Math-Net.Ru

All Russian mathematical portal

V. V. Aseev, Some remarks on Möbius structures, *Sib. Èlektron. Mat. Izv.*, 2021, Volume 18, Issue 1, 160–167

DOI: <https://doi.org/10.33048/semi.2021.18.013>

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use

<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 54.84.234.179

April 26, 2023, 17:39:02

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 18, №1, стр. 160–167 (2021)

DOI 10.33048/semi.2021.18.013

УДК 517.54

MSC 30C65

SOME REMARKS ON MÖBIUS STRUCTURES

V.V. ASEEV

ABSTRACT. Some elementary relations between semimetrics in a Möbius structure allow us to give a correct definition of subspaces of a Möbius space and obtain the solutions in the problem of junction for Ptolemaic Möbius spaces as well as the problem of extending the Möbius structure from a subset to the whole set.

Keywords: Möbius structure, Möbius space, Ptolemaic semimetric, Ptolemaic Möbius structure, Möbius equivalence of semimetrics.

1. MÖBIUS STRUCTURES

Let us remind that a *semimetric* $\rho(.,.)$ on a set X is a real non-negative function on $X \times X$ such that $\rho(x, y) \equiv \rho(y, x)$ and $\rho(x, y) = 0$ iff $x = y$.

The concept of Möbius structure has been introduced by S.V. Buyalo (see [1]) as a useful tool in the theory of hyperbolic spaces and has the following definition:

Definition 1. Let the set X contain at least four distinct points. Two semimetrics $\rho(x, y)$ and $\sigma(x, y)$ on X are Möbius equivalent to each other iff the equality

$$\frac{\rho(x_1, x_2)\rho(x_3, x_4)}{\rho(x_1, x_3)\rho(x_2, x_4)} = \frac{\sigma(x_1, x_2)\sigma(x_3, x_4)}{\sigma(x_1, x_3)\sigma(x_2, x_4)} \quad (1.1)$$

holds for each four of distinct points x_1, x_2, x_3, x_4 in X . The family of all semimetrics on X falls into the classes of equivalence under the relation of Möbius equivalency, and each of them is called Möbius structure on X .

If $\mathbf{M}$ is a Möbius structure on X and $\rho \in \mathbf{M}$, we say that $\mathbf{M}$ is generated by ρ . And the pair $(X, \mathbf{M})$ will be called *Möbius space*.

ASEEV, V.V., SOME REMARKS ON MÖBIUS STRUCTURES.

© 2021 ASEEV V.V.

The work was carried out in the framework of the State Task to the Sobolev Institute of Mathematics (Project 0314-2019-007).

Received February, 3, 2020, published March, 5, 2021.

Proposition 1. *Let semimetrics ρ and σ on X be Möbius equivalent, and (a, b) be a pair of distinct points in X . Then the constant*

$$k = \frac{\rho(b, c)\sigma(a, c)}{\rho(a, c)\sigma(b, c)} \quad (1.2)$$

does not depend on the choice of point $c \in X \setminus \{a, b\}$, and the equality

$$\frac{\rho(x, b)}{\rho(x, a)} = k \frac{\sigma(x, b)}{\sigma(x, a)} \quad (1.3)$$

holds for every $x \in X \setminus \{a\}$.

Proof. Applying the condition of Möbius equivalence (1.1) to distinct points $\{x_1, x_2, x_3, x_4\} = \{x, b, a, c\}$ we obtain the desired equality (1.3):

$$\frac{\rho(x, b)}{\rho(x, a)} = \left(\frac{\rho(x, b)\rho(a, c)}{\rho(x, a)\rho(b, c)} \right) \frac{\rho(b, c)}{\rho(a, c)} = \left(\frac{\sigma(x, b)\sigma(a, c)}{\sigma(x, a)\sigma(b, c)} \right) \frac{\rho(b, c)}{\rho(a, c)} = \frac{\sigma(x, b)}{\sigma(x, a)} \cdot k$$

for $x \in X \setminus \{a, b, c\}$. For $x = c$ the equality (1.3) follows from (1.2).

