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ON ALMOST OMEGA-CATEGORICITY OF WEAKLY O-MINIMAL THEORIES

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ABSTRACT. Here we prove that weakly o-minimal theories of finite convexity rank having less than 2^ω countable models are almost ω -categorical.

Keywords: almost ω -categoricity, weak o-minimality, convexity rank, binary theory.

1. PRELIMINARIES

Let L be a countable first order language. Throughout this paper we consider L -structures and suppose that L contains a binary relation symbol $<$ which is interpreted as a linear order in these structures. An *open interval* in such a structure M is a parametrically definable subset of M of the form $I = \{c \in M : M \models a < c < b\}$ for some $a, b \in M \cup \{-\infty, \infty\}$ with $a < b$. Similarly, we may define *closed*, *half open-half closed*, etc., *intervals* in M . We can also represent an arbitrary point $a \in M$ as a closed interval $[a, a]$. By an *interval* in M we shall mean, ambiguously, any of the above types of intervals in M . A subset A of a linearly ordered structure M is *convex* if for all $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$.

This paper concerns the notion of *weak o-minimality* which was initially deeply studied by D. Macpherson, D. Marker and C. Steinhorn in [1]. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \dots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M . We recall that such a structure M is said to be *o-minimal* if any definable (with parameters) subset of M is a union of finitely many intervals and points in M .

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Thus, weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring [2] provide an important example of weakly o-minimal (not o-minimal) structures.

Let A and B be arbitrary subsets of a linearly ordered structure M . Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$, and $A < \{b\}$ means that $A < \{b\}$. For an arbitrary subset A of M we introduce the following notations: $A^+ := \{b \in M \mid A < b\}$ and $A^- := \{b \in M \mid b < A\}$. For an arbitrary one-type p we denote by $p(M)$ the set of realizations of p in M . For an arbitrary tuple $\bar{b} = \langle b_1, b_2, \dots, b_n \rangle$ of length n we denote by $\bar{b}_i$ the tuple $\langle b_1, b_2, \dots, b_i \rangle$ for each $1 \leq i \leq n-1$. If $B \subseteq M$ and E is an equivalence relation on M then we denote by B/E the set of equivalence classes (E -classes) which have representatives in B . If f is a function on M then we denote by $\text{Dom}(f)$ the domain of f . A theory T is said to be *binary* if every formula of T is equivalent in T to a boolean combination of formulas with at most two free variables.

Further throughout the paper we consider an arbitrary complete theory T (if unless otherwise stated), where M is a sufficiently saturated model of T .

Definition 1. Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic. We say that p is not *weakly orthogonal* to q (denoting this by $p \not\perp^w q$) if there exist an A -definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

In other words, p is *weakly orthogonal* to q (denoting this by $p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A .

Lemma 1. [3] Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. Then the relation of non weak orthogonality $\not\perp^w$ is an equivalence relation on $S_1(A)$.

Definition 2. [4] We say that p is *quite orthogonal* to q ($p \perp^q q$) if there is no A -definable bijection $f : p(M) \rightarrow q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

Obviously, any o-minimal theory is quite o-minimal. In [5], countably categorical quite o-minimal theories were completely described, and in [6], algebras of distributions of binary isolating formulas were described for quite o-minimal theories with a small number of countable models.

We extend the definition of the rank of convexity of a formula [7] on arbitrary (non-necessarily definable) sets:

Definition 3. Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. The *rank of convexity of the set A* ($RC(A)$) is defined as follows:

- 1) $RC(A) = -1$ if $A = \emptyset$.
- 2) $RC(A) = 0$ if A is finite and non-empty.
- 3) $RC(A) \geq 1$ if A is infinite.
- 4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:
 - For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
 - For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A .
- 5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha \leq \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α , we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes. Clearly, each o-minimal theory has convexity rank 1.

We say that a theory T has κ (less than κ) countable models if T has κ (less than κ) countable pairwise non-isomorphic models. We say that a complete theory T is *Ehrenfeucht* if T is not countably categorical and T has finitely many countable models. Recall that a complete theory T is *small* if $|S(T)| = \omega$, where $S(T) = \bigcup_{k=1}^{\omega} S_k(\emptyset)$. It is known that any countable theory having less than 2^{ω} countable models (in particular, any countably categorical and any Ehrenfeucht theory) is small.

