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Hilbert Space with Reproducing Kernel and Uniform Distribution Preserving Maps. I

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Dedicated to the memory of Anatolii Alekseevich Karatsuba

1. Introduction

A map $\Phi: [0, 1]^s \rightarrow [0, 1]^s$ is called uniform distribution preserving (u.d.p.) map if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, the image $\Phi(\mathbf{x}_n)$ is again u.d. Let H be a Hilbert space of real functions $f(\mathbf{x})$, $\mathbf{x} \in [0, 1]^s$, with a reproducing kernel $K(\mathbf{x}, \mathbf{y})$, and let $\mathbf{x}_0, \dots, \mathbf{x}_{N-1} \in [0, 1]^s$ be an N -terms sequence. In the quasi-Monte Carlo (QMC) integration the following *mean square worst-case error*

$$\int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n \oplus \boldsymbol{\sigma}) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \quad (1)$$

is studied for digital shift $\mathbf{x} \oplus \boldsymbol{\sigma}$.¹ This quantity is the mean value of the L^2 discrepancy in QMC integration in a Hilbert space, see Section 5. For some u.d.p. $\Phi(\mathbf{x})$ the following mean square worst-case error

$$\int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \quad (2)$$

is less than (1). For example, Cristea, Dick, Leobacher and Pillichshammer [1] consider a Sobolev space of functions for which the second partial derivatives are square integrable, van der Corput sequence x_n , $n = 0, 1, \dots, N-1$, with base $b = 2$ and $N = 2^\beta$ and the u.d.p. map² $\Phi_0(x) = 1 - |2x - 1|$. They found (1) as $O(1/N^2)$ and (2) as $O(1/N^4)$.³ A similar result for $x_n = n/N$ and $x_n + \sigma \bmod 1$ instead of $x_n \oplus \sigma$ has previously been shown by Hickernell [2]. The aim of this work is to analyze a role of $\Phi_0(x) = 1 - |2x - 1|$ in this improvement $O(1/N^4)$. Generally, we study an influence of u.d.p. maps $\Phi(\mathbf{x})$ on the mean square worst-case error (2). Applying the method of Fourier–Walsh expansion⁴ we express⁵ the mean square worst-case error (2) in the form

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}_1(\mathbf{k}, \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2, \quad (3)$$

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¹This is also u.d.p. map.

²It called the tent or baker’s transformation.

³Precisely, for any $\varepsilon > 0$ there exists a digital net $x_0, \dots, x_{N-1}$ such that (1) is $O(1/N^{2-\varepsilon})$, and a digital net such that (2) is $O(1/N^{4-\varepsilon})$.

⁴The idea of using Walsh functions stems from Larcher [3].

⁵Partially known formula, see [1; Theorem 4].

where $\widehat{K}_1(\mathbf{k}, \mathbf{k})$ are Fourier–Walsh coefficients of $K(\Phi(\mathbf{x}), \Phi(\mathbf{y}))$. Here $K(\mathbf{x}, \mathbf{y})$, $\Phi(\mathbf{x})$ and $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ are arbitrary. Thus, in (3) the role of kernel, u.d.p. map and sequence are separated. In Section 3 we apply (3) to the $\Phi(\mathbf{x}) = \mathbf{x}$, $\Phi(\mathbf{x}) = \mathbf{b}^\alpha \mathbf{x} \bmod 1$ and to a generalized tent function $\Phi(\mathbf{x})$. In Section 4.1, for dimension $s = 1$, for kernel $K(x, y) = 1 - \max(x, y)$ and for the van der Corput sequence $x_0, x_1, \dots, x_{N-1}$, we give explicit formulas of the mean square worst-case errors. Theorem 8 shows that the best result is attained for shift $x \oplus \sigma$. In Section 4.2⁶ for a Sobolev space with reproducing kernel

$$K(x, y) = 1 + \gamma B_1(x)B_1(y) + \frac{\gamma^2}{4} B_2(x)B_2(y) - \frac{\gamma^2}{24} B_4(|x - y|)$$

we see that the rate $O(1/N^4)$ of mean square worst-case error for the tent u.d.p. map $\Phi_0(x)$ is not only given by special properties of $\Phi_0(x)$, but it is also a consequence of a selection of exactly $N = 2^\beta$ initial terms of the van der Corput sequence. Finally, in Theorem 10, for $N = 2^\beta + 1$ we obtain the mean square worst-case error only $O(1/N^2)$.

Note that in all parts of this paper we carefully compute mean square worst-case errors exactly.

2. Basic objects

2.1. Notations. In this paper we use the following notations:

- s is a dimension of the unit cube $[0, 1]^s$;
- $\mathbf{x} = (x_1, x_2, \dots, x_s)$, $\mathbf{y} = (y_1, y_2, \dots, y_s)$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_s)$ are vectors in $[0, 1]^s$;
- $\mathbf{k} = (k_1, k_2, \dots, k_s)$ is a vector with nonnegative integer coordinates;
- $\mathbf{b} = (b_1, b_2, \dots, b_s)$ is a vector with positive integer coordinates; we call it a vector of bases;
- $\mathbf{b}^\alpha \mathbf{x} \bmod 1 = (b_1^{\alpha_1} x_1, \dots, b_s^{\alpha_s} x_s) \bmod 1$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$ has nonnegative integers coordinates;
- $\mathbf{b}^\alpha \mid \mathbf{k}$ is equivalent to $b_i^{\alpha_i} \mid k_i$ for $i = 1, 2, \dots, s$;
- $\mathbf{k}/\mathbf{b}^\alpha = (k_1/b_1^{\alpha_1}, \dots, k_s/b_s^{\alpha_s})$;
- $x = x_0/b + x_1/b^2 + \dots$ is a b -adic representation of $x \in [0, 1]$;
- $\sigma = \sigma_0/b + \sigma_1/b^2 + \dots$;
- $k = k_0 + k_1b + k_2b^2 + \dots + k_nb^n$ is a b -adic expression of the integer k , $k_n \neq 0$;
- $\text{wal}_k(x) = e^{(2\pi i/b)(k_0x_0 + k_1x_1 + \dots + k_nx_n)}$ is the k -th Walsh function $\text{wal}_k: [0, 1] \rightarrow \mathbb{C}$ in the base b ;
- $\text{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^s \text{wal}_{k_i}(x_i)$;
- $x \oplus \sigma = (x_0 + \sigma_0 \bmod b)/b + (x_1 + \sigma_1 \bmod b)/b^2 + \dots$;
- $\mathbf{x} \oplus \boldsymbol{\sigma} = (x_1 \oplus \sigma_1, x_2 \oplus \sigma_2, \dots, x_s \oplus \sigma_s)$;
- $\mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (\{x_1 + \sigma_1\}, \{x_2 + \sigma_2\}, \dots, \{x_s + \sigma_s\})$, where $\{x\}$ is the fractional part of x ;
- $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ is an N -terms sequence in $[0, 1]^s$;
- $\mathbb{N}_0 = \{0, 1, 2, \dots\}$;
- if $k = k_0 + k_1b + \dots + k_nb^n$ then

$$-k = -k_0 - k_1b - \dots - k_nb^n, \quad \ominus k = k'_0 + k'_1b + \dots + k'_nb^n,$$

where $k_0 + k'_0 = k_1 + k'_1 = \dots = k_n + k'_n = 0 \pmod{b}$.