Thus (1.3) is valid for all $x \in X \setminus \{a, b\}$, and so

$$k = \frac{\rho(x, b)\sigma(x, a)}{\rho(x, a)\sigma(x, b)}$$

does not depend on $x \in X \setminus \{a, b\}$. Besides, (1.3) is true for $x = b$, for it turns to $0 = 0$. $\square$

The following theorem gives a slightly more convenient definition of Möbius equivalence.

Theorem 1. *Two semimetrics ρ and σ on X are Möbius equivalent if and only if there exists a positive real function $f(x)$ on X such that*

$$\rho(x, y) = f(x)f(y)\sigma(x, y) \quad (1.4)$$

for all $x, y \in X$. Besides, the function f in (1.4) is unique and may be represented by formula

$$f(x) = \frac{\rho(x, a)}{\sigma(x, a)} \sqrt{k \cdot \frac{\sigma(a, b)}{\rho(a, b)}} \quad \text{for } x \neq a;$$

$$f(a) = \sqrt{\frac{1}{k} \cdot \frac{\rho(a, b)}{\sigma(a, b)}} \quad (1.5)$$

with arbitrarily chosen pair (a, b) of distinct points, where

$$k = \frac{\rho(c, b)\sigma(c, a)}{\rho(c, a)\sigma(c, b)} \quad (1.6)$$

does not depend on $c \in X \setminus \{a, b\}$.

Proof. The sufficiency. The immediate calculation shows that (1.4) involves (1.1) for any four (x_1, x_2, x_3, x_4) of distinct points in X , i.e. the Möbius equivalence of σ and ρ .

The necessity. Let a, b be distinct points in X . By Proposition 1 the number k in (1.6) does not depend on $c \in X \setminus \{a, b\}$, so

$$k = \frac{\rho(b, x)\sigma(a, x)}{\rho(a, x)\sigma(b, x)} \quad (1.7)$$

for every $x \in X \setminus \{a, b\}$. Then, taking into account (1.7) and Möbius equivalence of ρ and σ , we have for any distinct $x, y \in X \setminus \{a, b\}$ the desired equality (1.4) with function f from (1.5):

$$\begin{aligned} f(x)f(y)\sigma(x, y) &= \frac{\rho(x, a)}{\sigma(x, a)} \cdot \frac{\rho(y, a)}{\sigma(y, a)} \cdot \left(\frac{\rho(b, x)\sigma(a, x)}{\rho(a, x)\sigma(b, x)} \right) \frac{\sigma(a, b)}{\rho(a, b)} \sigma(x, y) = \\ &= \left(\frac{\sigma(a, b)\sigma(x, y)}{\sigma(x, b)\sigma(y, a)} \right) \frac{\rho(y, a)\rho(b, x)}{\rho(a, b)} = \left(\frac{\rho(a, b)\rho(x, y)}{\rho(x, b)\rho(y, a)} \right) \frac{\rho(y, a)\rho(b, x)}{\rho(a, b)} = \rho(x, y) . \end{aligned}$$

In the case where $y = b$, $x \in X \setminus \{a\}$ the equality (1.4) is also true:

$$\begin{aligned} f(x)f(b)\sigma(x, b) &= \frac{\rho(x, a)}{\sigma(x, a)} \left(k \frac{\sigma(a, b)}{\rho(a, b)} \right) \frac{\rho(b, a)}{\sigma(b, a)} \sigma(x, b) = \frac{\rho(x, a)}{\sigma(x, a)} \cdot k \cdot \sigma(x, b) = \\ &= \frac{\rho(x, a)}{\sigma(x, a)} \frac{\rho(b, x)}{\rho(a, x)} \frac{\sigma(a, x)}{\sigma(b, x)} \sigma(x, b) = \rho(x, b) . \end{aligned}$$

Finally, if $y = a$ and $x \neq a$ we also obtain (1.4):

$$f(x)f(a)\sigma(x, a) = \frac{\rho(x, a)}{\sigma(x, a)} \sqrt{\frac{k \cdot \sigma(a, b)}{\rho(a, b)}} \sqrt{\frac{\rho(a, b)}{k \cdot \sigma(a, b)}} \sigma(x, a) = \rho(x, a) .$$

In the case $x = y$ (1.4) is trivially true.