Definition 4. [8, 9] Let T be a complete theory, and $p_1(x_1), \dots, p_n(x_n) \in S_1(\emptyset)$. A type $q(x_1, \dots, x_n) \in S_n(\emptyset)$ is said to be a $(p_1, \dots, p_n)$ -type if $q(x_1, \dots, x_n) \supseteq \bigcup_{i=1}^n p_i(x_i)$. The set of all $(p_1, \dots, p_n)$ -types of the theory T is denoted by $S_{p_1, \dots, p_n}(T)$. A countable theory T is said to be *almost ω -categorical* if for any types $p_1(x_1), \dots, p_n(x_n) \in S_1(\emptyset)$ there are only finitely many types $q(x_1, \dots, x_n) \in S_{p_1, \dots, p_n}(T)$.

Almost ω -categoricity is closely connected with the notion of Ehrenfeuchtness of a theory. So in [8] it was proved that if T is an almost ω -categorical theory with $I(T, \omega) = 3$ then a dense linear order is interpreted in T . In [10] the authors established almost ω -categoricity of Ehrenfeucht quite o-minimal theories and that Exchange Principle for algebraic closure holds in almost ω -categorical quite o-minimal theories. Recently binarity of both almost ω -categorical quite o-minimal theories and almost ω -categorical weakly o-minimal theories of convexity rank 1 was proved in [11] and [12] respectively.

Definition 5. [13] The *disjunct union* $\bigsqcup_{n \in \omega} \mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ with predicate signatures Σ_n , $n \in \omega$, such that for any $n_1, n_2 < \omega$ with $n_1 \neq n_2$ we have $\Sigma_{n_1} \cap \Sigma_{n_2} = \{=, <\}$, is said to be the structure of the signature $\bigcup_{n \in \omega} \Sigma_n \cup \{P_n^{(1)} \mid n \in \omega\}$ with the universe $\bigcup_{n \in \omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n , coinciding with their interpretations in the structures $\mathcal{M}_n$, $n \in \omega$. The *disjunct union of theories* T_n of predicate signatures Σ_n , $n \in \omega$, such that for any $n_1, n_2 < \omega$ with $n_1 \neq n_2$ we have $\Sigma_{n_1} \cap \Sigma_{n_2} = \{=, <\}$, is said to be the theory $\bigsqcup_{n \in \omega} T_n \equiv \text{Th} \left(\bigsqcup_{n \in \omega} \mathcal{M}_n \right)$, where $\mathcal{M}_n \models T_n$, $n \in \omega$.

Clearly, the disjunct union of theories $\bigsqcup_{n \in \omega} T_n$ does not depend on the choice of models M_n of theories T_n .

Observe that almost ω -categorical weakly o-minimal theories are not Ehrenfeucht in general. As an example of such a theory we can take a disjunct union of countably many copies of the Ehrenfeucht example with three countable models, ordered by type ω . This theory has countably many pairwise weakly orthogonal non-isolated 1-types over $\emptyset$, and therefore has the maximal number of countable models.

Observe also that almost ω -categorical weakly o-minimal theories are not small in general. As an example of such a theory we can consider the structure $M = \langle \mathbb{Q}, <, q \rangle_{q \in \mathbb{Q}}$. Obviously, $Th(M)$ is o-minimal and has continuum of 1-types over $\emptyset$, i.e. it is not small.

At last, observe that there exist almost ω -categorical weakly o-minimal theories having a non-algebraic type $p \in S_1(\emptyset)$ with an infinite convexity rank. As an example of such a theory we can consider any ω -categorical 2-indiscernible non-3-indiscernible weakly o-minimal structure constructed in [14].

Recently, Vaught's conjecture was confirmed for some classes of complete theories: for quite o-minimal theories in [15], for weakly o-minimal theories of convexity rank 1 in [16], and for weakly o-minimal theories of finite convexity rank in [17]. Also, a criterion for the countable spectrum to be maximal in small binary quite o-minimal theories of finite convexity rank was obtained in [18]. Here we prove that weakly o-minimal theories of finite convexity rank having less than 2^ω countable models are almost ω -categorical.

