Math-Net.Ru

All Russian mathematical portal

F. A. Shamoyan, V. A. Bednazh, V. A. Kustova, Blaschke product in Privalov classes,
Sib. Èlektron. Mat. Izv., 2021, Volume 18, Issue 1, 168–175

DOI: <https://doi.org/10.33048/semi.2021.18.014>

Use of the all-Russian mathematical portal Math-Net.Ru implies that you have read and agreed to these terms of use

<http://www.mathnet.ru/eng/agreement>

Download details:

IP: 34.229.45.116

April 26, 2023, 07:13:59

SIBERIAN ELECTRONIC
MATHEMATICAL REPORTS

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 18, №1, смп. 168–175 (2021)

УДК 517.53

DOI 10.33048/semi.2021.18.014

MSC 30E05, 30H99

BLASCHKE PRODUCT IN PRIVALOV CLASSES

F.A. SHAMOYAN, V.A. BEDNAZH, V.A. KUSTOVA

ABSTRACT. In terms of zeros, an exact characterization is given of those Blaschke products for which the n th-order derivative belongs to the class of I.I. Privalov in a single circle.

Keywords: Blaschke product, Privalov class, derivative.

1. INTRODUCTION

Let $\mathbb{C}$ be a complex plane and consider a unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ on $\mathbb{C}$, by $H(D)$ denote the set of all functions that are analytic in D .

Given $0 < q < +\infty$, define a Privalov class as follows:

$$\Pi_q = \{f \in H(D) : \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\varphi})|)^p d\varphi < +\infty\}.$$

The case of $q > 1$ for Π_q was studied by I.I. Privalov in his monograph [1].

It is clear that the class Π_1 is in fact the class N of functions of bounded type on the unit disk, i.e. the Nevalinna class for D (see [2]).

In complex analysis, a well-known Bloch-Nevalinna conjecture, which was proposed by R. Nevalinna in 1929 in [2], states that the class N is invariant with respect to a differential operator. The first example, that confirmed that said conjecture was wrong was presented by Frostman, who constructed the Blaschke product $B(z, z_k)$, such that its derivative does not belong to N , while the closure of its zero-set coincides with the boundary of a circle.

Therefore, a natural question arose: given a Blaschke product $B(z, z_k)$, does its n -th order derivative belong to N ? Interestingly, by considering the problem in terms of limit points of a zero-set on the unit circle T , it is possible to obtain a

SHAMOYAN, F.A., BEDNAZH, V.A., KUSTOVA, V.A., BLASCHKE PRODUCT IN PRIVALOV CLASSES.

© 2021 SHAMOYAN F.A., BEDNAZH V.A., KUSTOVA V.A.

Received October, 20, 2020, published March, 5, 2021.

sufficiently complete characterisation of those products, whose derivative belongs not only to N but also to Π_q for all $0 < q < +\infty$,

The following statement holds.

Theorem 1. *Let $\{z_k\}_1^\infty$ be a sequence of points, inside the unit disk D , that satisfies the Blaschke condition, i.e.*

$$\sum_1^\infty (1 - |z_k|) < +\infty, \quad (1)$$

and let E be the set of limit points of the sequence $\{z_k\}_1^\infty$ on the unit circle. By $\{e^{i\alpha_k}e^{i\beta_k}\}_{k=1}^\infty$ denote a sequence of supplementary arcs of a set E on T . Then, if

$$\sum_{k=1}^\infty (\beta_k - \alpha_k) \ln^q \frac{1}{\beta_k - \alpha_k} < +\infty,$$

where $z_n = r_n e^{i\theta_n}$, $n = 1, 2, \dots$ and the amount of points in every interval (α_k, β_k) , $k = 1, 2, \dots$, is uniformly bounded, then for any $n \in N$ the n -th ordered derivative of the Blaschke product belongs to the Privalov class Π_q .