The uniqueness. Now let us assume the existence of some other positive real function $g(x)$ for which the equality $\rho(x, y) = g(x)g(y)\sigma(x, y)$ is true for all $x, y \in X$. Then

$$\rho(x, y) = f(x)f(y)\sigma(x, y) = g(x)g(y)\sigma(x, y)$$

for all $x, y \in X$. For any arbitrarily chosen pair (x_0, y_0) of distinct points in X we have $\sigma(x_0, y_0) \neq 0$ and

$$\frac{f(x_0)}{g(x_0)} = \frac{g(y_0)}{f(y_0)} = \lambda$$

Then for all $x \neq y_0$ we have $\sigma(x, y_0) \neq 0$ and

$$\frac{f(x)}{g(x)} = \frac{g(y_0)}{f(y_0)} = \lambda$$

Simultaneously, for all $x \neq x_0$ we have $\sigma(x, x_0) \neq 0$ and

$$\frac{f(x)}{g(x)} = \frac{g(x_0)}{f(x_0)} = \frac{1}{\lambda}$$

It means that $\lambda = 1$ and $f(x) = g(x)$ for all $x \in X$. Thus the uniqueness of f in (1.4) is proved. $\square$

Corollary 1. *Let the subset X_0 of the set X contain at least four distinct points, and $\mathbf{M}$ be a Möbius structure on X . Then the family of all restrictions $\rho_0 = \rho|_{X_0}$ of semimetrics $\rho \in \mathbf{M}$ is the Möbius structure on X_0 .*

Proof. Obviously, for any semimetrics $\rho, \sigma \in \mathbf{M}$ their restrictions $\rho_0 = \rho|_{X_0}$ and $\sigma_0 = \sigma|_{X_0}$ to X_0 are Möbius equivalent to each other. Let λ be arbitrarily chosen semimetric in $\mathbf{M}$ and $\lambda_0 = \lambda|_{X_0}$. Let $\mathbf{M}_0$ be Möbius structure on X_0 generated by λ_0 . We have to show that every semimetric $\mu_0 \in \mathbf{M}_0$ is the restriction to X_0 of some semimetric $\mu \in \mathbf{M}$.

Since μ_0 is Möbius equivalent to λ_0 there exists (by Theorem 1) a real positive function f_0 on X_0 for which the equality $\mu_0(x, y) = f_0(x)f_0(y)\lambda_0(x, y)$ is valid for all $x, y \in X_0$. Let $f(x)$ be an arbitrarily constructed extension of f_0 from X_0 to the

set X . (For example, we may assume $f(x) = 1$ for $x \in X \setminus X_0$.) Then the semimetric $\mu(x, y) = f(x)f(y)\lambda(x, y)$ belongs to $\mathbf{M}$ and $\mu_0 = \mu|_{X_0}$. Thus the Möbius structure $\mathbf{M}_0$ on X_0 coincides with the family of all restrictions $\rho|_{X_0}$ of semimetrics from $\mathbf{M}$. $\square$

Corollary 1 provides the correctness of the following definition

Definition 2. *Let the subset X_0 of the set X have at least four distinct points and $\mathbf{M}$ be a Möbius structure on X . Then there exists the unique Möbius structure $\mathbf{M}_0$ on X_0 consisting of all restrictions to X_0 of semimetrics from $\mathbf{M}$. This Möbius structure is said to be the restriction to X_0 of the Möbius structure $\mathbf{M}$, and we shall use the notation $\mathbf{M}_0 = \mathbf{M}|_{X_0}$. In that case the Möbius space $(X_0, \mathbf{M}_0)$ is said to be the subspace of the Möbius space $(X, \mathbf{M})$.*

2. NORMALIZATION OF SEMIMETRICS

We shall consider a special subfamily of semimetrics in a Möbius structure.

Definition 3. *Let $(X, \mathbf{M})$ be a Möbius space. Given a pair of distinct points (a, b) in X , the semimetric $\mu \in \mathbf{M}$ is called (a, b) -normalized iff*

$$\mu(x, a) + \mu(x, b) = 1 \quad (2.1)$$

for all $x \in X$. In this case, it's clear that $\mu(a, b) = 1$.

