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An Optimal Choice of Dirichlet Polynomials for the Nyman–Beurling Criterion

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*In memory of Professor A. A. Karatsuba
on the 75th anniversary of his birth*

1. Introduction

The Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true if and only if

$$\lim_{N \rightarrow \infty} d_N^2 = 0,$$

where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta A_N \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{1/4 + t^2}$$

and the infimum is over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^N a_n/n^s$ of length N (see [1] for a nice account of this).

An open question is to determine what the rate of convergence of d_n to zero is, assuming the Riemann hypothesis. Balazard and de Roton showed that, if the Riemann hypothesis is true, then

$$d_N^2 \ll \frac{(\log \log N)^{5/2+\varepsilon}}{\sqrt{\log N}},$$

for all $\varepsilon > 0$. On the other hand Báez-Duarte, Balazard, Landreau and Saias [2], [3] showed (unconditionally) that d_N^2 can not decay faster than a constant times $1/\log N$. More precisely, they showed that

$$\liminf_{N \rightarrow \infty} d_N^2 \log N \geq \sum_{\operatorname{Re}(\rho)=1/2} \frac{1}{|\rho|^2},$$

where the sum is restricted to distinct zeros of the Riemann zeta function on the critical line. The constant was later improved by Burnol [4] who showed

$$\liminf_{N \rightarrow \infty} d_N^2 \log N \geq \sum_{\operatorname{Re}(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of ρ . This lower bound is believed to be optimal and one expects that

$$d_N^2 \sim \frac{1}{\log N} \sum_{\operatorname{Re}(\rho)=1/2} \frac{m(\rho)^2}{|\rho|^2}. \quad (1)$$

Notice that under the Riemann hypothesis, one has

$$\sum_{\operatorname{Re}(\rho)=1/2} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi$$

and in particular, if all the non-trivial zeros of $\zeta(s)$ are simple, then (1) can be rewritten as

$$d_N^2 \sim \frac{2 + \gamma - \log 4\pi}{\log N}.$$

It is the purpose of this note to prove (1) under the Riemann Hypothesis and assuming a mild condition on the growth of the mean value of $1/|\zeta'(\rho)|^2$ over the non-trivial zeros $|\rho| \leq T$ of $\zeta(s)$. This will be achieved by using the Dirichlet polynomial

$$V_N(s) := \sum_{n=1}^N \left(1 - \frac{\log n}{\log N}\right) \frac{\mu(n)}{n^s}.$$

THEOREM 1. *If the Riemann hypothesis is true and if*

$$\sum_{|\operatorname{Im}(\rho)| \leq T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{3/2-\delta} \quad (2)$$

for some $\delta > 0$, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta V_N \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{1/4 + t^2} \sim \frac{2 + \gamma - \log 4\pi}{\log N}.$$

The condition (2) implicitly assumes that the zeros of the Riemann zeta function are all simple. Moreover, this upper bound is “mild” in the sense that a conjecture, due to Gonek and recovered by a different heuristic method of Hughes, Keating, and O’Connell [5], predicts that

$$\sum_{|\rho| \leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{6}{\pi^3} T.$$

We remark that Theorem 1 is in contrast to what one might have expected after viewing the graphs of Landreau and Richards [6] which at first sight suggest that V_N is not optimal.

This behaviour of the Riemann zeta function resembles that of polynomials. In fact, Grenander and Rosenblatt [7] (see also [4, Theorem 2.1]) showed that for a polynomial $P(z)$ one has that the zeros of P are all located outside or on the unit circle if and only if $\lim_{N \rightarrow \infty} \delta_N = 0$, where

$$\delta_N^2 = \frac{1}{2\pi} \inf_{Q_N} \int_0^{2\pi} |1 - P(z)Q_N(z)|^2 d\theta,$$

where $z = e^{i\theta}$ and the infimum is over polynomials Q_N of degree at most N . Moreover, if this happens, then

$$\lim_{N \rightarrow \infty} N \delta_N^2 = \sum_{|\rho|=1} m(\rho)^2,$$

where the sum is restricted to the distinct zeros ρ of $P(z)$ lying on the unit circle and $m(\rho)$ is again the multiplicity of ρ .

This analogy seems to apply also to the choices of optimal polynomials.

THEOREM 2. *Let $P(z)$ be a polynomial whose zeros are all simple and lie outside or on the unit circle. Let*

$$W_N(z) := \sum_{n=0}^N \left(1 - \frac{n}{N}\right) a_n z^n, \quad (3)$$

where

$$\frac{1}{P(z)} = \sum_{n \geq 0} a_n z^n$$

is the Taylor expansion in $x = 0$ of the inverse of $P(z)$ (i.e. it is the formal power series inverse of $P(z)$). Then

$$\frac{1}{2\pi} \int_0^{2\pi} |1 - P(z)W_N(z)|^2 d\theta \sim \frac{1}{N} \sum_{|\rho|=1} m(\rho)^2,$$

where $z = e^{i\theta}$.