2.2. Hilbert space with reproducing kernel $K(\mathbf{x}, \mathbf{y})$. The basic properties of Hilbert spaces can be found in monographs [4] and [5]. In this paper we consider the following two Hilbert spaces:

$$H_1 = \left\{ f: [0, 1] \rightarrow \mathbb{R}, f \text{ absolutely continuous, } f(1) = 0, \int_0^1 (f'(x))^2 dx \text{ exists} \right\}$$

⁶Using a Fourier–Walsh expansion of the kernel $K(\Phi_0(x), \Phi_0(y))$ in [1; p. 447].

[5; pp. 30–31] with scalar product $f(x) \cdot g(x) = \int_0^1 f'(x)g'(x) dx$ and reproducing kernel $K(x, y) = 1 - \max(x, y)$;

$$H_2 = \left\{ f: [0, 1] \rightarrow \mathbb{R}, f \text{ absolutely continuous, } \int_0^1 (f''(x))^2 dx \text{ exists} \right\}$$

[1] with scalar product

$$f(x) \cdot g(x) = \int_0^1 f(x) dx \int_0^1 g(x) dx + \frac{1}{\gamma} (f(1) - f(0))(g(1) - g(0)) + \frac{1}{\gamma^2} \int_0^1 f''(x)g''(x) dx$$

and reproducing kernel

$$K(x, y) = 1 + \gamma B_1(x)B_1(y) + \frac{\gamma^2}{4} B_2(x)B_2(y) - \frac{\gamma^2}{24} B_4(|x - y|),$$

where B_n is the n -th Bernoulli polynomial⁷ and $\gamma \in [0, 1]$ is the weight. This H_2 is called the weighted Sobolev space of the order $\alpha = 2$.

2.3. Uniform distribution preserving map $\Phi: [0, 1]^s \rightarrow [0, 1]^s$. Map $\Phi: [0, 1]^s \rightarrow [0, 1]^s$ is called *uniform distribution preserving* (u.d.p.) *map* if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n, n = 1, 2, \dots$, the image $\Phi(\mathbf{x}_n)$ is again u.d. For one-dimensional case basic properties of u.d.p. maps can be found in [6; 2.1, (VII)] and [7; 2.5.1].

For example, if $\Phi(x), \Psi(x)$ are u.d.p. transformations and α is a real number, then $\Psi(\Phi(x)), 1 - \Phi(x), \Phi(x) + \alpha \bmod 1$ are also u.d.p. Also, as we mentioned in the Introduction,

$$\Phi_0(x) = 1 - |2x - 1|, \quad x \in [0, 1], \quad (4)$$

is u.d.p. map called *tent map* or *baker's map*, see [2].

Examples of multi-dimensional u.d.p. functions:

- (i) $\Phi(\mathbf{x}) = \mathbf{x} \oplus \boldsymbol{\sigma}$;
- (ii) $\Phi(\mathbf{x}) = (\Phi_1(x_1), \dots, \Phi_s(x_s))$, where $\Phi_n(x)$ are one-dimensional u.d.p. map;
- (iii) $\Phi(\mathbf{x}) = \mathbf{b}^\alpha \mathbf{x} \bmod 1 = (b_1^{\alpha_1} x_1, \dots, b_s^{\alpha_s} x_s) \bmod 1$;
- (iv) $\Phi(\mathbf{x}) = \mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (x_1 + \sigma_1, \dots, x_s + \sigma_s) \bmod 1$;
- (v) $\Phi(\mathbf{x}) = (A\mathbf{x})^T \bmod 1$, where A is an $s \times s$ nonsingular integer matrix, cf. Steinerberger [8; Theorem 2];
- (vi) $\Phi(\mathbf{x}) = \pi(\mathbf{x})$, where $\pi(\mathbf{x}) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ is a permutation.

The main criterion of u.d.p. map directly follows from Weyl's limit relation (see [7; pp. 1–62]).

THEOREM 1. *A map $\Phi(\mathbf{x})$ is u.d.p. if and only if $\Phi(\mathbf{x})$ is Riemann integrable and for every Riemann integrable $f: [0, 1]^s \rightarrow \mathbb{R}$ we have*

$$\int_{[0, 1]^s} f(\Phi(\mathbf{x})) d\mathbf{x} = \int_{[0, 1]^s} f(\mathbf{x}) d\mathbf{x}. \quad (5)$$

3. General results

3.1. The mean square worst-case error and u.d.p. maps $\Psi(\mathbf{x}, \boldsymbol{\sigma})$. By the Sloan and Woźniakowski [9] formula the square worst-case quasi-Monte Carlo error can be expressed as

$$\begin{aligned} & \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0, 1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 \\ &= \int_{[0, 1]^s} \int_{[0, 1]^s} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} - \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0, 1]^s} K(\mathbf{x}_n, \mathbf{y}) d\mathbf{y} + \frac{1}{N^2} \sum_{n, m=0}^{N-1} K(\mathbf{x}_m, \mathbf{x}_n). \end{aligned} \quad (6)$$

⁷Bernoulli polynomials: $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$, $\dots$

Replacing the sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in (6) by $\Psi(\mathbf{x}_0, \boldsymbol{\sigma}), \dots, \Psi(\mathbf{x}_{N-1}, \boldsymbol{\sigma})$, where $\Psi(\mathbf{x}, \boldsymbol{\sigma})$ is a u.d.p. map with respect to both $\mathbf{x}$ and $\boldsymbol{\sigma}$, and integrating (6) by $\boldsymbol{\sigma}$ the mean square worst-case error can be expressed as

$$\begin{aligned} & \int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Psi(\mathbf{x}_n, \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \\ &= \frac{1}{N^2} \sum_{n,m=0}^{N-1} \int_{[0,1]^s} K(\Psi(\mathbf{x}_m, \boldsymbol{\sigma}), \Psi(\mathbf{x}_n, \boldsymbol{\sigma})) d\boldsymbol{\sigma} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \end{aligned} \quad (7)$$

This follows from

$$\frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} \int_{[0,1]^s} K(\Psi(\mathbf{x}_n, \boldsymbol{\sigma}), \mathbf{y}) d\mathbf{y} d\boldsymbol{\sigma} = 2 \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}, \quad (8)$$

which holds for every u.d.p. map $\Psi(\mathbf{x}, \boldsymbol{\sigma})$ and arbitrary $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in $[0, 1]^s$, where we apply the u.d.p. characteristic integral equation (5).

