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On a Class of Monic Binomials

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Dedicated to the memory of Professor A. A. Karatsuba

1. If $f(X)$ is a polynomial with coefficients in a field K , then the sequence $a_1, a_2, \dots, a_k$ of distinct elements of K is called an f -cycle of length k , if $f(a_k) = a_1$ and for $j = 1, 2, \dots, k-1$ one has $f(a_j) = a_{j+1}$. The truth of a conjecture of Morton and Silverman [1] would imply that if K is an algebraic number field of degree N , then the length of any such cycle is bounded by a constant depending only on N and the degree of the polynomial f . It was shown in [2] that if $f(X) = X^n + c$ is a binomial of an odd prime-power degree $n = p^m$, and K is an algebraic number field of degree N not containing the p th roots of unity, then the lengths of cycles of F in K are bounded by a certain constant $c = c(K, p)$, and in the case when the prime p exceeds 2^N , then this bound depends only on N . The purpose of this note is to show that a modification of the argument in [2] leads to a similar result for polynomials of the form $X^n + c \in K[X]$ when n is odd and for every prime divisor q of n one has $q-1 \nmid N$. Actually we establish this for a larger class of fields K (see Theorem 2 in Sect. 3). Note that it suffices to consider the case of totally complex fields K as in a real field the polynomial $X^n + c$ is increasing hence cannot have cycles of length ≥ 2 .

2. We start with an auxiliary result established earlier ([2]) in the case when n is a prime power. Recall that a monic polynomial has bad reduction at a prime ideal $\mathfrak{p}$ if at least one of its coefficients is not $\mathfrak{p}$ -integral.

THEOREM 1. *Let K be an algebraic number field, $\mathbb{Z}_K$ its ring of integers and $f(X) = X^n + c$ with odd n and $c \in K \setminus \mathbb{Z}_K$. If f has in K a cycle of length ≥ 3 and $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ at which f has bad reduction, then either $\mathfrak{p}$ divides $n\mathbb{Z}_K$ or there exists a prime divisor q of n satisfying*

$$N(\mathfrak{p}) \equiv 1 \pmod{q}.$$

To prove it we shall use the following auxiliary result proved in [2, Lemmas 2, 3], which generalizes slightly the results obtained in [3, Corollary 4] and [4, Corollary 6.7].

LEMMA 1. *Let $f(X) = X^n + c$ with $n \geq 2$ and $c \in K \setminus \mathbb{Z}_K$, and assume that $r_1 \mapsto \dots \mapsto r_k \mapsto r_1$ is an f -cycle of length $k \geq 3$ lying in K .*

(i) *Put $I_0 = c\mathbb{Z}_K$, $I_j = r_j\mathbb{Z}_K$, $j = 1, 2, \dots, k$, and write*

$$I_j = A_j B_j^{-1}, \quad j = 0, 1, \dots, k, \quad (1)$$

where $A_0, A_1, \dots, A_k, B_0, B_1, \dots, B_k$ are ideals of $\mathbb{Z}_K$ satisfying $(A_j, B_j) = 1$ for $j = 0, 1, \dots, k$. Then the ideal B_0 is an n th power, say $B_0 = B^n$, and for $j = 1, 2, \dots, k$ one has $B_j = B$.

(ii) *There is a class $\mathbb{X}$ in the class-group of ideals of $\mathbb{Z}_K$ such that if the ideal I lies in $\mathbb{X}$ and is prime to B , then there exist $a, b, N_1, \dots, N_k \in \mathbb{Z}_K$ such that*

$$c = \frac{a}{b^n}, \quad r_j = \frac{N_j}{b}, \quad (a\mathbb{Z}_K, b^n\mathbb{Z}_K) = I^n, \quad (N_j\mathbb{Z}_K, b\mathbb{Z}_K) = I, \quad j = 1, 2, \dots, k.$$

If we extend the sequence N_j by periodicity, putting $N_{j+k} = N_j$ for $j \geq 1$, then one has

$$(N_j\mathbb{Z}_K, B) = 1 \quad (2)$$

and

$$\prod_{i=1}^k \frac{N_{i+1}^n - N_i^n}{N_{i+1} - N_i} = b^{k(n-1)}. \quad (3)$$

We need also the following well-known result.