We remark that the proofs of Theorems 1 and 2 are very similar, the main difference being that the Riemann zeta function has infinitely many zeros. This generates some issues concerning the convergence of certain sums of $1/\zeta'(\rho)$, which force us to assume condition (2).

2. Polynomials

LEMMA 1. Let $P(s)$ be a polynomial with $P(0) \neq 0$. We have

$$W_N(s) = \frac{1}{P(s)} \left(1 + \frac{s}{N} \frac{P'}{P}(s) \right) - \frac{s}{N} Y_N(s),$$

where $W_N(s)$ is defined in (3),

$$Y_N(s) := \sum_{\rho} \operatorname{Res}_{z=\rho} \frac{s^N}{P(z)(z-s)^2 z^N}$$

and the sum is over distinct zeros ρ of $P(z)$.

Proof. Since $P(0) \neq 0$, we can take an $\varepsilon > 0$ such that all the zeros of $P(z)$ lie outside of the circle $|z| = \varepsilon$. Now, observe that we can assume $0 < |s| < \varepsilon$, since the result will then extend to all $\mathbb{C}$ by analytic continuation. Denoting by $\mathcal{C}_y$ the circle of radius $y > 0$ (oriented in the positive direction), by the residue theorem we have that

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{dz}{z^{n+1}},$$

therefore

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \sum_{n=0}^N \left(1 - \frac{n}{N} \right) \left(\frac{s}{z} \right)^n \frac{dz}{z}.$$

Now,

$$\sum_{n=0}^N \left(1 - \frac{n}{N} \right) z^n = -\frac{1}{N} \frac{z - z^{N+1}}{(1-z)^2} + \frac{1}{1-z}$$

and thus

$$W_N(s) = \frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \left(-\frac{1}{N} \frac{s z^N - s^{N+1}}{(z-s)^2 z^N} + \frac{1}{z-s} \right) dz.$$

Now, by the residue theorem

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \left(-\frac{1}{N} \frac{s}{(z-s)^2} + \frac{1}{z-s} \right) dz = \frac{1}{P(s)} \left(1 + \frac{s}{N} \frac{P'(s)}{P(s)} \right),$$

whereas, moving the line of integration to $\mathcal{C}_y$ and letting y tend to infinity, one has that

$$\frac{1}{2\pi i N} \int_{\mathcal{C}_\varepsilon} \frac{1}{P(z)} \frac{s^{N+1}}{(z-s)^2 z^N} dz = -\frac{s}{N} Y_N(s)$$

and the Lemma follows.

Proof of Theorem 2. Let $\delta > 1$ be such that $P(s)$ does not have any zero on $1 < |s| \leq \delta$. We have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |1 - P(z)W_N(z)|^2 d\theta &= \frac{1}{2\pi i} \int_{\mathcal{C}_1} (1 - P(s)W_N(s)) \left(1 - \overline{P}\left(\frac{1}{s}\right)\overline{W}_N\left(\frac{1}{s}\right)\right) \frac{ds}{s} \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_\delta} (1 - P(s)W_N(s)) \left(1 - \overline{P}\left(\frac{1}{s}\right)\overline{W}_N\left(\frac{1}{s}\right)\right) \frac{ds}{s}. \end{aligned}$$

Therefore, by Lemma 1, this is

$$\frac{1}{2\pi i N^2} \int_{\mathcal{C}_\delta} \left(\frac{P'}{\overline{P}}(s) - P(s)Y_N(s) \right) \left(\frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) - \overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right) \right) \frac{ds}{s}.$$

Now, for $|s| = \delta$ one has

$$Y_N(s)\overline{Y}_N\left(\frac{1}{s}\right) = O(1),$$

therefore

$$\frac{1}{2\pi i N^2} \int_{\mathcal{C}_\delta} \left(\frac{P'}{\overline{P}}(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) + P(s)Y_N(s)\overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right) \right) \frac{ds}{s} = O\left(\frac{1}{N^2}\right).$$

Moreover for $s \in \mathcal{C}_\delta$ one has that $\overline{Y}_N(1/s) = O(\delta^{-N})$, thus

$$-\frac{1}{2\pi i N^2} \int_{\mathcal{C}_\delta} \left(\frac{P'}{\overline{P}}(s)\overline{P}\left(\frac{1}{s}\right)\overline{Y}_N\left(\frac{1}{s}\right) \right) \frac{ds}{s} = O\left(\frac{\delta^{-N}}{N^2}\right).$$