3.2. The mean square worst-case error expressed by Fourier–Walsh coefficients. For u.d.p. function of the type $\mathbf{x} \oplus \boldsymbol{\sigma}$ we compute (7) analogously as in [2] using Fourier–Walsh expansion of the kernel $K(\mathbf{x}, \mathbf{y})$, see [5; Theorem 12.4]:⁸

$$K(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \hat{K}(\mathbf{k}, \mathbf{k}') \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}'}(\mathbf{y}),$$

where

$$\hat{K}(\mathbf{k}, \mathbf{k}') = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}'}(\mathbf{y})} d\mathbf{x} d\mathbf{y}. \quad (9)$$

THEOREM 2. *For every sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in the unit cube $[0, 1]^s$ and an arbitrary kernel $K(\mathbf{x}, \mathbf{y})$ with Fourier–Walsh expansion (9)⁹ we have*

$$\int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n \oplus \boldsymbol{\sigma}) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \hat{K}(\mathbf{k}, \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2, \quad (10)$$

where

$$\hat{K}(\mathbf{k}, \mathbf{k}) = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{y})} d\mathbf{x} d\mathbf{y}.$$

Proof. Using the following elementary properties of Walsh functions

$$\begin{aligned} \text{wal}_{\mathbf{k}}(\mathbf{x} \oplus \boldsymbol{\sigma}) &= \text{wal}_{\mathbf{k}}(\mathbf{x}) \text{wal}_{\mathbf{k}}(\boldsymbol{\sigma}), \\ \text{wal}_{\mathbf{k} \oplus \mathbf{k}'}(\mathbf{x}) &= \text{wal}_{\mathbf{k}}(\mathbf{x}) \text{wal}_{\mathbf{k}'}(\mathbf{x}), \\ \int_{[0,1]^s} \overline{\text{wal}_{\mathbf{k}}(\boldsymbol{\sigma})} \text{wal}_{\mathbf{k}'}(\boldsymbol{\sigma}) d\boldsymbol{\sigma} &= \begin{cases} 0 & \text{if } \mathbf{k} \neq \mathbf{k}', \\ 1 & \text{if } \mathbf{k} = \mathbf{k}', \end{cases} \end{aligned}$$

⁸In the literature (see [5; p. 364]) it is preferably used opposite

$$K(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \hat{K}(\mathbf{k}, \mathbf{k}') \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}'}(\mathbf{y})}.$$

⁹Assuming that the convergence rate of (9) admits integration by terms.

we express

$$\begin{aligned}
& \int_{[0,1]^s} K(\mathbf{x} \oplus \boldsymbol{\sigma}, \mathbf{y} \oplus \boldsymbol{\sigma}) d\boldsymbol{\sigma} \\
&= \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \widehat{K}(\mathbf{k}, \mathbf{k}') \int_{[0,1]^s} \overline{\text{wal}_{\mathbf{k}}(\mathbf{x} \oplus \boldsymbol{\sigma})} \text{wal}_{\mathbf{k}'}(\mathbf{y} \oplus \boldsymbol{\sigma}) d\boldsymbol{\sigma} \\
&= \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \widehat{K}(\mathbf{k}, \mathbf{k}') \begin{cases} 0 & \text{if } \mathbf{k} \neq \mathbf{k}', \\ \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}}(\mathbf{y}) & \text{if } \mathbf{k} = \mathbf{k}'. \end{cases} \quad (11)
\end{aligned}$$

Thus in (7) we have

$$\int_{[0,1]^s} K(\mathbf{x}_m \oplus \boldsymbol{\sigma}, \mathbf{x}_n \oplus \boldsymbol{\sigma}) d\boldsymbol{\sigma} = \sum_{\mathbf{k} \in \mathbb{N}_0^s} \widehat{K}(\mathbf{k}, \mathbf{k}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{x}_m)} \text{wal}_{\mathbf{k}}(\mathbf{x}_n)$$

and $\widehat{K}(\mathbf{0}, \mathbf{0}) = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$, and the result (10) follows.

In the following theorem we extend [1; Theorem 4] or [5; Theorem 12.7], which was proved for Sobolev weighted spaces.

THEOREM 3. *For every sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in the unit cube $[0,1]^s$ and every u.d.p. map $\Phi(\mathbf{x})$ and an arbitrary kernel $K(\mathbf{x}, \mathbf{y})$ with Fourier–Walsh expansion (9) we have*

$$\int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}_1(\mathbf{k}, \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2, \quad (12)$$

where

$$\widehat{K}_1(\mathbf{k}, \mathbf{k}) = \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{y})} d\mathbf{x} d\mathbf{y}.$$

Proof. We express $K(\Phi(\mathbf{x}), \Phi(\mathbf{y}))$ to the Fourier–Walsh series

$$K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) = \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \widehat{K}_1(\mathbf{k}, \mathbf{k}') \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}'}(\mathbf{y}),$$

where

$$\widehat{K}_1(\mathbf{k}, \mathbf{k}') = \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}'}(\mathbf{y})} d\mathbf{x} d\mathbf{y}$$

and as in (11) we express¹⁰

$$\begin{aligned}
& \int_{[0,1]^s} K(\Phi(\mathbf{x} \oplus \boldsymbol{\sigma}), \Phi(\mathbf{y} \oplus \boldsymbol{\sigma})) d\boldsymbol{\sigma} \\
&= \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \widehat{K}_1(\mathbf{k}, \mathbf{k}') \int_{[0,1]^s} \overline{\text{wal}_{\mathbf{k}}(\mathbf{x} \oplus \boldsymbol{\sigma})} \text{wal}_{\mathbf{k}'}(\mathbf{y} \oplus \boldsymbol{\sigma}) d\boldsymbol{\sigma} = \sum_{\mathbf{k} \in \mathbb{N}_0^s} \widehat{K}_1(\mathbf{k}, \mathbf{k}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}}(\mathbf{y})
\end{aligned}$$

and then by (7) now we find (12).