LEMMA 2. *If $a \neq b$ are non-zero elements of $\mathbb{Z}_K$ and $n \geq 2$, then for every prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ containing $(a^n - b^n)/(a - b)$ either $\mathfrak{p}$ divides both $a\mathbb{Z}_K$ and $b\mathbb{Z}_K$, or $\mathfrak{p} \mid n\mathbb{Z}_K$ and $a \equiv b \pmod{\mathfrak{p}}$, or, finally, for some prime divisor q of n one has $N\mathfrak{p} \equiv 1 \pmod{q}$.*

Proof of Theorem 1. Assume that the polynomial $f(X) = X^n + c$ has a cycle of length $k \geq 3$. Let $N_1, \dots, N_k$ and B be as in Lemma 1. Since $c \notin \mathbb{Z}_K$ we have $B \neq \mathbb{Z}_K$. If $\mathfrak{p}$ is a prime ideal divisor of B , then $b \in \mathfrak{p}$ and the equality (3) implies that for some i we have

$$\frac{N_{i+1}^n - N_i^n}{N_{i+1} - N_i} \in \mathfrak{p}.$$

Lemma 2 implies now that either $\mathfrak{p}$ divides both $N_i\mathbb{Z}_K$ and $N_{i+1}\mathbb{Z}_K$, or $\mathfrak{p} \mid n\mathbb{Z}_K$ and $N_i \equiv N_{i+1} \pmod{\mathfrak{p}}$, or for some prime divisor q of n one has $N(\mathfrak{p}) \equiv 1 \pmod{q}$, but the first possibility is excluded by (2).

3. Let $n \geq 3$ be an odd integer and denote by $\mathcal{X}_n$ the family of algebraic number fields K of finite degree such that almost all (i.e., all except finitely many) prime ideals of the first degree in the ring $\mathbb{Z}_K$ of integers of K have their norms congruent to unity mod q for a prime q dividing n . Note that the family $\mathcal{X}_n$ depends only on the set of prime divisors of n .

Now we state our results.

THEOREM 2. *Let n be an odd integer, let K be an algebraic number field of degree N , not belonging to the family $\mathcal{X}_n$ and let $f(X) = X^n + c$ with $c \in K \setminus \mathbb{Z}_K$.*

(i) *There exists a constant $B = B(K, n)$ depending only on K and n such that the length of a cycle of $f(X)$ lying in K does not exceed $B(k, n)$.*

(ii) *If all prime divisors q of n satisfy $q > 2^N$, then $B(K, n)$ depends only on the degree N of K . If 2 splits in K , then $B(K, n) \leq 2$.*

COROLLARY 1. *If n is odd and $K \notin \mathcal{X}_n$ is an Abelian extension of the rationals of conductor F , then for every $\epsilon > 0$ one has $B(K, n) \ll_{\epsilon} (FM)^{L+1+\epsilon}$, where L denotes the Linnik constant and M is the product of prime divisors of n .*

COROLLARY 2. *If n is a product of two odd prime-powers, K is an algebraic number field of degree N , and for primes $q \mid n$ that field does not contain roots of unity of order q and $f(X) = x^n + c$ with $c \in K \setminus \mathbb{Z}_K$, then the lengths of f -cycles in K are bounded by a value depending only on K and n .*

COROLLARY 3. *If n is odd, $f(X) = X^n + c$ with $c \in K \setminus \mathbb{Z}_K$, where K is an algebraic number field of degree N with trivial class-group, and for primes $q \mid n$ that field does not contain roots of unity of order q , then the lengths of f -cycles in K are bounded by a value depending only on K and n .*

COROLLARY 4. *If n is odd and square-free, $f(X) = X^n + c$ with $c \in K \setminus \mathbb{Z}_K$, where $K \notin \mathcal{X}_n$ is a Galois extension of the rationals of degree N and for prime divisors q of n we have $q - 1 \nmid N$, then the lengths of f -cycles in K are bounded by a value depending only on K and n .*

4. To simplify the check whether a given field K lies in $\mathcal{X}_n$ we provide first a few simple assertions about the family $\mathcal{X}_n$. We shall henceforth assume that every non-zero element of K has a positive norm. This holds in particular in totally complex fields.

LEMMA 3. Let K be an algebraic number field, n a positive rational integer, $\mathfrak{f} = n\mathbb{Z}_K$ and denote by H_n the class-group mod $\mathfrak{f}$. If I, J are two ideals lying in the same class in H_n , then $N(I) \equiv N(J) \pmod{n}$.