2. MAIN THEOREM

Further we need the notion of a (p, q) -splitting formula, which was introduced in [19] for non-algebraic isolated 1-types. Let $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic, $p \not\leq^w q$. Extending the notion of a (p, q) -splitting formula to the non-isolated case, we say that an A -definable formula $\phi(x, y)$ is a (p, q) -splitting formula if there is $a \in p(M)$ such that

$$\phi(a, M) \cap q(M) \neq \emptyset, \neg \phi(a, M) \cap q(M) \neq \emptyset,$$

$$\phi(a, M) \cap q(M) \text{ is convex, and } [\phi(a, M) \cap q(M)]^- = [q(M)]^-.$$

If $\phi_1(x, y)$, $\phi_2(x, y)$ are (p, q) -splitting formulas we say that $\phi_1(x, y)$ is *less than* $\phi_2(x, y)$ if there exists $a \in p(M)$ such that

$$\phi_1(a, M) \cap q(M) \subset \phi_2(a, M) \cap q(M).$$

It is obvious that if $p, q \in S_1(A)$ are non-algebraic and $p \not\leq^w q$, then there exists a (p, q) -splitting formula, and the set of all (p, q) -splitting formulas is linearly ordered. It is also obvious that for every (p, q) -splitting formula $\phi(x, y)$ the function $f(x) := \sup \phi(x, M)$ is not constant on $p(M)$.

Lemma 2. [17] *Let T be a weakly o-minimal theory having less than 2^ω countable models, $M \models T$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, $p_1 \not\leq^w p_2$. Then there are only finitely many (p_1, p_2) -splitting formulas.*

Theorem 1. *Any weakly o-minimal theory of finite convexity rank having less than 2^ω countable models is almost ω -categorical.*

Proof. Let T be a weakly o-minimal theory of finite convexity rank having less than 2^ω countable models, and let M be a countable saturated model for T .

We prove by induction on $n \geq 2$ that for any family of non-algebraic types $p_1, \dots, p_n \in S_1(\emptyset)$ there exist only finitely many $(p_1, \dots, p_n)$ -types.

Step $n = 2$.

Case 1. $p_1 \perp^w p_2$. Then the set of formulas $p_1(x_1) \cup p_2(x_2)$ determines a complete 2-type over $\emptyset$, and consequently, the number of (p_1, p_2) -types is equal to 1.

Case 2. $p_1 \not\leq^w p_2$. Then by Lemma 2 there are only finitely many (p_1, p_2) -splitting formulas. Denote them by $\Phi_1(x, y), \Phi_2(x, y), \dots, \Phi_s(x, y)$ for some $s < \omega$, and suppose that for any $a \in p_1(M)$ we have

$$\Phi_1(a, M) \cap p_2(M) \subset \Phi_2(a, M) \cap p_2(M) \subset \dots \subset \Phi_s(a, M) \cap p_2(M).$$

Then we assert that uniquely possible extensions of the set $p_1(x_1) \cup p_2(x_2)$ are the following:

$$\begin{aligned} & p_1(x_1) \cup p_2(x_2) \cup \{\Phi_1(x_1, x_2)\} \\ & p_1(x_1) \cup p_2(x_2) \cup \{\Phi_2(x_1, x_2) \wedge \neg \Phi_1(x_1, x_2)\} \\ & \dots\dots\dots \\ & p_1(x_1) \cup p_2(x_2) \cup \{\Phi_s(x_1, x_2) \wedge \neg \Phi_{s-1}(x_1, x_2)\} \\ & p_1(x_1) \cup p_2(x_2) \cup \{\neg \Phi_s(x_1, x_2)\} \end{aligned}$$

Indeed, we show that $p_1(x_1) \cup p_2(x_2) \cup \{\Phi_1(x_1, x_2)\}$ determines a complete type over $\emptyset$. If this is not true then there is an $\emptyset$ -definable formula $\Psi(x_1, x_2)$ such that

$$M \models \Phi_1(a_1, a_2) \wedge \Psi(a_1, a_2) \wedge \Phi_1(a'_1, a'_2) \wedge \neg \Psi(a'_1, a'_2)$$

for some $a_1, a'_1 \in p_1(M), a_2, a'_2 \in p_2(M)$.