¹⁰ In [1] or [5; Theorem 12.7] the authors use a new kernel

$$K^*(\mathbf{x}, \mathbf{y}) = \int_{[0,1]^s} K(\Phi(\mathbf{x} \oplus \boldsymbol{\sigma}), \Phi(\mathbf{y} \oplus \boldsymbol{\sigma})) d\boldsymbol{\sigma}.$$

We have $\widehat{K}^*(\mathbf{k}, \mathbf{k}) = \widehat{K}_1(\mathbf{k}, \mathbf{k})$ and thus Theorem 3 is a consequence of Theorem 2.

3.3. U.d.p. map $\Phi(\mathbf{x} \oplus \sigma)$, where $\Phi(\mathbf{x}) = \mathbf{b}^\alpha \mathbf{x} \bmod 1$. Here $\mathbf{x} \oplus \sigma$ and $\text{wal}_{\mathbf{k}}(\mathbf{x})$ are in the same base $\mathbf{b}$.

THEOREM 4. For $\Phi(\mathbf{x}) = \mathbf{b}^\alpha \mathbf{x} \bmod 1$ and for a kernel $K(\mathbf{x}, \mathbf{y})$ of the type

$$K(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^s K_j(x_j, y_j)$$

and for an arbitrary sequence $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ in $[0, 1]^s$ we have

$$\begin{aligned} & \int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \sigma)) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\sigma \\ &= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}, \mathbf{b}^\alpha | \mathbf{k}}} \hat{K}\left(\frac{\mathbf{k}}{\mathbf{b}^\alpha}, \frac{\mathbf{k}}{\mathbf{b}^\alpha}\right) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \end{aligned} \quad (13)$$

$$= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \hat{K}(\mathbf{k}, \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\Phi(\mathbf{x}_n)) \right|^2 \quad (14)$$

where

$$\hat{K}(\mathbf{k}, \mathbf{k}) = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{y})} d\mathbf{x} d\mathbf{y}.$$

Proof. We use Theorem 3. For this we need to compute Fourier–Walsh series

$$K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) = \sum_{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s} \hat{K}_1(\mathbf{k}, \mathbf{k}') \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}'}(\mathbf{y}),$$

what follows from the one-dimensional Fourier–Walsh series for the kernel

$$K(\Phi(x), \Phi(y)) = \sum_{k, k' \in \mathbb{N}_0} \hat{K}_1(k, k') \overline{\text{wal}_k(x)} \text{wal}_{k'}(y),$$

where u.d.p. $\Phi(x) = b^i x \bmod 1$. For coefficient $\hat{K}_1(k, k')$ we have

$$\begin{aligned} \hat{K}_1(k, k') &= \int_0^1 \int_0^1 K(\Phi(x), \Phi(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \begin{cases} K\left(\frac{k}{b^i}, \frac{k'}{b^i}\right) = \int_0^1 \int_0^1 K(x, y) \text{wal}_{k/b^i}(x) \overline{\text{wal}_{k'/b^i}(y)} dx dy & \text{if } b^i \mid k \text{ and } b^i \mid k', \\ 0 & \text{if } b^i \nmid k \text{ or } b^i \nmid k'. \end{cases} \end{aligned} \quad (15)$$

Proof of (15). We have

$$\begin{aligned} & \int_0^1 \int_0^1 K(\Phi(x), \Phi(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \sum_{\alpha=0}^{b^i-1} \sum_{\beta=0}^{b^i-1} \int_{\alpha/b^i}^{(\alpha+1)/b^i} \int_{\beta/b^i}^{(\beta+1)/b^i} K(b^i x - \alpha, b^i y - \beta) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \end{aligned}$$

and by substitutions $b^i x - \alpha = u$, $b^i x - \beta = v$ we find

$$\sum_{\alpha=0}^{b^i-1} \sum_{\beta=0}^{b^i-1} \int_0^1 \int_0^1 K(u, v) \frac{1}{b^{2i}} \text{wal}_k \left(\frac{u + \alpha}{b^i} \right) \overline{\text{wal}_{k'} \left(\frac{v + \beta}{b^i} \right)} du dv.$$

For

$$\frac{\alpha}{b^i} = \frac{\alpha_0}{b} + \frac{\alpha_1}{b^2} + \cdots + \frac{\alpha_{i-1}}{b^i},$$

$$u = \frac{u_0}{b} + \frac{u_1}{b^2} + \cdots, \quad k = k_0 + k_1 b + \cdots + k_{n-1} b^{n-1}$$

we have

$$\frac{u + \alpha}{b^i} = \frac{\alpha_0}{b} + \frac{\alpha_1}{b^2} + \cdots + \frac{\alpha_{i-1}}{b^i} + \frac{u_0}{b^{i+1}} + \frac{u_1}{b^{i+2}} + \cdots,$$

which for Walsh function $\text{wal}_k(x)$ in base b gives

$$\begin{aligned} \text{wal}_k \left(\frac{u + \alpha}{b^i} \right) &= e^{(2\pi i/b)(k_0 \alpha_0 + \cdots + k_{i-1} \alpha_{i-1} + k_i u_0 + \cdots + k_{n-1} u_{n-1-i})} \\ &= e^{(2\pi i/b)(k_0 \alpha_0 + \cdots + k_{i-1} \alpha_{i-1})} \text{wal}_{[k/b^i]}(u), \end{aligned}$$

where $[k/b^i]$ is the integer part of k/b^i . Thus

$$\begin{aligned} &\sum_{\alpha=0}^{b^i-1} \sum_{\beta=0}^{b^i-1} \int_0^1 \int_0^1 K(u, v) \frac{1}{b^{2i}} \text{wal}_k \left(\frac{u + \alpha}{b^i} \right) \overline{\text{wal}_{k'} \left(\frac{v + \beta}{b^i} \right)} du dv \\ &= \frac{1}{b^{2i}} \sum_{\alpha=0}^{b^i-1} e^{(2\pi i/b)(k_0 \alpha_0 + \cdots + k_{i-1} \alpha_{i-1})} \sum_{\beta=0}^{b^i-1} e^{-(2\pi i/b)(k'_0 \beta_0 + \cdots + k'_{i-1} \beta_{i-1})} \\ &\quad \times \int_0^1 \int_0^1 K(u, v) \text{wal}_{[k/b^i]}(u) \overline{\text{wal}_{[k'/b^i]}(v)} du dv. \end{aligned}$$

Since

$$\sum_{\alpha=0}^{b^i-1} e^{(2\pi i/b)(k_0 \alpha_0 + \cdots + k_{i-1} \alpha_{i-1})} = \left(\sum_{\alpha_0=0}^{b-1} e^{(2\pi i/b)k_0 \alpha_0} \right) \cdots \left(\sum_{\alpha_{i-1}=0}^{b-1} e^{(2\pi i/b)k_{i-1} \alpha_{i-1}} \right) \quad (16)$$

and the right hand side of (16) is non-zero if and only if $b \mid k_0, \dots, b \mid k_{i-1}$, which implies $k_0 = \cdots = k_{i-1} = 0$, and this is equivalent $b^i \mid k$, which gives (15).

