ρ_1 on m_{n-1} -almost all segments $l(x')$, $x' \in \tilde{\Pi}'$.

Let $t = (t_1, x') \in l(x') \setminus e$, where (28) holds for $l(x')$. It is easy to verify that the ε -girth condition holds for ρ , ρ_1 on $l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\}$ (see Remark to Definition 1) for all such t on $l(x')$. We will call this property the $[t, \varepsilon]$ -condition for the function ρ on the segment $l(x')$. So the implication 3) $\rightarrow$ 4) is true.

Let's go to the proof of implication 4) $\rightarrow$ 1) which we will present in two steps.

Step 1. Let $E \in \mathcal{G}_{1,\omega}(\Omega)$. By the definition 3, $m_n(E) = 0$ and Ω does not partitioned locally by E . Let $u \in L_{1,\omega}^1(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ and u be bounded

function on $\Omega \setminus E$. Then u can be extended to a function $g \in L^1_{1,\omega}(\Omega)$ for which $g|_{\Omega \setminus E} = u$, as follows.

Next we use constructions from the proof of Theorem 2. Let the sequence $\{B_j\}$ be a locally finite covering of Ω by balls $B_j = B(a_j, r_j) \Subset \Omega$, $j \geq 1$. Let $\{\varphi_j\}$ be a C^∞ -partition of unity, subordinated to the covering $\{B_j\}$. Besides that let $B'_j = B(a_j, r'_j)$ be a ball such that $0 < r'_j < r_j$ and $\text{supp } \varphi_j \subset B'_j$. Set $e_j = \overline{B'_j} \cap E$ and let $v_j = u\varphi_j$ on $\overline{B'_j} \setminus e_j$ and $v_j = 0$ on $R^n \setminus \overline{B'_j}$.

It is easy to see (proof of Lemma 4) that $v_j \in W^{1,1}_\omega(R^n \setminus e_j) \cap C^\infty(R^n \setminus e_j)$ and so, $v_j|_{\Omega \setminus E} = u\varphi_j$, $|\nabla v_j|$ is a continuous function on $R^n \setminus e_j$.

Take some coordinate rectangle Π_j such that $\overline{B'_j} \subset \Pi_j$ and verify that v_j can be extended to a function $g_j \in L^1_{1,\omega}(R^n)$, when $g|_{R^n \setminus e_j} = v_j$.

Since $v_j = 0$ on $R^n \setminus \overline{B'_j}$, it suffices to prove that $v_j|_{\Pi_j \setminus e_j}$ can be extended to function $g_j \in L^1_{1,\omega}(\Pi_j)$. Fix $j \geq 1$.

In the implication 3) $\rightarrow$ 4) proof given above, we will put $e = e_j$, $\rho = |\nabla v_j|$ on $R^n \setminus e$, $\Pi_j = \Pi$ and $\rho = |\nabla v_j| = 0$ on e . In addition, we set $\frac{\partial v_j}{\partial x_i} = 0$ on e , $i = 1, \dots, n$, and we use notation from the proof of implication 3) $\rightarrow$ 4).

Show that v_j is an absolutely continuous function on m_{n-1} -almost all segments $l(x') = \{(x_1, x') : a_1 \leq x_1 \leq b_1\}$, $x' \in \Pi' = \{(x_2, \dots, x_n) : a_i < x_i < b_i, i = 2, \dots, n\}$, if we define v_j properly on e .

By Proposition 1, we also note that $\int_{\Pi} |\nabla v_j| dx < \infty$. This implies

$$(29) \quad \int_{l(x')} |\nabla v_j| dx_1 < \infty$$

for m_{n-1} -almost all $l(x')$, $x' \in \Pi'$. In addition, if $l(x') \cap e = \emptyset$ then v_j is infinitely differentiable, and therefore obviously absolutely continuous on $l(x')$.