Show that there is $a''_2 \in p_2(M)$ such that

$$M \models \Phi_1(a_1, a''_2) \wedge \neg \Psi(a_1, a''_2).$$

Assume the contrary: for any $a''_2 \in p_2(M)$ with $\Phi_1(a_1, a''_2)$ we have $\Psi(a_1, a''_2)$.

Let $q(x_1, x_2) := tp(\langle a_1, a_2 \rangle / \emptyset)$. Consider for any $\theta(y) \in p_2$ and $\phi(x, y) \in q$ the following formula:

$$K_{\theta, \phi}(x) := \forall y [\theta(y) \wedge \phi(x, y) \rightarrow \Psi(x, y)].$$

Since we assumed that for any $a''_2 \in p_2(M)$ with $\Phi_1(a_1, a''_2)$ we have $\Psi(a_1, a''_2)$, we have $M \models K_{\theta, \Phi_1}(a_1)$ for any $\theta(y) \in p_2$. Since $a_1, a'_1 \in p_1(M)$, we have $M \models K_{\theta, \Phi_1}(a'_1)$ for any $\theta(y) \in p_2$. This contradicts $\neg \Psi(a'_1, a'_2)$.

Consequently, there is $a''_2 \in p_2(M)$ such that $M \models \Phi_1(a_1, a''_2) \wedge \neg \Psi(a_1, a''_2)$.

Without loss of generality, suppose $p_1(M) < p_2(M)$ and $a_2 < a''_2$. Also, by weak o-minimality we can assume that $\Psi(a_1, M)$ is convex.

Consider the following formula:

$$\Theta(a_1, y) := a_1 \leq y \wedge \exists t [\Psi(a_1, t) \wedge y \leq t].$$

Obviously, $\Theta(x, y)$ is a (p_1, p_2) -splitting formula and

$$\Theta(a_1, M) \cap p_2(M) \subset \Phi_1(a_1, M) \cap p_2(M).$$

The last contradicts that $\Phi(x, y)$ is the least (p_1, p_2) -splitting formula. Similarly, we can prove that the remaining extensions of the set $p_1(x_1) \cup p_2(x_2)$ determine a complete type over $\emptyset$.

Thus, there are only finitely many (p_1, p_2) -types.

Suppose that we already have established finiteness of the number of $(p_1, \dots, p_k)$ -types for all $k \leq n$ and we prove it for $n + 1$.

Step $n + 1 > 2$. Consider arbitrary non-algebraic types $p_1, \dots, p_n, p_{n+1} \in S_1(\emptyset)$. By Theorem 4.4 of [17] T is binary, and therefore we have only the following cases.

Case 1. $p_{n+1} \perp^w p_i$ for every $1 \leq i \leq n$. In this case the number of $(p_1, \dots, p_n, p_{n+1})$ -types coincides with the number of $(p_1, \dots, p_n)$ -types.

Case 2. $p_{n+1} \not\leq^w p_i$ for every $1 \leq i \leq n$. In this case by Lemma 1 the types $p_1, \dots, p_n, p_{n+1}$ are pairwise non-weakly orthogonal. By the induction hypothesis

the number of $(p_1, \dots, p_n)$ -types is finite. Denote this number by $S_{p_1, \dots, p_n}$. Since $p_i \not\leq^w p_{n+1}$ for every $1 \leq i \leq n$, by Lemma 2 there exist only finitely many (p_i, p_{n+1}) -splitting formulas. Denote this number by m_i , and let $\{\Phi_1^i(x, y), \Phi_2^i(x, y), \dots, \Phi_{m_i}^i(x, y)\}$ be a complete list of (p_i, p_{n+1}) -splitting formulas so that for any $a_i \in p_i(M)$ the following holds:

$$\Phi_1^i(a_i, M) \cap p_{n+1}(M) \subset \Phi_2^i(a_i, M) \cap p_{n+1}(M) \subset \dots \subset \Phi_{m_i}^i(a_i, M) \cap p_{n+1}(M).$$