Conversely, there exists a sequence $\{z_k\}_1^\infty \subset D$, satisfying the Blaschke condition, such that if

$$\sum_{k=1}^\infty (\beta_k - \alpha_k) \ln^q \frac{1}{\beta_k - \alpha_k} = +\infty,$$

then $B^{(n)}(z, z_k) \notin \Pi_q$, $n \in N$, $0 < q < +\infty$.

2. PROOF OF SUPPORTING STATEMENTS

Prior to the proof of the theorem, we will prove some supporting statements. By $\rho(F, E)$ denote the distance between sets F and E .

Lemma 1. *Let $Z = \{r_n e^{i\theta_n}\}$ be a sequence of points inside the unit disk D , that satisfies the conditions of Theorem 1, and let E be the set of limit points on the unit circle T . Then the following statements are equivalent:*

$$1) \int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, E)} d\theta < +\infty, \quad (2)$$

$$2) \int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta < +\infty. \quad (3)$$

Proof. Note that $\rho(e^{i\theta}, Z) \leq \rho(e^{i\theta}, E)$, and since E is a set of limit points on the unit circle, statement 1) immediately follows from 2).

We will now show that 2) follows from 1). For this note that $\frac{2\pi}{\rho(z, E)} \geq 1$ for all $z \in D$, moreover, it follows from 1) that the linear measure of E equals zero, also $\lim_{r \rightarrow 1-0} \rho(re^{i\theta}, E) = \rho(e^{i\theta}, E)$ (see [4]) everywhere on $(-\pi, \pi)$.

We will show that, if condition (1) holds, then

$$I = \int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta < +\infty.$$

Let $z_n = r_n e^{i\theta_n}$, $n = 1, 2, \dots$. It is clear that

$$I = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta.$$

Moreover,

$$\rho(e^{i\theta}, Z)^2 = \inf_{n \geq 1} |e^{i\theta} - r_n e^{i\theta_n}|^2 = \inf_{n \geq 1} \left((1 - r_n)^2 + 4r_n \sin^2 \frac{\theta - \theta_n}{2} \right).$$

It can be seen that

$$|z_n - e^{i\theta}|^2 \geq c((1 - r_n)^2 + (\theta - \theta_n)^2) > c|\theta - \theta_n|^2. \quad (4)$$

Therefore,

$$\rho(e^{i\theta}, Z) \geq c \inf_{n \geq 1} |\theta_n - \theta|.$$

From this we derive that

$$\int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta \leq \int_{\alpha_k}^{\beta_k} \sup_{n \geq 1} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta. \quad (5)$$

Now we fix k and get the boundary on

$$I_k = \int_{\alpha_k}^{\beta_k} \sup_{n \geq 1} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta.$$

Let $\gamma_k = \frac{\alpha_k + \beta_k}{2}$ be the center of an interval (α_k, β_k) ,

$$\Delta_k = \{\theta : |\theta - \gamma_k| < (\beta_k - \alpha_k)\}.$$

If $\theta_n \notin \Delta_k$, then, given $\theta \in [\alpha_k, \beta_k]$ $|\theta_n - \theta| \geq \frac{\beta_k - \alpha_k}{2}$, we have

$$\inf_{\theta_n \notin \Delta_k} |\theta - \theta_n| \geq \frac{(\beta_k - \alpha_k)}{2}.$$

Therefore,

$$\int_{\alpha_k}^{\beta_k} \sup_{n \geq 1, \theta_n \notin \Delta_k} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta \leq (\beta_k - \alpha_k) \ln^q \frac{4\pi}{\beta_k - \alpha_k}.$$

Now we will obtain bounds on the same integral, when $\theta_n \in \Delta_k$. Then the following cases are possible: $\theta_n \notin [\alpha_k, \beta_k]$ and $\theta_n \in [\alpha_k, \beta_k]$.