Now, using (15), assumption $K(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^s K_j(x_j, y_j)$ and that u.d.p. map has the form $\Phi(\mathbf{x}) = (\Phi_1(x_1), \dots, \Phi_s(x_s))$, then for Fourier-Walsh coefficients $\hat{K}_1(\mathbf{k}, \mathbf{k}')$ of $K(\Phi(\mathbf{x}), \Phi(\mathbf{y}))$ we have

$$\begin{aligned} \hat{K}_1(\mathbf{k}, \mathbf{k}') &= \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}'}(\mathbf{y})} d\mathbf{x} d\mathbf{y} \\ &= \prod_{j=1}^s \hat{K}_{1,j}(x_j, y_j) = \prod_{j=1}^s \hat{K}_j \left(\frac{x_j}{b_j^{\alpha_j}}, \frac{y_j}{b_j^{\alpha_j}} \right) = \hat{K} \left(\frac{\mathbf{k}}{\mathbf{b}^\alpha}, \frac{\mathbf{k}'}{\mathbf{b}^\alpha} \right) \end{aligned} \quad (17)$$

if $\mathbf{b}^\alpha \mid \mathbf{k}$ and $\mathbf{b}^\alpha \mid \mathbf{k}'$. Others $\hat{K}_1(\mathbf{k}, \mathbf{k}') = 0$. This implies

$$K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) = \sum_{\substack{\mathbf{k}, \mathbf{k}' \in \mathbb{N}_0^s \\ \mathbf{b}^\alpha \mid \mathbf{k}, \mathbf{b}^\alpha \mid \mathbf{k}'}} \hat{K} \left(\frac{\mathbf{k}}{\mathbf{b}^\alpha}, \frac{\mathbf{k}'}{\mathbf{b}^\alpha} \right) \overline{\text{wal}_{\mathbf{k}}(\mathbf{x})} \text{wal}_{\mathbf{k}'}(\mathbf{y}).$$

Then by (12) from (17) follows (13).

The expression (14) directly follows from that $\mathbf{b}^\alpha \mid \mathbf{k}$ implies

$$\text{wal}_{\mathbf{k}}(\mathbf{x}) = \text{wal}_{\mathbf{k}/\mathbf{b}^\alpha}(\mathbf{b}^\alpha \mathbf{x} \bmod 1). \quad (18)$$

Proof of (18). Assume that for $s = 1$

$$k = k_0 + k_1 b + \cdots + k_n b^n, \quad x = \frac{x_0}{b} + \frac{x_1}{b^2} + \cdots,$$

then $\text{wal}_k(x) = e^{(2\pi i/b)(k_0 x_0 + k_1 x_1 + \cdots + k_n x_n)}$, $b^i \mid k$ equal to $k_0 = k_1 = \cdots = k_{i-1} = 0$,

$$b^i x = x_0 b^{i-1} + x_1 b^{i-2} + \cdots + x_{i-1} + \frac{x_i}{b} + \cdots,$$

then

$$e^{(2\pi i/b)(k_i x_i + k_{i+1} x_{i+1} + \cdots)} = \text{wal}_{k/b^i}(\{b^i x\}).$$

This completes the proof of Theorem 4.

EXAMPLE. Let $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ be a sequence in $[0, 1]^s$ such that every coordinate $x_{i,n}$ of $\mathbf{x}_n$ has the form

$$x_{i,n} = \frac{a_{i,n}}{b_i^{\alpha_i}}, \quad 0 \leq a_{i,n} < b_i^{\alpha_i}, \quad a_{i,n} \in \mathbb{N}_0. \quad (19)$$

Then (13) has the form

$$\begin{aligned} & \int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \\ &= \int_{[0,1]^s} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \end{aligned} \quad (20)$$

Proof. For an arbitrary term $\mathbf{x}_n$ of the sequence (19) we have $(\mathbf{b}^\alpha \mathbf{x}_n \bmod 1) = \mathbf{0}$ and thus for an arbitrary $\mathbf{k}$ we have $\text{wal}_{\mathbf{k}}(\mathbf{b}^\alpha \mathbf{x}_n \bmod 1) = 1$. Thus, for the sum (13) we have

$$\begin{aligned} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}(\mathbf{k}, \mathbf{k}) &= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}(\mathbf{k}, \mathbf{k}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{0})} \text{wal}_{\mathbf{k}}(\mathbf{0}) \\ &= \int_{[0,1]^s} K(\mathbf{0} \oplus \boldsymbol{\sigma}, \mathbf{0} \oplus \boldsymbol{\sigma}) d\boldsymbol{\sigma} - \widehat{K}(\mathbf{0}, \mathbf{0}) = \int_{[0,1]^s} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} - \widehat{K}(\mathbf{0}, \mathbf{0}) \end{aligned} \quad (21)$$

and $\widehat{K}(\mathbf{0}, \mathbf{0}) = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$.

REMARK 1. Hickernell [10] notes that the mean square worst-case error with respect to all sequences $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in $[0, 1]^s$ becomes

$$\begin{aligned} & \int_{[0,1]^{sN}} \left(\sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 \right) d\mathbf{x}_0 \dots d\mathbf{x}_{N-1} \\ &= \frac{1}{N} \left(\int_{[0,1]^s} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \right). \end{aligned} \quad (22)$$

Note that the error (20) of the sequence (19) is much worse than the mean square worst-case error in (22).