Obviously, for every $1 \leq i < j \leq n$ and for any $1 \leq s \leq m_i$, $1 \leq l \leq m_j$ we have that either

$$\begin{aligned} \Phi_s^i(a_i, M) \cap p_{n+1}(M) &\subseteq \Phi_l^j(a_j, M) \cap p_{n+1}(M) \\ \text{or } \Phi_l^j(a_j, M) \cap p_{n+1}(M) &\subset \Phi_s^i(a_i, M) \cap p_{n+1}(M). \end{aligned}$$

Consequently, (if necessary) there is some re-notation of all (p_1, p_{n+1}) -splitting formulas $\Phi_1^1(x, y), \dots, \Phi_{m_1}^1(x, y)$, (p_2, p_{n+1}) -splitting formulas $\Phi_1^2(x, y), \dots, \Phi_{m_2}^2(x, y), \dots$, and (p_n, p_{n+1}) -splitting formulas $\Phi_1^n(x, y), \dots, \Phi_{m_n}^n(x, y)$ by $F_1(x, y), F_2(x, y), \dots, F_{m_1+m_2+\dots+m_n}(x, y)$ so that

$$F_1(c_1, M) \cap p_{n+1}(M) \subseteq F_2(c_2, M) \cap p_{n+1}(M) \subseteq$$

$$\dots \subseteq F_{m_1+m_2+\dots+m_n}(c_{m_1+m_2+\dots+m_n}, M) \cap p_{n+1}(M),$$

where for any $1 \leq k \leq m_1 + m_2 + \dots + m_n$ there is $1 \leq i_k \leq n$ such that $c_k = a_{i_k}$.

Take an arbitrary $q(x_1, x_2, \dots, x_n) \in S_{p_1, \dots, p_n}(T)$. We assert that there exist at most $m_1 + m_2 + \dots + m_n$ extensions of the set of formulas $q(x_1, x_2, \dots, x_n) \cup p(x_{n+1})$:

$$\begin{aligned} &q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_1(x^{(1)}, x_{n+1})\} \\ &q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_2(x^{(2)}, x_{n+1}) \wedge \neg F_1(x^{(1)}, x_{n+1})\} \\ &\dots \dots \dots \\ &q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_{m_1+m_2+\dots+m_n}(x^{(m_1+m_2+\dots+m_n)}, x_{n+1}) \wedge \\ &\quad \neg F_{m_1+m_2+\dots+m_n-1}(x^{(m_1+m_2+\dots+m_n-1)}, x_{n+1})\} \\ &q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{\neg F_{m_1+m_2+\dots+m_n}(x^{(m_1+m_2+\dots+m_n)}, x_{n+1})\}, \end{aligned}$$

where for every $1 \leq i \leq m_1 + m_2 + \dots + m_n$ there exists $1 \leq i_k \leq n$ such that $x^{(i)} = x_{i_k}$.

Show that $q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_1(x^{(1)}, x_{n+1})\}$ determines a complete type over $\emptyset$.

Without loss of generality, suppose that $x^{(1)} = x_1$, i.e. for every $2 \leq j \leq n$

$$\Phi_1^1(a_1, M) \cap p_{n+1}(M) \subseteq \Phi_1^j(a_j, M) \cap p_{n+1}(M).$$

Assume the contrary: there exists an $\emptyset$ -definable formula $\psi(x_1, \dots, x_n, x_{n+1})$ such that both

$$q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_1(x^{(1)}, x_{n+1})\} \cup \{\psi(x_1, \dots, x_n, x_{n+1})\}$$

$$\text{and } q(x_1, x_2, \dots, x_n) \cup p(x_{n+1}) \cup \{F_1(x^{(1)}, x_{n+1})\} \cup \{\neg \psi(x_1, \dots, x_n, x_{n+1})\}$$

are consistent.