Given the conditions of this Lemma,

$$\begin{aligned} I_k &= \int_{\alpha_k}^{\beta_k} \sup_{\theta_n \in \Delta_k} \ln^q \frac{2\pi}{|\theta_n - \theta|} d\theta = \\ &= \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{|\theta_{n_0} - \theta|} d\theta \end{aligned}$$

for some n_0 (it is assumed that θ_{n_0} is positioned to the left of the interval (α_k, β_k)). Therefore,

$$I_k = \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{|\theta_{n_0} - \theta|} d(\theta - \theta_{n_0}).$$

Using integration by parts, we obtain

$$\begin{aligned} I_k &= (\theta - \theta_{n_0}) \ln^q \frac{2\pi}{(\theta - \theta_{n_0})} \Big|_{\alpha_k}^{\beta_k} + \int_{\alpha_k}^{\beta_k} (\theta - \theta_{n_0}) q \ln^{q-1} \frac{2\pi}{(\theta - \theta_{n_0})} \frac{2\pi}{(\theta - \theta_{n_0})} d\theta = \\ &= (\beta_k - \theta_{n_0}) \ln^q \frac{2\pi}{\beta_k - \theta_{n_0}} - (\alpha_k - \theta_{n_0}) \ln^q \frac{2\pi}{\alpha_k - \theta_{n_0}} + q \int_{\alpha_k}^{\beta_k} \ln^{q-1} \frac{2\pi}{\theta - \theta_{n_0}} d\theta. \end{aligned}$$

From this we get

$$\int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\theta - \theta_n} \left(1 - \frac{2\pi q}{\ln \frac{2\pi}{\theta - \theta_n}} \right) d\theta \leq c(\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k}.$$

Since $\beta_k - \alpha_k \rightarrow 0$ as $k \rightarrow +\infty$, one can use the last obtained bound to show that

$$I_k \leq c(\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k}. \quad (6)$$

The case $\theta_{n_0} \in [\alpha_k, \beta_k]$ is checked in a similar way.

Finally, from (5) and (6) we derive

$$\int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta \leq c \int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, E)} d\theta \leq c \sum_{k=1}^{\infty} (\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k},$$

where c is some positive constant.

The lemma is proved. $\square$

One may be familiar with the statement of the following lemma, but we will give it for the sake of completeness.

Lemma 2. *Let $Z = \{z_k\}_1^\infty$ be the sequence of points inside the unit disk D , that satisfies the Blaschke condition. Also let $B(z, z_k) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \bar{z}_k z} \frac{\bar{z}_k}{|z_k|}$ be the Blaschke product with Z as its zero-set. Then there exists a positive number C , depending only on n , such that*

$$|B^{(n)}(z, z_k)| \leq \frac{C(n)}{\rho^{2n}(z, Z)}, \quad z \in D.$$

Proof. We will prove this lemma by induction.

First consider $n = 1$.

We have that

$$\frac{B^{(1)}(z, z_k)}{B(z, z_k)} = \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{(z - z_k)(1 - \bar{z}_k z)}, \quad z \in D.$$

Let $b_k(z) = \frac{z_k - z}{1 - \bar{z}_k z} \frac{\bar{z}_k}{|z_k|}$, $B_k = B/b_k$. Therefore,

$$B^{(1)}(z) = B(z) \cdot \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{(z - z_k)(1 - \bar{z}_k z)}.$$

Since $B(z, z_k) = B_k(z, z_k) \frac{z - z_k}{1 - \bar{z}_k z}$, we can use the last equality to derive that

$$B^{(1)}(z, z_k) = \sum_{k=1}^{\infty} \frac{B_k(z, z_k)(z - |z_k|^2)}{(1 - \bar{z}_k z)^2}, \quad (7)$$

i.e. $|B^{(1)}(z, z_k)| \leq \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)}{|1 - \bar{z}_k z|^2}$. But since $|1 - \bar{z}_k z| \geq |z_k - z|$, $z, z_k \in D$, we can use the last bound to show that

$$|B^{(1)}(e^{i\theta}, z_k)| \leq \frac{\sum_{k=1}^{\infty} (1 - |z_k|)}{\rho(e^{i\theta}, Z)^2}.$$