Finally, by the residue theorem,

$$\begin{aligned} &-\frac{1}{2\pi i N^2} \int_{\mathcal{C}_\delta} P(s)Y_N(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) \frac{ds}{s} \\ &= -\frac{1}{N^2} \sum_{|\rho|=1} \operatorname{Res}_{s=\rho} P(s)Y_N(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) \frac{1}{s} - \frac{1}{2\pi i N^2} \int_{\mathcal{C}_{1/\delta}} P(s)Y_N(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) \frac{ds}{s} \\ &= -\frac{1}{N^2} \sum_{|\rho|=1} \operatorname{Res}_{s=\rho} P(s)Y_N(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) + O\left(\frac{\delta^{-N}}{N^2}\right). \end{aligned}$$

The Theorem then follows by observing that

$$\operatorname{Res}_{s=\rho} P(s)Y_N(s) \frac{\overline{P}'}{\overline{P}}\left(\frac{1}{s}\right) \frac{1}{s} = -N + O(1).$$

3. The Riemann zeta-function

We start with the following lemma, which is the analogue of Lemma 1. We remark that this lemma is unconditional.

LEMMA 2. *If $0 < \operatorname{Re}(s) < 1$, then*

$$V_N(s) = \frac{1}{\zeta(s)} \left(1 - \frac{1}{\log N} \frac{\zeta'}{\zeta}(s) \right) + \frac{1}{\log N} \sum_{\rho} R_N(\rho, s) + \frac{1}{\log N} F_s\left(\frac{1}{N}\right),$$

where the sum is over distinct non-trivial zeros ρ of $\zeta(s)$ with

$$R_N(\rho, s) = \operatorname{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},$$

and where

$$F_s(z) = \pi z^s \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n+1} z^{2n}}{(2n)! \zeta(2n+1) (2n+s)^2}$$

is an entire function of z .

Proof. We have

$$V_N(s) = \frac{1}{\log N} \frac{1}{2\pi i} \int_{(2)} \frac{N^w}{\zeta(s+w)} \frac{dw}{w^2},$$

where we use the notation $\int_{(c)}$ to mean an integration up the vertical line from $c - i\infty$ to $c + i\infty$. Now we move the path of integration to $\operatorname{Re}(w) = -\operatorname{Re}(s) - 2M - 1$ for a large integer M . The residue at $w = \rho - s$ is $R_N(\rho, s)/\log N$. The residue at $s + w = -2n$ is

$$\frac{N^{-2n-s}}{\zeta'(-2n)(2n+s)^2 \log N}$$

and the integral on the new path is $\ll N^{-2M-1}$. Letting $M \rightarrow \infty$ and using

$$\zeta'(-2n) = \frac{(-1)^n \pi (2n)! \zeta(2n+1)}{(2\pi)^{2n+1}}$$

we obtain the result.

LEMMA 3. *Let $\varepsilon > 0$. Assume the Riemann hypothesis and that all the zeros of $\zeta(s)$ are simple. Then, if condition (2) holds, for $\operatorname{Re}(s) = 1/2 \pm \varepsilon$ one has*

$$\sum_{\rho} R_N(\rho, s) \ll N^{\mp\varepsilon} |s|^{3/4-\delta/2+\varepsilon}. \quad (4)$$

Proof. Firstly observe that, by the Cauchy-Schwartz inequality, (2) implies

$$\sum_{|\rho| \leq T} \frac{1}{|\zeta'(\rho)|} \ll \sqrt{N(T) \sum_{|\rho| \leq T} \frac{1}{|\zeta'(\rho)|^2}} \ll T^{5/4-\delta/2} \sqrt{\log T},$$

since

$$N(T) := \frac{1}{2} \sum_{|\rho| \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Therefore, by partial summation, we have that the series

$$\sum_{\rho} \frac{1}{|\zeta'(\rho)| |\rho|^{\alpha}}$$

is convergent for any $\alpha > 5/4 - \delta/2$. Now, for a simple zero ρ , we have

$$R_N(\rho, s) = \sum_{\rho} \frac{N^{\rho-s}}{\zeta'(\rho)(\rho-s)^2}.$$

Therefore

$$\begin{aligned}
N^{\pm\epsilon} \sum_{\rho} R_N(\rho, s) &\ll \sum_{|\rho-s| < |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} + \sum_{|\rho-s| \geq |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} \\
&\ll \sum_{|\rho-s| < |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} + \sum_{|\rho-s| \geq |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho|^2} \\
&\ll \sum_{|\rho-s| < |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} + 1.
\end{aligned} \tag{5}$$

Now, by the Cauchy–Schwartz inequality,

$$\sum_{|\rho-s| < |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} \ll \sqrt{\left(\sum_{|\rho| < 2|s|} \frac{1}{|\zeta'(\rho)|^2} \right) \left(\sum_{|\rho| < 2|s|} \frac{1}{|\rho-s|^4} \right)} \ll |s|^{3/4-\delta/2+\epsilon},$$

since, by partial summation,

$$\sum_{|\rho| < 2|s|} \frac{1}{|\rho-s|^4} \ll \log(|s|+2).$$

This completes the proof of the lemma.