3.4. U.d.p. map $\Phi(\mathbf{x} \oplus \sigma)$, where $\Phi(\mathbf{x})$ is the tent map. The simple tent map $\Phi_0(x)$ in (4) can be extended to the map $\Phi(x)$ with the graph in Fig. 1 and defined by following: Putting

$$I_i = \left[\frac{i-1}{b}, \frac{i}{b} \right), \quad i = 1, 2, \dots, b,$$

we define

$$\Phi(x) = \begin{cases} \Phi_i(x) = bx - (i-1) & \text{if } x \in I_i, 2 \mid i, \\ \Psi_i(x) = -bx + i & \text{if } x \in I_i, 2 \nmid i. \end{cases} \quad (23)$$

Fig. 1. Graph of $\Phi(x)$ tent function

Remind that if $k = k_0 + k_1b + \dots + k_nb^n$, then

$$-k = -k_0 - k_1b - \dots - k_nb^n, \quad \ominus k = k'_0 + k'_1b + \dots + k'_nb^n,$$

where $k_0 + k'_0 = k_1 + k'_1 = \dots = k_n + k'_n = 0 \pmod{b}$. If

$$x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots + \frac{x_n}{b^{n+1}} + \dots,$$

then

$$\text{wal}_{-k}(x) = e^{(2\pi i/b)(-k_0x_0 - k_1x_1 - \dots - k_nx_n)} = \text{wal}_{\ominus k}(x) = \overline{\text{wal}_k(x)} \quad (24)$$

and in the following we use, e.g., $\widehat{K}(k, -k)$ instead of $\widehat{K}(k, \ominus k)$.

THEOREM 5. *Let $K(x, y)$ be an arbitrary continuous kernel with a Fourier–Walsh series*

$$K(x, y) = \sum_{k, k' \in \mathbb{N}_0} \widehat{K}(k, k') \overline{\text{wal}_k(x)} \text{wal}_{k'}(y)$$

in base b , $2 \mid b$, and let $\Phi(x)$ be the tent function (23). Then for every sequence $x_0, \dots, x_{N-1}$ in the unit interval $[0, 1)$ we have

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \sum_{k=1}^{\infty} \widehat{K}_1(k, k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2,$$

where¹¹

$$\begin{aligned} \widehat{K}_1(k, k) = \frac{1}{4} \left(\widehat{K} \left(\left[\frac{k}{b} \right], \left[\frac{k}{b} \right] \right) + \widehat{K} \left(-\left[\frac{k}{b} \right], -\left[\frac{k}{b} \right] \right) + \widehat{K} \left(\left[\frac{k}{b} \right], -\left[\frac{k}{b} \right] \right) e^{(2\pi i/b)(k_0+k_1+\dots+k_n)} \right. \\ \left. + \widehat{K} \left(-\left[\frac{k}{b} \right], \left[\frac{k}{b} \right] \right) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} \right) \end{aligned} \quad (25)$$

for $(b/2) \mid k$ and $\widehat{K}_1(k, k) = 0$ otherwise. Here $k = k_0 + k_1b + \dots + k_nb^n$, $k_n \neq 0$ and $x \oplus \sigma$ and $\Phi(x)$ are in the same even base b .

Proof. We start with Fourier–Walsh expansion

$$K(\Phi(x), \Phi(y)) = \sum_{k, k' \in \mathbb{N}_0} \widehat{K}_1(k, k') \overline{\text{wal}_k(x)} \text{wal}_{k'}(y),$$

where

$$\widehat{K}_1(k, k') = \int_0^1 \int_0^1 K(\Phi(x), \Phi(y)) \overline{\text{wal}_k(x)} \text{wal}_{k'}(y) dx dy.$$

Then by the graph in Fig. 1 of the tent function $\Phi(x)$ we express

$$\begin{aligned} \int_0^1 \int_0^1 K(\Phi(x), \Phi(y)) \overline{\text{wal}_k(x)} \text{wal}_{k'}(y) dx dy \\ = \sum_{\substack{i,j=1 \\ i,j \text{ odd}}}^b \iint_{I_i \times I_j} + \sum_{\substack{i,j=1 \\ i,j \text{ even}}}^b \iint_{I_i \times I_j} + \sum_{\substack{i,j=1 \\ i \text{ odd}, j \text{ even}}}^b \iint_{I_i \times I_j} + \sum_{\substack{i,j=1 \\ i \text{ even}, j \text{ odd}}}^b \iint_{I_i \times I_j}. \end{aligned} \quad (26)$$

Thus, we have distinguish four cases, namely I–IV.

I) If i, j both odd, the first integral in (26) is of the form

$$\begin{aligned} \iint_{I_i \times I_j} K(\Phi_i(x), \Phi_j(y)) \overline{\text{wal}_k(x)} \text{wal}_{k'}(y) dx dy \\ = \int_0^1 \int_0^1 K(x, y) \overline{\text{wal}_k(\Phi_i^{-1}(x))} \text{wal}_{k'}(\Phi_j^{-1}(y)) |d\Phi_i^{-1}(x) d\Phi_j^{-1}(y)| \\ = \int_0^1 \int_0^1 K(x, y) e^{(2\pi i/b)k_0(i-1)} \overline{\text{wal}_{[k/b]}(x)} e^{(2\pi i/b)k'_0(j-1)} \overline{\text{wal}_{[k'/b]}(x)} \frac{dx}{b} \frac{dy}{b}. \end{aligned} \quad (27)$$

In (27), for $k = k_0 + k_1b + k_2b^2 + \dots + k_nb^n$, we use

$$\overline{\text{wal}_k(\Phi_i^{-1}(x))} = e^{(2\pi i/b)k_0(i-1)} \overline{\text{wal}_{[k/b]}(x)}. \quad (28)$$

Proof of (28). By Fig. 2, $\text{wal}_k(\Phi_i^{-1}(x)) = \text{wal}_k((i-1)/b + x/b)$ and from

$$\begin{aligned} x = \frac{x_0}{b} + \frac{x_1}{b^2} + \frac{x_2}{b^3} + \dots, \quad \frac{x}{b} = \frac{x_0}{b^2} + \frac{x_1}{b^3} + \frac{x_2}{b^4} + \dots, \\ \frac{i-1}{b} + \frac{x}{b} = \frac{i-1}{b} + \frac{x_0}{b^2} + \frac{x_1}{b^3} + \frac{x_2}{b^4} + \dots, \end{aligned}$$

¹¹Alternately

$$\widehat{K}_1(k, k) = \frac{1}{2} \left(\widehat{K} \left(\left[\frac{k}{b} \right], \left[\frac{k}{b} \right] \right) + \text{Re} \left(\widehat{K} \left(\left[\frac{k}{b} \right], -\left[\frac{k}{b} \right] \right) e^{(2\pi i/b)(k_0+k_1+\dots+k_n)} \right) \right).$$