Now, by considering (7) we get that

$$B^{(m)}(z, z_k) = \sum_{k=1}^{\infty} (1 - |z_k|^2) (B_k(z, z_k) \times \frac{1}{(1 - \bar{z}_k z)^2})^{(m-1)}.$$

By using the Leibniz formula we obtain the following:

$$\begin{aligned} B^{(m)}(z, z_k) &= \sum_{k=1}^{\infty} (1 + |z_k|) \sum_{s=0}^{m-1} C_{m-1}^s \times B_k^{(s)}(z) \cdot \frac{\bar{z}_k^{m-1-s} (1 + m - s)!}{(1 - \bar{z}_k z)(m - 1 - s)} = \\ &= \sum_{k=1}^{\infty} (1 - |z_k|) \sum_{s=0}^{m-1} C_{m-1}^s B_k^{(s)}(z) \cdot z_k^{m-1-s} \times \frac{(1 + m - s)!}{(1 - \bar{z}_k z)^{1+m-s}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |B^{(m)}(z, z_k)| &\leq A(m) \sum_{k=1}^{\infty} \left(\sum_{s=0}^{m-1} \frac{|B_k^{(s)}(z_k)|}{|z - z_k|^m} \right) \times (1 - |z_k|) \leq \\ &\leq A_1(m) \sum_{s=0}^{m-1} \frac{1}{\rho(z, Z)^{2s}} \cdot \frac{1}{\rho(z, Z)^{m+1-s}}. \end{aligned}$$

From this, one can immediately derive the bound

$$|B^{(m)}(z, z_k)| \leq \frac{C(m)}{\rho(z, Z)^{2m}}, \quad z \in D, \quad m \in N.$$

The lemma is proved. $\square$

Note. By using the proof of Lemma 2, one can also obtain the following bound:

$$|B^{(n)}(re^{i\varphi}, z_k)| \leq \frac{C(n)}{\rho(e^{i\varphi}, Z)^{2n}}, \quad r \in [0, 1], \quad (8).$$

3. PROOF OF THE MAIN RESULT

Proof of theorem. The first part of the theorem follows directly from Lemma 1, Lemma 2 and (8).

We will now prove the second part, i.e. if

$$\sum_{k=1}^{+\infty} (\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k} = +\infty,$$

then there exists a sequence $\{z_k\}_1^\infty \subset D$, such that

$$\sum_{k=1}^{+\infty} (1 - |z_k|) < +\infty,$$

at the same time $B^{(n)}(z, z_k) \notin \Pi_q$ for all $0 < q < +\infty$.

We will first prove this statement for $n = 1$. From (7), we can derive that

$$B^{(1)}(e^{i\theta}, z_k) = B(e^{i\theta}, z_k) e^{-i\theta} \sum_{k=1}^{+\infty} \frac{(1 - |z_k|^2)}{|e^{i\theta} - z_k|^2}.$$

Hence

$$|B^{(1)}(e^{i\theta}, z_k)| = \sum_{k=1}^{+\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2}$$

almost everywhere on the segment $[-\pi, \pi]$.

Now we construct the sequence $\{z_k\}_1^\infty$. Set $z_n = (1 - \varepsilon_n)e^{i\alpha_n}$, $\tilde{z}_n = (1 - \varepsilon_n)e^{i\beta_n}$, $\varepsilon_n = \beta_n - \alpha_n$, $n = 1, 2, \dots$. It is clear that $1 - |\tilde{z}_n| = 1 - |z_n| = \varepsilon_n$, hence the sequences $\{z_n\}_1^\infty$ and $\{\tilde{z}_n\}_1^\infty$ satisfy the Blaschke condition. Let $w_{2k-1} = z_k$, $w_{2k} = \tilde{z}_k$. It can be seen that the sequence $\{w_k\}_{k=1}^{+\infty}$ also satisfies the Blaschke condition.