Proof of Theorem 1. We have

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta V_N \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{1/4 + t^2} &= \frac{1}{2\pi i} \int_{(1/2)} (1 - \zeta V_N(s))(1 - \zeta V_N(1-s)) \frac{ds}{s(1-s)} \\
&= \frac{1}{2\pi i} \int_{(1/2-\epsilon)} (1 - \zeta V_N(s))(1 - \zeta V_N(1-s)) \frac{ds}{s(1-s)}.
\end{aligned}$$

By Lemma 2, this is

$$\begin{aligned}
&\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\epsilon)} \left(\frac{\zeta'}{\zeta^2}(s) - \sum_{\rho} R_N(\rho, s) - F_s \left(\frac{1}{N} \right) \right) \\
&\quad \times \left(\frac{\zeta'}{\zeta^2}(1-s) - \sum_{\rho} R_N(\rho, 1-s) - F_{1-s} \left(\frac{1}{N} \right) \right) \frac{\zeta(s)\zeta(1-s)}{s(1-s)} ds.
\end{aligned} \tag{6}$$

Now, we have

$$\begin{aligned}
&\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\epsilon)} \sum_{\rho_1, \rho_2} R_N(\rho_1, s) R_N(\rho_2, 1-s) \frac{\zeta(s)\zeta(1-s)}{s(1-s)} ds \\
&\ll \frac{1}{\log^2 N} \int_{(1/2-\epsilon)} \sum_{|\rho-s| < |\rho|/2} \frac{1}{|\zeta'(\rho)| |\rho-s|^2} \frac{|ds|}{|s|^{5/4+\delta/2-5\epsilon}} + O\left(\frac{1}{\log^2 N} \right),
\end{aligned}$$

where we used (4), (5) and the bound $\zeta(1/2 \pm \epsilon \pm it) \ll |t|^{2\epsilon}$ (which is a consequence of the Lindelöf hypothesis). Reversing the order of summation and integration, we have that this is bounded by

$$\begin{aligned}
&\frac{1}{\log^2 N} \sum_{\rho} \frac{1}{|\zeta'(\rho)|} \int_{1/2-\epsilon+i(\operatorname{Im}(\rho)-|\rho|/2)}^{1/2-\epsilon+i(\operatorname{Im}(\rho)+|\rho|/2)} \frac{|ds|}{|\rho-s|^2 |s|^{5/4+\delta/2-5\epsilon}} + O\left(\frac{1}{\log^2 N} \right) \\
&\ll \frac{1}{\log^2 N} \sum_{\rho} \frac{1}{|\zeta'(\rho)| |\rho|^{5/4+\delta/2-5\epsilon}} \ll \frac{1}{\log^2 N},
\end{aligned}$$

if $\epsilon < \delta/10$.

Now, by Lemma 3 and the trivial estimate $F_s(z) = O(N^{-5/2})$, all the other terms in (6) are trivially $O(1/\log^2 N)$ apart from

$$-\frac{1}{\log^2 N} \frac{1}{2\pi i} \int_{(1/2-\varepsilon)} \frac{\zeta'}{\zeta} (1-s) \sum_{\rho} R_N(\rho, s) \frac{\zeta(s)}{s(1-s)} ds. \quad (7)$$

The integrand has a double pole at every zero ρ of residue

$$\begin{aligned} \operatorname{Res}_{s=\rho} \left(\frac{\zeta'}{\zeta} (1-s) \sum_{\rho} R_N(\rho, s) \frac{\zeta(s)}{s(1-s)} \right) &= \left(\log N - \frac{1}{2} \frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{\chi'}{\chi}(\rho) + \frac{1-2\rho}{|\rho|^2} \right) \frac{1}{|\rho|^2} \\ &= \frac{\log N}{|\rho|^2} + O\left(\frac{1}{|\rho|^{2-\varepsilon} |\zeta'(\rho)|} + \frac{1}{|\rho|^2} \right), \end{aligned}$$

where we used the bound $\zeta''(1/2 + it) \ll |t|^\varepsilon$, which follows from the Lindelöf hypothesis and Cauchy's estimate for the derivatives of a holomorphic function. It follows that moving the line of integration in (7) to $\operatorname{Re}(s) = 1/2 + \varepsilon$ we get that the integral is equal to

$$\frac{1}{\log N} \sum_{\rho} \frac{1}{|\rho|^2} + O\left(\frac{1}{\log^2 N} \right),$$

and Theorem 1 then follows.

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