Thus, we consider only those segments $l(x')$, $x' \in \Pi'$, that satisfy the inequality (29), $l(x') \cap E \neq \emptyset$, $\mathcal{H}^1(l(x') \cap E) = 0$ and on which the ε -girth condition holds for $\rho = |\nabla v_j|$ for all $\varepsilon > 0$ (see Remark to Definition 1). The family of all such segments $l(x')$ is denoted by Γ_1 . Take $l(x') \in \Gamma_1$ and given $\eta > 0$, choose $\beta > 0$ such that $\int_F |\nabla v_j| dx_1 < \eta$ for every $F \subset l(x')$ with $\mathcal{H}^1(F) < \beta$.

On the segment $l(x')$ consider an absolutely continuous function

$$(30) \quad q(t_1, x') = \int_{a_1}^{t_1} \frac{dv_j}{dx_1} dx_1,$$

where $t = (t_1, x') \in l(x')$. We state that $q(t) = v_j(t)$ for all $t = (t_1, x') \in l(x') \setminus e$.

By the choice of $l(x')$ and $[t, \varepsilon]$ -girth condition for $|\nabla v_j|$ on $l(x')$ (see proof of implication 3) $\rightarrow$ 4) above), there exist intervals $U_k = (c_k, d_k) \subset l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\}$, $k = 1, \dots, k_1$, with $\bigcup_{k=1}^{k_1} U_k \supset e \cap l(t_1, x')$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$ joining the endpoints of U_k such that

$$(31) \quad \sum_{k=1}^{k_1} \int_{\gamma_k} ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{\gamma_k} |\nabla v_j| ds < \varepsilon.$$

Without loss of generality, let $c_k = (r_k, x')$, $d_k = (\tilde{r}_k, x')$, $k = 1, \dots, k_1$, $a_1 < r_1 < \tilde{r}_1 < \dots < r_k < \tilde{r}_k < t_1$. Set $c = (a_1, x')$, $0 < \varepsilon < \eta$. Consider a rectifiable curve $\gamma = [c, c_1] \cup \gamma_1 \cup [d_1, c_2] \cup \gamma_2 \dots \cup [d_{k-1}, c_k] \cup \gamma_k \cup [d_k, t_1]$.

Since $v_j \in C^\infty(R^n \setminus e)$ we have $\int_\gamma \frac{dv_j}{ds} ds = v_j(t) - v_j(c)$. On the other hand, due to absolute continuity of the integrals in (30) and (31) we deduce

$$v_j(t) - v_j(c) = \int_{a_1}^{t_1} \frac{dv_j}{dx_1} dx_1 + o(1),$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $v_j(c) = 0$, taking $\varepsilon \rightarrow 0$, we have

$$v_j(t) = \int_{a_1}^{t_1} \frac{dv_j}{dx_1} dx_1 \text{ for all } t = (t_1, x') \in l(x') \setminus e.$$

Define $v_j(x_1, x')$ as $\int_{a_1}^{x_1} \frac{dv_j}{dx_1} dx_1$ if $x \in l(x') \cap e$, $l(x') \in \Gamma_1$, and $v_j(x) = 0$ if $x = (x_1, x') \in l(x') \cap e$, $l(x') \notin \Gamma_1$.

Hence $v_j(x)$ is an absolutely continuous function in R^n on m_{n-1} -almost all straight lines l , parallel to the x_1 -axis.

In addition, using integration by parts and Fubini's theorem, we obtain

$$(32) \quad \int_{R^n} v_j \frac{\partial \varphi}{\partial x_1} dx_1 = - \int_{R^n} \varphi \frac{\partial v_j}{\partial x_1} dx_1$$

for all $\varphi \in C_0^\infty(R^n)$. Now note that in (32) it is possible to redefine the values of v_j , $\frac{\partial v_j}{\partial x_1}$ on $e = e_j$. Then we get $g_j = v_j$ on $R^n \setminus e$, $g_j = 0$ on e , and $\frac{\partial g_j}{\partial x_1} = \frac{\partial v_j}{\partial x_1}$ on $R^n \setminus e_j$, $\frac{\partial g_j}{\partial x_1} = 0$ on e_j , $j = 1, \dots, n$.