The (a, b) -normalization of any semimetric $\rho \in \mathbf{M}$ is realized by formula

$$\rho_0(x, y) := \frac{\rho(x, y)\rho(a, b)}{(\rho(x, a) + \rho(x, b))(\rho(y, a) + \rho(y, b))} \quad (2.2)$$

Since $\rho_0(x, y) = g(x)g(y)\rho(x, y)$ with $g(x) = \sqrt{\rho(a, b)}/(\rho(x, a) + \rho(x, b))$, then by Theorem 1 the semimetric ρ_0 is Möbius equivalent to ρ , i.e. (2.2) defines the operator of (a, b) -normalization $\rho \mapsto \rho_0$ on $\mathbf{M}$. For a (a, b) -normalized semimetric ρ_0 we have $\rho_0(a, b) = 1$.

Recall that a semimetric $\mu(x, y)$ on a set X is called to be *Ptolemaic* if the Ptolemy's Inequality

$$\mu(x_1, x_2)\mu(x_3, x_4) \leq \mu(x_1, x_3)\mu(x_2, x_4) + \mu(x_1, x_4)\mu(x_2, x_3)$$

is valid for all $x_1, x_2, x_3, x_4 \in X$. It is easy to check up that if any one of Möbius equivalent semimetrics is Ptolemaic then each of them is Ptolemaic as well.

A Möbius structure $\mathbf{M}$ on X is called *Ptolemaic* if any one (and then every one) of $\rho \in \mathbf{M}$ is a Ptolemaic semimetric. In that case $(X, \mathbf{M})$ is said to be Ptolemaic Möbius space.

It was proved in [2] that any (a, b) -normalized Ptolemaic semimetric μ on X satisfies the Triangle Inequality, i.e. is a metric on X . Besides, $\mu(x, y) \leq 1$ for all $x, y \in X$. Indeed, from (2.1) we have $\mu(a, b) = 1$, and $\mu(z, a) \leq 1$, $\mu(z, b) \leq 1$ for all $z \in X$. Then Ptolemy's Inequality gives: $\rho_0(x, y) = \rho_0(x, y)\rho_0(a, b) \leq \rho_0(x, a)\rho_0(y, b) + \rho_0(x, b)\rho_0(y, a) \leq \rho_0(x, a) + \rho_0(x, b) = 1$.

Now we can reformulate the Theorem 1 for the case of (a, b) -normalized semimetrics.

Theorem 2. *Let a pair of distinct points (a, b) be chosen in a Möbius space $(X, \mathbf{M})$. Given (a, b) -normalized semimetrics ρ_0 and σ_0 in $\mathbf{M}$, the equality*

$$\rho_0(x, y) = f_0(x)f_0(y)\sigma_0(x, y) \quad (2.3)$$

holds for all $x, y \in X$, where

$$f_0(x) = \frac{\sqrt{k}}{\sigma_0(x, a) + k \cdot \sigma_0(x, b)} \quad (2.4)$$

and the constant

$$k = \frac{\rho_0(b, c)\sigma_0(a, c)}{\rho_0(a, c)\sigma_0(b, c)} \quad (2.5)$$

does not depend on the choice of a point $c \in X \setminus \{a, b\}$.

Proof. By Theorem 1 there exists a unique real positive function $f_0(x)$ on X such that (2.3) holds for all $x, y \in X$, and the function f_0 is

$$f_0(x) = \frac{\rho_0(x, a)}{\sigma_0(x, a)}\sqrt{k} \text{ for } x \neq a; \quad f_0(a) = \sqrt{1/k} \quad (2.6)$$

with a constant k defined by (2.5). By Proposition 1 the equality

$$\frac{\rho_0(x, b)}{\rho_0(x, a)} = k \frac{\sigma_0(x, b)}{\sigma_0(x, a)}$$

is valid on $X \setminus \{a\}$. It means that for all $x \neq a$

$$k \cdot \frac{\sigma_0(x, b)}{\sigma_0(x, a)} = \frac{1 - \rho_0(x, a)}{\rho_0(x, a)} = \frac{1}{\rho_0(x, a)} - 1,$$

and therefore

$$\frac{1}{\rho_0(x, a)} = 1 + k \frac{\sigma_0(x, b)}{\sigma_0(x, a)} = \frac{\sigma_0(x, a) + k \cdot \sigma_0(x, b)}{\sigma_0(x, a)}.$$