Fig. 2. $\Phi_i(x)$ and inverse function of $\Phi_i(x)$

and by the definition of Walsh functions we have

$$\begin{aligned} \text{wal}_k\left(\frac{i-1}{b} + \frac{x}{b}\right) &= e^{(2\pi i/b)(k_0(i-1)+k_1x_0+k_2x_1+k_3x_2+\dots)} \\ &= e^{(2\pi i/b)k_0(i-1)} e^{(2\pi i/b)(k_1x_0+k_2x_1+\dots+k_nx_{n-1})} = e^{(2\pi i/b)k_0(i-1)} \text{wal}_{[k/b]}(x). \end{aligned}$$

Summing (27) over $\sum_{\substack{i,j=1 \\ i,j \text{ odd}}}^b$ and from

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^b e^{(2\pi i/b)k_0(i-1)} = \begin{cases} \frac{b}{2} & \text{if } \frac{b}{2} \mid k_0 \quad \left(\Leftrightarrow \frac{b}{2} \mid k\right), \\ 0 & \text{otherwise,} \end{cases}$$

we find

$$\begin{aligned} &\sum_{\substack{i,j=1 \\ i,j \text{ odd}}}^b \iint_{I_i \times I_j} K(\Phi_i(x), \Phi_j(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \begin{cases} \frac{1}{4} \widehat{K}\left(\left[\frac{k}{b}\right], \left[\frac{k'}{b}\right]\right) & \text{if } \frac{b}{2} \mid k, \frac{b}{2} \mid k', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (29)$$

This gives the first term in the parenthesis (25).

II) If i, j both even, then

$$\begin{aligned} &\iint_{I_i \times I_j} K(\Psi_i(x), \Psi_j(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \int_0^1 \int_0^1 K(x, y) \text{wal}_k(\Psi_i^{-1}(x)) \overline{\text{wal}_{k'}(\Psi_j^{-1}(y))} |d\Psi_i^{-1}(x) d\Psi_j^{-1}(y)| \\ &= \int_0^1 \int_0^1 K(x, y) e^{(2\pi i/b)k_0(i-1)} e^{-(2\pi i/b)(k_1+k_2+\dots+k_n)} \overline{\text{wal}_{[k/b]}(x)} \\ &\quad \times e^{-(2\pi i/b)k'_0(j-1)} e^{(2\pi i/b)(k'_1+k'_2+\dots+k'_{n'})} \text{wal}_{[k'/b]}(y) \left| \frac{dx}{(-b)} \frac{dy}{(-b)} \right|. \end{aligned} \quad (30)$$

In (30) we use

$$\text{wal}_k(\Psi_i^{-1}(x)) = e^{(2\pi i/b)k_0(i-1)} e^{-(2\pi i/b)(k_1+k_2+\dots+k_n)} \overline{\text{wal}_{[k/b]}(x)}. \quad (31)$$

Proof of (31). By Fig. 3, $\text{wal}_k(\Psi_i^{-1}(x)) = \text{wal}_k(i/b - x/b)$, and from

$$\begin{aligned} x &= \frac{x_0}{b} + \frac{x_1}{b^2} + \frac{x_2}{b^3} + \dots, \\ \frac{i}{b} - \frac{x}{b} &= \frac{i}{b} - \frac{x_0}{b^2} - \frac{x_1}{b^3} - \frac{x_2}{b^4} - \dots \\ &= \frac{i-1}{b} + \left(\frac{1}{b} - \frac{x_0}{b} - \frac{x_1}{b^2} - \frac{x_2}{b^3} - \dots \right) \\ &= \frac{i-1}{b} + \left(\frac{b-1}{b^2} + \frac{b-1}{b^3} + \frac{b-1}{b^4} + \dots - \frac{x_0}{b} - \frac{x_1}{b^2} - \frac{x_2}{b^3} - \dots \right) \\ &= \frac{i-1}{b} + \left(\frac{b-1-x_0}{b^2} + \frac{b-1-x_1}{b^3} + \frac{b-1-x_2}{b^4} + \dots \right), \end{aligned}$$

and by the definition of Walsh functions we have

$$\begin{aligned} \text{wal}_k\left(\frac{i}{b} - \frac{x}{b}\right) &= e^{(2\pi i/b)(k_0(i-1)+k_1(b-1-x_0)+k_2(b-1-x_1)+\dots)} \\ &= e^{(2\pi i/b)k_0(i-1)} e^{-(2\pi i/b)(k_1+k_2+\dots+k_n)} \overline{\text{wal}_{[k/b]}(x)}. \end{aligned}$$

Fig. 3. $\Psi_i(x)$ and inverse function of $\Psi_i(x)$

Submitting in (30) the sum

$$\sum_{\substack{i=1 \\ i \text{ even}}}^b e^{(2\pi i/b)k_0(i-1)} = \begin{cases} e^{-(2\pi i/b)k_0} \frac{b}{2} & \text{if } \frac{b}{2} \mid k_0 \quad \left(\Leftrightarrow \frac{b}{2} \mid k \right), \\ 0 & \text{otherwise,} \end{cases} \quad (32)$$

and

$$\int_0^1 \int_0^1 K(x, y) \overline{\text{wal}_{[k/b]}(x)} \text{wal}_{[k'/b]}(y) dx dy = \widehat{K}(i, j), \quad (33)$$

where $i \oplus [k/b] = 0, j \oplus [k'/b] = 0$, we find

$$\begin{aligned} & \sum_{\substack{i,j=1 \\ i,j \text{ even}}}^b \iint_{I_i \times I_j} K(\Psi_i(x), \Psi_j(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \begin{cases} \frac{1}{4} \widehat{K}(i, j) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} e^{(2\pi i/b)(k'_0+k'_1+\dots+k'_{n'})} \\ \quad \text{if } \frac{b}{2} \mid k, \frac{b}{2} \mid k', i \oplus \left[\frac{k}{b}\right] = 0, j \oplus \left[\frac{k'}{b}\right] = 0, \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned} \quad (34)$$

This gives the second term in the parenthesis (25).

Proof of (33). Starting with $K(x, y) = \sum_{i,j=0}^{\infty} \widehat{K}(i, j) \overline{\text{wal}_i(x)} \text{wal}_j(y)$ we find

$$\begin{aligned} & \int_0^1 \int_0^1 K(x, y) \overline{\text{wal}_{[k/b]}(x)} \text{wal}_{[k'/b]}(y) dx dy \\ &= \sum_{i,j=0}^{\infty} \widehat{K}(i, j) \int_0^1 \overline{\text{wal}_i(x)} \overline{\text{wal}_{[k/b]}(x)} dx \int_0^1 \text{wal}_j(y) \text{wal}_{[k'/b]}(y) dy. \end{aligned}$$

Because

$$\text{wal}_j(y) \text{wal}_{[k'/b]}(y) = \text{wal}_{j \oplus [k'/b]}(y)$$

and if $\int_0^1 \text{wal}_j(y) \text{wal}_{[k'/b]}(y) dy \neq 0$, then $j \oplus [k'/b] = 0$.