Similarly, let's make sure that the function g_j has a weak derivative $\frac{\partial g_j}{\partial x_i}$ in R^n for $i = 2, \dots, n$. This implies, by construction, $g_j \in W_{\omega}^{1,1}(R^n)$, $g_j = 0$ on $R^n \setminus \overline{B'_j}$, $g_j = u\varphi_j$ on $\Omega \setminus E$. Set $g = \sum_j g_j$ on Ω . Using technique from the proof of Theorem 2 (see (22)–(24)), we deduce that $\|u - g\|_{W_{\omega}^{1,1}(\Omega' \setminus E)} = 0$ for all open $\Omega' \Subset \Omega$, $\|u - g\|_{L_{1,\omega}^1(\Omega \setminus E)} = 0$, $\|u\|_{L_{1,\omega}^1(\Omega \setminus E)} = \|g\|_{L_{1,\omega}^1(\Omega)}$.

In other words, $g \in L_{1,\omega}^1(\Omega)$ and $g|_{\Omega \setminus E} = u$.

Step 2. Now let u be an arbitrary function in $L_{1,\omega}^1(\Omega \setminus E)$ and $\{D_k\}$ be a sequence (maybe finite) of all mutually disjoint connected components D_k of Ω . Then, by the definition of E , $D_k \setminus E$ is the connected component of $\Omega \setminus E$, $k \geq 1$, and $\Omega \setminus E = \bigcup_k (D_k \setminus E)$. By Proposition 7, there exists a sequence of bounded functions $u_j \in L_{1,\omega}^1(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ such that

$$(33) \quad \lim_{j \rightarrow \infty} \|u_j - u\|_{L_{1,\omega}^1(\Omega \setminus E)} = 0,$$

$$(34) \quad \lim_{j \rightarrow \infty} \|u_j - u\|_{L_{1,\omega}(\Omega')} = 0 \text{ for all } \Omega' \Subset \Omega \setminus E.$$

Accordingly to Step 1, we assume that $u_j \in L^1_{1,\omega}(\Omega)$ for all $j \geq 1$. In view of (33) and $m_n(E) = 0$ it follows that $\{u_j\}$ is the Cauchy sequence $L^1_{1,\omega}(\Omega)$. Then, by Proposition 2, $\{u_j\}$ converges in $L^1_{1,\omega}(D_k)$ to some function v_k , $k \geq 1$, as $j \rightarrow \infty$.

Moreover, from (33) $|\nabla(u - v_k)| = 0$ a.e. on $D_k \setminus e$ and therefore $u = v_k + c_k$ (see [5, Sec. 1.1.5]) on $D_k \setminus E$. Using (34), it is easily to show that $c_k = 0$, $k \geq 1$. For all $x \in \Omega$ we set $v(x) = v_k(x)$ if $x \in D_k$. By construction,

$$\|u\|_{L^1_{1,\omega}(\Omega \setminus E)} = \|v\|_{L^1_{1,\omega}(\Omega)}, \quad v(x)|_{\Omega \setminus E} = u(x).$$

Thus, E is a removable set for $L^1_{1,\omega}(\Omega)$, that completes the proof of the Theorem. $\square$

A simple modification of the arguments in the proof of Theorem 3 gives another assertion.

Theorem 4. *If E is an $NC_{1,\omega}$ -set in Ω then $L^m_{1,\omega}(\Omega \setminus E) = L^m_{1,\omega}(\Omega)$, $W^m_{1,\omega}(\Omega \setminus E) = W^m_{1,\omega}(\Omega)$, $W^{m,1}_{\omega}(\Omega \setminus E) = W^{m,1}_{\omega}(\Omega)$.*

6. $(1, 1, \omega)$ -EQUIVALENT DOMAINS IN R^n

Following Vodop'yanov and Gol'dstein [8] domains G_1 and G_2 ($G_1 \supset G_2$) in R^n will be called $(1, 1, \omega)$ -equivalent, if the restriction operator $\theta : L^1_{1,\omega}(G_1) \rightarrow L^1_{1,\omega}(G_2)$ ($\theta u = u|_{G_2}$) is the isomorphism of the vector spaces $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$.