The substitution

$$\rho_0(x, a) = \frac{\sigma_0(x, a)}{\sigma_0(x, a) + k \cdot \sigma_0(x, b)}$$

into (2.6) gives the required formula (2.4) for all $x \neq a$. In the case where $x = a$ we obtain the true equality $f_0(a) = \sqrt{k}/k$ as well. $\square$

Corollary 2. *Given a pair of distinct points (a, b) in a Möbius space $(X, \mathbf{M})$, if (a, b) -normalized semimetrics $\rho_0, \sigma_0 \in \mathbf{M}$ coincide to each other on a triple $\{a, b, c\}$, $c \in X \setminus \{a, b\}$, then $\rho_0(x, y) \equiv \sigma_0(x, y)$ on X .*

Proof. In that case $k = 1$ in formula (2.4), and because of the semimetrics ρ_0 and σ_0 being (a, b) -normalized, the function $f_0(x)$ in (2.4) is

$$f_0(x) = \frac{1}{\sigma_0(x, a) + \sigma_0(x, b)} = 1,$$

i.e. $\rho_0(x, y) = \sigma_0(x, y)$ for all $x, y \in X$. $\square$

3. DISJUNCTED UNION OF MÖBIUS SPACES

Let J be a set of indices, and $\{(X_j, \mathbf{M}_j)\}_{j \in J}$ be a collection of Möbius spaces. Considering the disjoint union $X := \sqcup_{j \in J} X_j$ we are to define a Möbius structure $\mathbf{M}$ on X in order to call $(X, \mathbf{M})$ the *disjoint union* of a given collection of Möbius spaces. This problem has two following settings.

Problem 1. To find a Möbius space $(X, \mathbf{M})$ such that $\mathbf{M}|_{X_j} = \mathbf{M}_j$ for every $j \in J$.

Problem 2. To find a Möbius space $(X, \mathbf{M})$ such that for any collection $\{\rho_j\}_{j \in J}$ of semimetrics $\rho_j \in \mathbf{M}_j$ there exists a semimetric $\rho \in \mathbf{M}$ such that $\rho|_{X_j} = \rho_j$ for every $j \in J$.

Besides, in the case where all Möbius structures $\mathbf{M}_j$ ($j \in J$) are Ptolemaic, the Möbius space $(X, \mathbf{M})$ must be Ptolemaic as well.

Proposition 2. *The Problems 1 and 2 are equivalent to each other. That is, every solution $(X, \mathbf{M})$ for any one of these Problems will be the solution of another one.*

Proof. Let $(X, \mathbf{M})$ be a solution to Problem 2. For an arbitrarily chosen collection of semimetrics $\{\rho_j \in \mathbf{M}_j\}_{j \in J}$ there exists $\rho \in \mathbf{M}$ such that $\rho_j = \rho|_{X_j}$ ($j \in J$). Since both Möbius structures $\mathbf{M}_j$ and $\mathbf{M}|_{X_j}$ on X_j are generated by semimetric $\rho_j = \rho|_{X_j}$, we have $\mathbf{M}_j = \mathbf{M}|_{X_j}$. Thus $(X, \mathbf{M})$ is a solution to Problem 1.

Now let $(X, \mathbf{M})$ be a solution to Problem 1 and $\{\rho_j\}$ be a given collection of semimetrics $\rho_j \in \mathbf{M}_j$ ($j \in J$). For an arbitrarily chosen semimetric $\sigma \in \mathbf{M}$ we have $\sigma_j := \sigma|_{X_j} \in \mathbf{M}|_{X_j} = \mathbf{M}_j$ ($j \in J$). By Theorem 1 there exists a real positive function f_j on X_j such that $\rho_j(x, y) = f_j(x)f_j(y)\sigma(x, y)$ for all $x, y \in X_j$ ($j \in J$). Consider the real positive function f which is defined on X by formula $f(x) = f_j(x)$ as $x \in X_j$ ($j \in J$). The semimetric $\rho(x, y) := f(x)f(y)\sigma(x, y)$ is Möbius equivalent to semimetric $\sigma \in \mathbf{M}$. Thus we obtain $\rho \in \mathbf{M}$ such that $\rho|_{X_i} = \rho_j$ for all $j \in J$. It means that $(X, \mathbf{M})$ is a solution to Problem 2. $\square$