Consequently, there are $a_1, a'_1 \in p_1(M), a_2, a'_2 \in p_2(M), \dots, a_n, a'_n \in p_n(M)$ and $a_{n+1}, a'_{n+1} \in p_{n+1}(M)$ such that $\langle a_1, \dots, a_n \rangle, \langle a'_1, \dots, a'_n \rangle \in q(M^n)$ and

$$M \models F_1(a_1, a_{n+1}) \wedge F_1(a'_1, a'_{n+1}) \wedge \psi(a_1, \dots, a_n, a_{n+1}) \wedge \neg \psi(a'_1, \dots, a'_n, a'_{n+1}).$$

By binarity of T we have that

$$\psi(x_1, \dots, x_n, x_{n+1}) \equiv \bigvee_{l=1}^s [\bigwedge_{1 \leq i < j \leq n+1} \psi_{ij}^l(x_i, x_j)]. \quad (*)$$

Since $tp(\langle a_1, \dots, a_n \rangle / \emptyset) = tp(\langle a'_1, \dots, a'_n \rangle / \emptyset)$, for every $1 \leq l \leq s$ the following holds:

$$M \models \bigwedge_{1 \leq i < j \leq n} \psi_{ij}^l(a_i, a_j) \Leftrightarrow M \models \bigwedge_{1 \leq i < j \leq n} \psi_{ij}^l(a'_i, a'_j).$$

Obviously, by $(*)$ we have

$$\neg\psi(x_1, \dots, x_n, x_{n+1}) \equiv \bigwedge_{l=1}^s [\bigvee_{1 \leq i < j \leq n+1} \neg\psi_{ij}^l(x_i, x_j)], \text{ i.e.}$$

$$\neg\psi(x_1, \dots, x_n, x_{n+1}) \equiv \bigwedge_{l=1}^s [\bigvee_{1 \leq i < j \leq n} \neg\psi_{ij}^l(x_i, x_j) \vee \bigvee_{m=1}^n \neg\psi_{m,n+1}^l(x_m, x_{n+1})],$$

and consequently,

$$\neg\psi(x_1, \dots, x_n, x_{n+1}) \equiv \bigwedge_{l=1}^s [\bigwedge_{1 \leq i < j \leq n} \psi_{ij}^l(x_i, x_j) \rightarrow \bigvee_{m=1}^n \neg\psi_{m,n+1}^l(x_m, x_{n+1})].$$

By similar arguments as in Step $n = 2$ (Case 2) we can show that there is $a''_{n+1} \in p_{n+1}(M)$ such that

$$M \models F_1(a_1, a''_{n+1}) \wedge \neg\psi(a_1, \dots, a_n, a''_{n+1}).$$

Consequently, there are $1 \leq l \leq s$ and $1 \leq j \leq n$ such that $M \models \neg\psi_{j,n+1}^l(a_j, a''_{n+1})$. Then we obtain a contradiction with the condition that $\Phi_1^j(x_j, x_{n+1})$ is the least (p_j, p_{n+1}) -splitting formula.

Similar arguments show that each of the extensions of the set of formulas $q(x_1, \dots, x_n, x_{n+1}) \cup p(x_{n+1})$ determines a complete type over $\emptyset$.

Case 3. $p_{n+1} \not\perp^w p_i$ and $p_{n+1} \perp^w p_j$ for some distinct $1 \leq i, j \leq n$. Then there exist (if necessary) some renumbering of the set $\{p_1, \dots, p_n\}$ and some $1 \leq k < n$ such that $p_{n+1} \perp^w p_j$ for all $1 \leq j \leq k$ and $p_{n+1} \not\perp^w p_l$ for all $k+1 \leq l \leq n$. By induction hypothesis both the number of $(p_1, \dots, p_k, p_{n+1})$ -types and the number of $(p_{k+1}, \dots, p_n, p_{n+1})$ -types are finite, and the number of $(p_1, \dots, p_k, p_{n+1})$ -types coincides with the number of $(p_1, \dots, p_k)$ -types. Denote these numbers by $S_{p_1, \dots, p_k}$ and $S_{p_{k+1}, \dots, p_n, p_{n+1}}$ respectively. Then we assert that the number of $(p_1, \dots, p_n, p_{n+1})$ -types is at most $S_{p_1, \dots, p_k} \times S_{p_{k+1}, \dots, p_n, p_{n+1}}$, i.e. it is also finite. $\square$