Theorem 5. *Domain G_1 and G_2 ($G_1 \supset G_2$) are $(1, 1, \omega)$ -equivalent iff the set $E = G_1 \setminus G_2$ is an $NC_{1,\omega}$ -set in G_1 .*

Proof. Necessity. Let the spaces $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$ be isomorphic as linear spaces for the restriction isomorphism $\theta u = u|_{G_2}$ and $u \in L^1_{1,\omega}(G_1)$. Passing to the factor-spaces $\check{L}^1_{1,\omega}(G_1)$ and $\check{L}^1_{1,\omega}(G_2)$ (see Sec. 2.3 and Proposition 3, 4) and using the Banach theorem, we obtain the boundedness of the operator θ^{-1} . Let us prove that $m_n(G_1 \setminus G_2) = 0$. Assume the converse. Then the set $G_1 \setminus G_2$ has at least one density point x_0 , which is also the Lebesgue point for weight ω .

Let us consider a sequence of open coordinate cubes $Q_m = Q\left(x_0, \frac{1}{m}\right)$ with the center x_0 and the edge of length $\frac{1}{m}$. Let us consider the function $u_m(x) = \text{dist}(x, R^n \setminus Q_m)$ on R^n . It is known (see [7]) that $|\nabla u_m| = 1$ a.e. on Q_m and $|\nabla u_m| = 0$ a.e. on $R^n \setminus Q_m$, $|u_m(x') - u_m(x'')| \leq |x' - x''|$ for any $x', x'' \in R^n$. Hence $u_m \in L^1_{1,\omega}(Q_m) \cap L^1_{1,\omega}$ for all $m \geq 1$. From the boundedness of the operator θ^{-1} we have

$$\int_{Q_m} \omega dx \leq \int_{G_1} |\nabla u_m| \omega dx \leq \|\theta^{-1}\| \int_{G_2} |\nabla u_m| \omega dx \leq \|\theta^{-1}\| \int_{G_2 \cap Q_m} \omega dx.$$

This implies

$$\frac{1}{|Q_m|} \int_{Q_m} \omega dx \leq \frac{\|\theta^{-1}\|}{|Q_m|} \int_{G_2 \cap Q_m} \omega dx.$$

For $m \rightarrow \infty$ the inequality is not valid. Consequently, $m_n(G_1 \setminus G_2) = 0$ and every function $v \in L^1_{1,\omega}(G_2)$ may be extended to function $u \in L^1_{1,\omega}(G_1)$ for which $u|_{G_2} = v$. Hence $E = G_1 \setminus G_2$ is a removable set for $L^1_{1,\omega}(G_1)$, and, by Theorem 3, E is an $NC_{1,\omega}$ -set in G_1 .

The sufficiency condition in the Theorem follows from the Theorem 3. Indeed, let $E = G_1 \setminus G_2$ be an $NC_{1,\omega}$ -set in G_1 . By Theorem 3, every function $v \in L^1_{1,\omega}(G_2)$ is extended to the function $u \in L^1_{1,\omega}(G_1)$ such that $u|_{G_2} = v$. In view of $m_n(E) = 0$, the extension is only one (the functions $u_1, u_2 \in L^1_{1,\omega}(G_1)$ with $m_n(\{x \in G_1 : u_1(x) \neq u_2(x)\}) = 0$ will be identified).

Consequently, $\theta : L^1_{1,\omega}(G_1) \rightarrow L^1_{1,\omega}(G_2)$ is the isomorphism of $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$. The Theorem is proved. $\square$

Remark 3. In the case $1 < p < \infty$ and $\omega \equiv 1$ the criterion of $(1, p)$ -equivalence of domains G_1 and G_2 ($G_1 \supset G_2$) in terms of NC_p -sets was established in [18].

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VLADIMIR ALEKSEEVICH SHLYK
 VLADIVOSTOK BRANCH OF RUSSIAN CUSTOMS ACADEMY,
 16V, STRELKOVAYA STR.,
 VLADIVOSTOK, 690034, RUSSIA
 INSTITUTE OF APPLIED MATHEMATICS, VLADIVOSTOK BRANCH OF THE RAS,
 7, RADIO STR.,
 VLADIVOSTOK, 690041, RUSSIA
 Email address: shlykva@yandex.ru