Proposition 3. *In the class of Ptolemaic Möbius structures, there exists a solution (not unique) to both Problems 1 and 2.*

Proof. Choose in each X_j ($j \in J$) an arbitrary pair (a_j, b_j) of distinct points and a (a_j, b_j) -normalized semimetric $\rho_j \in \mathbf{M}_j$. Let the semimetric ρ on $X = \sqcup_{j \in J} X_j$ be defined by formula: $\rho(x, y) = \rho_j(x, y)$ as $x, y \in X_j$; $\rho(x, y) = 1$ as $x \in X_j, y \in X_i, i \neq j$. Since each of Möbius spaces $(X_j, \mathbf{M}_j)$ is Ptolemaic, it suffices to verify the Ptolemaic inequality

$$\alpha(a, b; c, d) := \frac{\rho(a, b)\rho(c, d)}{\rho(a, c)\rho(b, d) + \rho(a, d)\rho(b, c)} \leq 1 \quad (3.1)$$

in the case where the points x_1, x_2, x_3, x_4 are distinct and do not belong to the same X_j .

Case 1. Let one of points a, b, c, d is in X_i while three other points are in X_j , $i \neq j$. Since

$$\alpha(a, b; c, d) = \alpha(b, a; d, c) = \alpha(c, d; a, b) = \alpha(d, c; b, a) \quad (3.2)$$

we may assume that $a \in X_i$ while $b, c, d \in X_j$. Since any normalised Ptolemaic semimetric is metric, we obtain the desired inequality (3.1):

$$\alpha(a, b; c, d) = \frac{1 \cdot \rho_j(c, d)}{1 \cdot \rho_j(b, d) + 1 \cdot \rho_j(b, c)} \leq 1$$

Case 2.1. Let $a, b \in X_i$ while $c \in X_j, d \in X_k$ where $i \neq j, i \neq k$. Since any normalized Ptolemaic semimetric is ≤ 1 , we have

$$\alpha(a, b; c, d) = \frac{\rho_i(a, b)\rho(c, d)}{1 + 1} < 1.$$

Case 2.2. Let $a, c \in X_i$ while $b \in X_j$, $d \in X_k$ where $i \neq j$, $i \neq k$. Then

$$\alpha(a, b; c, d) = \frac{1 \cdot 1}{\rho_i(a, c)\rho(b, d) + 1 \cdot 1} < 1.$$

Case 2.3. Let $a, d \in X_i$ while $b \in X_j$, $c \in X_k$ where $i \neq j$, $i \neq k$. Then

$$\alpha(a, b; c, d) = \frac{1 \cdot 1}{1 \cdot 1 + \rho_i(a, d)\rho(b, c)} < 1.$$

Case 3. If points a, b, c, d are in distinct X_i, X_j, X_k, X_s then $\alpha(a, b, c, d) = 1/2 < 1$. Thus the semimetric ρ on X is Ptolemaic. Consider the Möbius structure $\mathbf{M}$ on X generated by ρ . Since the restriction $\mathbf{M}|_{X_j}$ is a Möbius structure on X_j (by Corollary 1) and $\rho_j \in \mathbf{M}|_{X_j} \cap \mathbf{M}_j$, we have the equality $\mathbf{M}_j = \mathbf{M}|_{X_j}$. So $(X, \mathbf{M})$ is a desired solution to Problem 1. According to Proposition 2 it is also the solution to Problem 2. $\square$

4. THE EXTENSION OF PTOLEMAIC MÖBIUS SPACE

Next consider the

Problem 3. Given a Ptolemaic Möbius space $(X, \mathbf{M})$ and a point $q \notin X$, to find a Ptolemaic Möbius structure $\mathbf{M}^*$ on $X^* = X \cup \{q\}$ such that $\mathbf{M} = \mathbf{M}^*|_X$.