Let $B(z, w_k) = \prod_{k=1}^\infty \frac{w_k - z}{1 - \bar{w}_k z} \frac{\bar{w}_k}{|w_k|}$, $z \in D$. Then, by considering (7), we obtain that

$$B^{(1)}(e^{i\theta}, w_k) = e^{-i\theta} B(e^{i\theta}, w_k) \sum_{k=1}^{+\infty} \frac{1 - |w_k|^2}{|e^{i\theta} - w_k|^2},$$

for almost all $\theta \in [-\pi, \pi]$. Therefore,

$$|B^{(1)}(e^{i\theta}, w_k)| = \sum_{k=1}^{+\infty} \frac{1 - |w_k|^2}{|e^{i\theta} - w_k|^2}.$$

Considering the definition of the sequence $\{w_k\}_1^\infty$, we can use this equality to derive that

$$|B^{(1)}(e^{i\theta}, w_k)| \geq \sum_{k=1}^{+\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2}$$

$$\theta \in [-\pi, \pi]$$

Therefore,

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q d\theta \geq \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{(1 - |z_k|^2)}{|e^{i\theta} - z_k|^2} \right) \right)^q d\theta, \quad (9)$$

Now note that $z_k = (1 - \varepsilon_k)e^{i\alpha_k}$,

$$|e^{i\theta} - z_k|^2 = (1 - r_k)^2 + 4r_k \sin^2 \frac{\theta - \alpha_k}{2} = \varepsilon_k^2 + 4(1 - \varepsilon_k) \sin^2 \frac{\theta - \alpha_k}{2}.$$

By the arguments similar to those used in the proof of Lemma 1, we get that $|e^{i\theta} - z_k|^2 \geq \delta(\varepsilon_k^2 + (\theta - \alpha_k)^2)$ for some positive $\delta > 0$.

Now, turning back to the inequality (9), we obtain

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q d\theta \geq \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta + c. \quad (10)$$

We will now present the lower bound on the last integral. Set

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta = \\ &= \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta \end{aligned} \quad (11).$$

Let

$$I_n = \int_{\alpha_n}^{\beta_n} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta.$$

Set $\gamma_n = \frac{\beta_n + \alpha_n}{2}$, $n = 1, 2, \dots$. Then

$$I_n \geq \int_{\alpha_n}^{\gamma_n} \left(\ln^+ \frac{\varepsilon_n}{(\theta - \alpha_n)^2 + \varepsilon_n^2} \right)^q d\theta \geq \left(\ln^+ \frac{\varepsilon_n}{\frac{\varepsilon_n^2}{4} + \varepsilon_n^2} \right)^q \cdot \frac{\varepsilon_n}{4},$$

therefore,

$$I_n \geq \varepsilon_n \left(\ln \frac{4}{5\varepsilon_n} \right)^q, \quad n = 1, 2, \dots$$

Now, considering (11) and the final bound, we get that

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q d\theta + c_1 = +\infty.$$

The theorem is proved.

REFERENCES

- [1] I.I. Privalov, *Boundary properties of single-valued analytic functions*, Moscow State University, Moscow, 1941.
- [2] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929. JFM 55.0773.03
- [3] O. Frostman, *Sur les produits de Blaschke*, Fysiog. Sällsk. Lund Förh., **12**:15 (1942), 169–182. Zbl 0061.15112
- [4] E.M. Stein, *Singular integrals and differentiability properties of functions*, Mir, Moscow, 1973. Zbl 0281.44003

FAIZO AGITOVICH SHAMOYAN
SARATOV STATE UNIVERSITY,
83, ASTRAKHANSKAYA STR.,
SARATOV, 410012, RUSSIA
Email address: shamoyanfa@yandex.ru

VERA ARKADIEVNA BEDNAZH
BRYANSK STATE UNIVERSITY,
14, BEZHITSKAYA STR.,
BRYANSK, 410012, RUSSIA
Email address: verabednazh@mail.ru

VICTORIYA ANDREEVNA KUSTOVA
SARATOV STATE UNIVERSITY,
83, ASTRAKHANSKAYA STR.,
SARATOV, 410012, RUSSIA
Email address: victoriya.kustova@yandex.ru