Proof. The assumption implies the existence of $a, b \in \mathbb{Z}_K$ with $a \equiv b \pmod{\mathfrak{f}}$, $(ab\mathbb{Z}_K, \mathfrak{f}) = 1$ and $aI = bJ$. Since $\mathfrak{f}$ has a rational generator we have

$$N(a) \equiv N(b) \pmod{n},$$

and since norms of a, b are positive we get

$$N(a)N(I) = N(aI) = N(bJ) = N(b)N(J)$$

and $N(I) \equiv N(J) \pmod{n}$.

COROLLARY 5. If $K \in \mathcal{X}_n$, I is an ideal of $\mathbb{Z}_K$, prime to n , then for some prime divisor q of n one has $N(I) \equiv 1 \pmod{q}$.

Proof. Recall that in every ideal class in H_n there are infinitely many prime ideals of first degree and apply the lemma.

By $\mathcal{Y}_n$ we shall denote the family of all fields K containing $Q(\zeta_q)$ for some prime divisor q of n . One sees easily that one has always $\mathcal{Y}_n \subset \mathcal{X}_n$ and it follows from an old theorem of Bauer [5] that if n is a prime, then $\mathcal{Y}_n = \mathcal{X}_n$. It is not difficult to show that the same is true if n is a product of two primes (see Theorem 3 (i) below), but the following example shows that $\mathcal{X}_n$ may contain fields without any roots of unity $\neq \pm 1$.

Let $K = Q(\zeta_n)$ with $n = 105$ and let k be its subfield corresponding to the subgroup $H = \{1, 29, 34, 41\}$ of the Galois group $G(n) = C_{12} \oplus C_2^2$ of K/Q .

The extension k/Q is of degree 12, has a cyclic Galois group and its discriminant equals $3^6 \cdot 5^9 \cdot 7^{10}$. If p is a rational prime splitting in k , then $p \pmod{105} \in H$, hence $p \equiv 1 \pmod{q}$ for some $q \in \{3, 5, 7\}$, and since every ideal-class mod $105\mathbb{Z}_k$ contains infinitely many such primes p the Corollary 5 implies $k \in \mathcal{X}_{105}$.

5. We show now that in many cases a field contained in $\mathcal{X}_n$ must contain a root of unity ζ_q , q being a prime divisor of n . Recall that we work under the standing assumption that K does not contain elements of negative norm. We need a simple lemma in which we denote by $\omega(m)$ the number of distinct prime divisors of m .

LEMMA 4. Let n be an integer with $\omega(n) \geq 3$. If $K \in \mathcal{X}_n$ and the class-number of K equals 1, then there is a divisor d of n with $\omega(d) < \omega(n)$ such that $K \in \mathcal{X}_d$.

Proof. Put $s = \omega(n)$, let $n = \prod_{j=1}^s p_j^{\alpha_j}$ with prime p_j and $\alpha_j \geq 1$, and put $\mathfrak{f} = n\mathbb{Z}_K$. If the assertion would fail, then by the Corollary 5 there would exist an ideal I in $\mathbb{Z}_K$ with $(I, \mathfrak{f}) = 1$ and

$$N(I) \equiv r_j \pmod{p_j}, \quad j = 1, 2, \dots, s,$$

where $r_1 = 1$ and $r_j \not\equiv 0, 1 \pmod{p_j}$ for $j = 2, 3, \dots, s$. Indeed, otherwise we would have $K \in \mathcal{X}_d$ with $d = n/p_1^{\alpha_1}$.

Since all ideals of K are principal one obtains the existence of $a \in \mathbb{Z}_K$ with $(a\mathbb{Z}_K, \mathfrak{f}) = 1$ and $N(a) \not\equiv 1 \pmod{p_1}$, as otherwise we would have $K \in \mathcal{X}_{p_1}$.

If $b \in \mathbb{Z}_K$ satisfies

$$b \equiv a \pmod{p_1}, \quad b \equiv 1 \pmod{p_2 p_3 \cdots p_s},$$

then

$$N(b) \not\equiv 1 \pmod{p_1}, \quad N(b) \equiv 1 \pmod{p_i}, \quad i = 2, 3, \dots, s,$$

hence with $J = b\mathbb{Z}_K$ one gets $N(IJ) \not\equiv 1 \pmod{p_1}$ and

$$N(IJ) \equiv r_j \not\equiv 1 \pmod{p_j}$$

for $j = 2, 3, \dots, s$, contradicting the Corollary 5.