An arbitrary pair (a, b) of distinct points in X being chosen, consider (a, b) -normalized metric $\sigma_0 \in \mathbf{M}$ and define its extension to X^* by formula: $\sigma_0^*(x, y) = \sigma_0(x, y)$ if $x, y \in X$ and $\sigma_0^*(x, q) = 1$ for all $x \in X$. Since $\sigma_0^*|_X = \sigma_0$ is Ptolemaic, we have to check up the Ptolemy's inequality only for a tetrad (x_1, x_2, x_3, q) . In that case we have

$$\frac{\sigma_0^*(x_1, x_2)\sigma_0^*(x_3, q)}{\sigma_0^*(x_1, x_3)\sigma_0^*(x_2, q) + \sigma_0^*(x_1, q)\sigma_0^*(x_2, x_3)} = \frac{\sigma_0(x_1, x_2)}{\sigma_0(x_1, x_3) + \sigma_0(x_2, x_3)} \leq 1,$$

since σ_0 is a metric on X . So the semimetric σ_0^* on X^* is Ptolemaic. Then σ_0^* generate a Ptolemaic Möbius structure $\mathbf{M}^*$ on X^* , which gives a solution to Problem 3.

Now we can obtain the extension ρ^* to X^* for any semimetric $\rho \in \mathbf{M}$. Since

$$\rho^*(x, y) = \begin{cases} \rho(x, y) & \text{as } x, y \in X \\ \varphi(x) & \text{as } x \in X, y = q \end{cases}$$

we are to find a positive function $\varphi(x)$ on X which provides that $\rho^* \in \mathbf{M}^*$.

For $x \neq a, b$ the equality

$$\frac{\rho^*(x, q)\rho^*(a, b)}{\rho^*(x, a)\rho^*(b, q)} = \frac{\sigma_0^*(x, q)\sigma_0^*(a, b)}{\sigma_0^*(x, a)\sigma_0^*(b, q)} = \frac{1}{\sigma_0(x, a)}$$

gives

$$\varphi(x) = \varphi(b) \frac{\rho(x, a)}{\sigma_0(x, a)\rho(a, b)}.$$

The constant

$$k = \frac{\rho(c, b)\sigma_0(c, a)}{\rho(c, a)\sigma_0(c, b)}$$

does not depend on $c \in X \setminus \{a, b\}$. Then the equality

$$\frac{\rho^*(a, q)\rho^*(c, b)}{\rho^*(a, c)\rho^*(b, q)} = \frac{\sigma_0^*(a, q)\sigma_0^*(c, b)}{\sigma_0^*(a, c)\sigma_0^*(b, q)} = \frac{\sigma_0(c, b)}{\sigma_0(c, a)}$$

gives

$$\varphi(a) = \varphi(b) \frac{\rho(a, c) \sigma_0(b, c)}{\rho(b, c) \sigma_0(a, c)} = \frac{\varphi(b)}{k}.$$

Thus we obtain the desired extension of ρ from X to X^* :

$$\rho^*(x, q) = \varphi(x) = \begin{cases} C & \text{as } x = b \\ C/k & \text{as } x = a \\ C \cdot \frac{\rho(x, a)}{\rho(a, b) \sigma_0(x, a)} & \text{as } x \in X \setminus \{a, b\} \end{cases}$$

where $C > 0$ may be arbitrarily chosen.

PS: The author is greatly thankful to Referee for his(her) important amendments and remarks on the text of this article. In response to these remarks the text of the paper has been considerably improved.

REFERENCES

- [1] S.V. Buyalo, *Möbius and submöbius structures*, St. Petersburg Math. J., **28**:5 (2017), 555-568. MR3637585
- [2] V.V.Aseev, *Generalized angles in Ptolemaic Möbius structures*, Siberian Math. J., **59**:2 (2018), 241-256. MR3869981

VLADISLAV VASILYEVICH ASEEV
 SOBOLEV INSTITUTE OF MATHEMATICS,
 4, KOPTYUGA AVE.,
 NOVOSIBIRSK, 630090, RUSSIA
Email address: `btp@math.nsc.ru`