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REFERENCES

- [1] D. Macpherson, D. Marker, C. Steinhorn, *Weakly o-minimal structures and real closed fields*, Trans.Am. Math. Soc., **352**:12 (2000), 5435–5483. Zbl 0982.03021
- [2] M. Dickmann, *Elimination of quantifiers for ordered valuation rings*, J. Symb. Log., **52** (1987), 116–128. Zbl 0624.03025
- [3] B.S. Baizhanov, *Expansion of a model of a weakly o-minimal theory by a family of unary predicates*, J. Symb. Log., **66**:3 (2001), 1382–1414. Zbl 0992.03047
- [4] B.Sh. Kulpeshov, *Convexity rank and orthogonality in weakly o-minimal theories*, News of National Academy of Sciences of the Republic of Kazakhstan, series physics-mathematics, **227** (2003), 26–31.
- [5] B.Sh. Kulpeshov, *Countably categorical quite o-minimal theories*, J. Math. Sci., **188**:4 (2013), 387–397. Zbl 1280.03043
- [6] D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of distributions of binary isolating formulas for quite o-minimal theories*, Algebra Logic, **57**:6 (2019), 429–444. Zbl 07140957
- [7] B.Sh. Kulpeshov, *Weakly o-minimal structures and some of their properties*, J. Symb. Log., **63**:4 (1998), 1511–1528. Zbl 0926.03041

- [8] K. Ikeda, A. Pillay, A. Tsuboi, *On theories having three countable models*, Math. Log. Q., **44**:2 (1998), 161–166. Zbl 0897.03035
- [9] S.V. Sudoplatov, *Classification of countable models of complete theories. Part 1*, Novosibirsk State Technical University Publishing House, Novosibirsk, 2018.
- [10] B.Sh. Kulpeshov, S.V. Sudoplatov, *Linearly ordered theories which are nearly countably categorical*, Math. Notes, **101**:3 (2017), 475–483. Zbl 1420.03067
- [11] A.B. Altayeva, B.Sh. Kulpeshov, *Binarity of almost ω -categorical quite o-minimal theories*, Sib. Math. J., **61**:3 (2020), 379–390. Zbl 07214646
- [12] B.Sh. Kulpeshov, T.S. Mustafin, *Almost ω -categorical weakly o-minimal theories of convexity rank 1*, Sib. Math. J., **62**:1 (2021), 52–65. Zbl 07304933
- [13] R.E. Woodrow, *Theories with a finite number of countable models and a small language*, Ph.D. Thesis, Simon Fraser University, 1976.
- [14] B. Herwig, H.D. Macpherson, G. Martin, A. Nurtazin, J.K. Truss, *On $\aleph_0$ -categorical weakly o-minimal structures*, Ann. Pure Appl. Logic, **101**:1 (2000), 65–93. Zbl 0945.03056
- [15] B.Sh. Kulpeshov, S.V. Sudoplatov, *Vaught's conjecture for quite o-minimal theories*, Ann. Pure Appl. Logic, **168**:1 (2017), 129–149. Zbl 06643768
- [16] A. Alibek, B.S. Baizhanov, B.Sh. Kulpeshov, T.S. Zambarnaya, *Vaught's conjecture for weakly o-minimal theories of convexity rank 1*, Ann. Pure Appl. Logic, **169**:11 (2018), 1190–1209. Zbl 1412.03017
- [17] B.Sh. Kulpeshov, *Vaught's conjecture for weakly o-minimal theories of finite convexity rank*, Izv. Math., **84**:2 (2020), 324–347. Zbl 07206515
- [18] B.Sh. Kulpeshov, *Maximality of the countable spectrum in small quite o-minimal theories*, Algebra Logic, **58**:2 (2019), 137–143. Zbl 1444.03132
- [19] B.Sh. Kulpeshov, *Criterion for binarity of $\aleph_0$ -categorical weakly o-minimal theories*, Ann. Pure Appl. Logic, **45**:3 (2007), 354–367. Zbl 1444.03132

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