THEOREM 3. (i) If $q_1 \neq q_2$ are primes and $n = q_1^{a_1} q_2^{a_2}$, then $\mathcal{X}_n = \mathcal{Y}_{q_1} \cup \mathcal{Y}_{q_2}$.

(ii) If $K \in \mathcal{X}_n$ and the class-number of K equals 1, then $K \in \bigcup_{q|n} \mathcal{Y}_q$, q ranging over prime divisors of n .

(iii) If $K \in \mathcal{X}_n$ is a Galois extension of the rationals and the fields K and $Q(\zeta_n)$ are linearly disjoint, then $K \in \bigcup_{q|n} \mathcal{Y}_q$, q ranging over prime divisors of n .

Proof. (i) If $K \in \mathcal{X}_{q_1 q_2} \setminus (\mathcal{Y}_{q_1} \cup \mathcal{Y}_{q_2})$, then there exist ideals $\mathfrak{p}_1, \mathfrak{p}_2$ not dividing $n\mathbb{Z}_K$ and satisfying

$$N(\mathfrak{p}_i) \equiv 1 \pmod{q_i}, \quad i = 1, 2,$$

and

$$N(\mathfrak{p}_1) \not\equiv 1 \pmod{q_2}, \quad N(\mathfrak{p}_2) \equiv 1 \pmod{q_1}.$$

Put $I = \mathfrak{p}_1 \mathfrak{p}_2$ and observe that $N(I) \equiv N(\mathfrak{p}_1)N(\mathfrak{p}_2) \not\equiv 1 \pmod{q_i}$ for $i = 1, 2$, contradicting Corollary 5.

(ii) Downward induction based on Lemma 4 shows the existence of a divisor d of n with $\omega(d) \leq 2$ and $K \in \mathcal{X}_d$. The assertion follows now in the case $d = 1$ from Bauer's theorem, and in the case $d = 2$ from (i).

(iii) It suffices to consider the case of n square-free. Let $n = \prod_{j=1}^s q_j$ be the factorization of n into primes, put $N = [K : Q]$ and for any divisor $d > 1$ of n denote by K_d the field $K(\zeta_d)$. Since K and $Q(\zeta_n)$ are linearly disjoint, the same applies to K and $Q(\zeta_d)$, hence

$$[K_d : Q] = [K : Q][Q(\zeta_d) : Q] = N\varphi(d) = N \prod_{q|d} (q-1).$$

For any extension k/Q denote by $P(k)$ the set of rational primes splitting in it, and let $d(k)$ be its density. Recall that if k/Q is Galois, then $d(k) = 1/[k : Q]$ (see e.g. [6, Corollary 4 to Proposition 7.16]).

For $1 < d \mid n$ denote by X_d the set of rational primes splitting in K/Q and satisfying $p \equiv 1 \pmod{d}$ and observe that X_d coincides with the set of rational primes splitting in both extensions K/Q and $Q(\zeta_d)/Q$, hence $X_d = P(K_d)$.

The density of $P(K)$ equals $1/N$, but using the equalities

$$P(K) = \bigcup_{j=1}^r X_{q_j} \cup F$$

with a finite set F , and

$$d(K_d) = \frac{1}{[K_d : Q]} = \frac{1}{N\varphi(d)}$$

and applying the inclusion-exclusion principle we get

$$\frac{1}{N} = d(K) = - \sum_{1 < d|n} \mu(d)d(K_d) = -\frac{1}{N} \sum_{1 < d|n} \frac{\mu(d)}{\varphi(d)},$$

hence

$$0 = 1 + \sum_{1 < d|n} \frac{\mu(d)}{\varphi(d)} = \prod_{j=1}^s \left(1 - \frac{1}{q_j - 1}\right),$$

and this shows that for a certain prime $q_j \mid n$ we have $q_j = 2$, which is not possible, as n is odd.

6. We shall need a result of Pezda [7].

LEMMA 5. *If R is the ring of integers in a finite extension of the p -adic field and $\mathfrak{P}$ is its unique maximal ideal, then the lengths of cycles in R of polynomials from $R[X]$ are bounded by*

$$A(A-1)p^{1+\log_2 e},$$

where $A = \#R/\mathfrak{P}$ and e is the ramification index, i.e. $pR = \mathfrak{P}^e$.

7. *Proof of Theorem 2.* We may assume that K is totally complex, since if K has a real conjugate K' , then $F(X)$ is increasing on K' , hence there are no K' -cycles of length exceeding 1, and the same applies to K .

Now we shall look for a prime ideal $\mathfrak{P}$ which does not divide B and has its norm bounded by a constant, depending only on n and the field K . By Theorem 1 every prime ideal dividing B either divides n or its norm is congruent to unity mod q for a prime divisor q of n . If all prime divisors of n are larger than 2^N , then prime ideals dividing $2\mathbb{Z}_K$ cannot divide B , as their norms are $\leq 2^N$. If 2 splits in K , then any prime ideal dividing $2\mathbb{Z}_K$ has norm 2, and Theorem 1 implies the cycles of f in K can have at most 2 elements.

If, however, n has a prime ideal divisor of norm $\leq 2^N$, then we have to proceed in another way. The assumption $K \notin \mathcal{X}_n$ implies that there exists a prime ideal $\mathfrak{P}$ of K with $(N(\mathfrak{P})(N(\mathfrak{P})-1), n) = 1$, and obviously its norm can be bounded by a constant depending only on n and K .

Let now L be the $\mathfrak{P}$ -completion of the field K and denote by R its ring of integers. The choice of $\mathfrak{P}$ guarantees that our cycle lies in R and $f \in R[X]$, hence we may apply Lemma 5 to get the assertions (i) and (ii) of the theorem.

Proof of Corollary 1. Assume that the extension K/Q is Abelian and let f be its conductor, i.e. f is the minimal integer satisfying $K \subset Q(\zeta_f)$. If H is the subgroup of the Galois group of $Q(\zeta_f)$ corresponding to K , then a prime p splits in K/Q if and only if $p \pmod{f}$ lies in H .

If $(f, M) = 1$, then choose a prime p satisfying

$$p \equiv 1 \pmod{f}, \quad p \equiv -1 \pmod{M}.$$

Then p splits in K and satisfies $(p(p-1), n) = 1$, hence prime ideals dividing of p in K do not divide B , and Linnik's theorem gives the assertion.

Now assume $(f, M) = d > 1$. If $h \in H$ satisfies $h \equiv 1 \pmod{q}$ for some prime $q \mid d$ and $p \equiv h \pmod{f}$, then $p \equiv 1 \pmod{q}$. In view of $K \notin \mathcal{X}_n$ this cannot hold for all $h \in H$. Therefore there exists $h_0 \in H$ such that for primes $q \mid d$ one has $h_0 \not\equiv 1 \pmod{q}$. Now let D be the product of all prime divisors of M not dividing f and choose a prime p satisfying

$$p \equiv h_0 \pmod{f}, \quad p \equiv -1 \pmod{D}.$$

Observe that p does not divide n , then otherwise p would divide f , hence $p \mid (f, h_0) = 1$. Therefore $(p(p-1), n) = 1$ and since p splits in K , the preceding argument applies and we get the assertion.

Proof of Corollary 2. Apply Theorem 2 (i) and Theorem 3 (i).

Proof of Corollary 3. This follows from Theorem 2 (i) and Theorem 3 (ii).

Proof of Corollary 4. The condition $q-1 \nmid N$ implies the linear disjointness of K and $Q(\zeta_n)$. If K would belong to $\mathcal{X}_n$, then by Theorem 3 (iii) there would exist a prime divisor q of n with $K \in \mathcal{Y}_q$, hence $Q(\zeta_q) \subset K$ and $q-1 \mid N$, contrary to our assumption. It remains to apply Theorem 2 (i).

Bibliography

- [1] P. Morton, J. H. Silverman, “Rational periodic points of rational functions”, *Internat. Math. Res. Notices*, 1994, № 2, 97–109.
- [2] W. Narkiewicz, “Cycle-lengths of a class of monic binomials”, *Funct. Approx. Comment. Math.*, **42**:2 (2010), 163–168.
- [3] R. Walde, P. Russo, “Rational periodic points of the quadratic function $Q_c(x) = x^2 + c$ ”, *Amer. Math. Monthly*, **101**:4 (1994), 318–331.
- [4] G. S. Call, S. W. Goldstine, “Canonical heights on projective space”, *J. Number Theory*, **63**:2 (1997), 211–243.
- [5] M. Bauer, “Zur Theorie der algebraischer Zahlkörper”, *Math. Ann.*, **77**:3 (1916), 353–356.
- [6] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 3rd ed., Springer Monogr. Math., Springer-Verlag, Berlin, 2004.
- [7] T. Pezda, “Polynomial cycles in certain local domains”, *Acta Arith.*, **66**:1 (1994), 11–22.