Similarly, if $\int_0^1 \overline{\text{wal}_i(x)} \overline{\text{wal}_{[k/b]}(x)} dx \neq 0$, then $i \oplus [k/b] = 0$.

III) For odd i and even j we have

$$\begin{aligned} & \sum_{\substack{i,j=1 \\ i \text{ odd}, j \text{ even}}}^b \iint_{I_i \times I_j} K(\Phi_i(x), \Psi_j(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \begin{cases} \frac{1}{4} \int_0^1 \int_0^1 K(x, y) \text{wal}_{[k/b]}(x) \text{wal}_{[k'/b]}(y) e^{(2\pi i/b)(k'_0+k'_1+\dots+k'_{n'})} dx dy & \text{if } \frac{b}{2} \mid k, \frac{b}{2} \mid k', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (35)$$

Now, using the Fourier–Walsh expansion $K(x, y) = \sum_{i,j=0}^{\infty} \widehat{K}(i, j) \overline{\text{wal}_i(x)} \text{wal}_j(y)$, we find

$$\begin{aligned} & \int_0^1 \int_0^1 K(x, y) \text{wal}_{[k/b]}(x) \text{wal}_{[k'/b]}(y) dx dy \\ &= \sum_{i,j=0}^{\infty} \widehat{K}(i, j) \int_0^1 \overline{\text{wal}_i(x)} \text{wal}_{[k/b]}(x) dx \int_0^1 \text{wal}_j(y) \text{wal}_{[k'/b]}(y) dy \\ &= \begin{cases} \widehat{K}\left(\left[\frac{k}{b}\right], j\right) & \text{if } j \oplus \left[\frac{k'}{b}\right] = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (36)$$

Proof of (36). If $\int_0^1 \overline{\text{wal}_i(x)} \text{wal}_{[k/b]}(x) dx \neq 0$, then $i = [k/b]$. If $\int_0^1 \text{wal}_j(y) \text{wal}_{[k'/b]}(y) dy \neq 0$, then $j \oplus [k'/b] = 0$.

Thus (35) is equal to

$$\begin{cases} \frac{1}{4} \widehat{K}\left(\left[\frac{k}{b}\right], j\right) e^{(2\pi i/b)(k'_0+k'_1+\dots+k'_{n'})} & \text{if } \frac{b}{2} \mid k, \frac{b}{2} \mid k', j \oplus \left[\frac{k'}{b}\right] = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

IV) Similarly, for even i and odd j we have

$$\begin{aligned} & \sum_{\substack{i,j=1 \\ i \text{ even}, j \text{ odd}}}^b \iint_{I_i \times I_j} K(\Psi_i(x), \Psi_j(y)) \text{wal}_k(x) \overline{\text{wal}_{k'}(y)} dx dy \\ &= \begin{cases} \frac{1}{4} \widehat{K}\left(i, \left[\frac{k'}{b}\right]\right) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} & \text{if } \frac{b}{2} \mid k, \frac{b}{2} \mid k', i \oplus \left[\frac{k}{b}\right] = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (38)$$

Summing up (29), (34), (37), and (38) we find

$$\begin{aligned} \widehat{K}_1(k, k') &= \frac{1}{4} \left(\widehat{K}\left(\left[\frac{k}{b}\right], \left[\frac{k'}{b}\right]\right) + \widehat{K}\left(-\left[\frac{k}{b}\right], -\left[\frac{k'}{b}\right]\right) + \widehat{K}\left(\left[\frac{k}{b}\right], -\left[\frac{k'}{b}\right]\right) e^{(2\pi i/b)(k'_0+k'_1+\dots+k'_n)} \right. \\ &\quad \left. + \widehat{K}\left(-\left[\frac{k}{b}\right], \left[\frac{k'}{b}\right]\right) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} \right). \end{aligned} \quad (39)$$

Inserting (39) into (12) we find (25).

Now, the proof of Theorem 5 is complete.

For some control of a Fourier–Walsh expansion of a kernel $K(\mathbf{x}, \mathbf{y})$, the following theorem can be used.

THEOREM 6. *For an arbitrary u.d.p. map $\Psi(\mathbf{x}, \boldsymbol{\sigma})$ and an arbitrary kernel $K(\mathbf{x}, \mathbf{y})$, for $N = 1$ we have*

$$\int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| f(\Psi(\mathbf{x}_0, \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} = \int_{[0,1]^s} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \quad (40)$$

Proof. For $N = 1$, the equation (7) implies

$$\begin{aligned} & \int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| f(\Psi(\mathbf{x}_0, \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \\ &= \int_{[0,1]^s} K(\Psi(\mathbf{x}_0, \boldsymbol{\sigma}), \Psi(\mathbf{x}_0, \boldsymbol{\sigma})) d\boldsymbol{\sigma} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \\ &= \int_{[0,1]^s} K(\mathbf{x}, \mathbf{x}) d\mathbf{x} - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \end{aligned}$$

REMARK 2. Theorem 6 implies the following.

By Theorem 2 and Theorem 3 we have that (40) is equal to

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}(\mathbf{k}, \mathbf{k}) = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}_1(\mathbf{k}, \mathbf{k}), \quad (41)$$

for an arbitrary u.d.p. map $\Phi(\mathbf{x})$ and a common kernel $K(\mathbf{x}, \mathbf{y})$.

By Theorem 4 we have

$$\sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}}} \widehat{K}(\mathbf{k}, \mathbf{k}) = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}, \mathbf{b}^\alpha | \mathbf{k}}} \widehat{K}\left(\frac{\mathbf{k}}{\mathbf{b}^\alpha}, \frac{\mathbf{k}}{\mathbf{b}^\alpha}\right), \quad (42)$$

for $\Phi(\mathbf{x}) = \mathbf{b}^\alpha \mathbf{x} \bmod 1$.¹²

By Theorem 5, for one-dimensional tent map $\Phi(x)$, we have (41) of the form

$$\begin{aligned} & \sum_{k=1}^{\infty} \widehat{K}(k, k) \\ &= \sum_{\substack{k=1 \\ (b/2) | k}}^{\infty} \frac{1}{4} \left(\widehat{K}\left(\left[\frac{k}{b}\right], \left[\frac{k}{b}\right]\right) + \widehat{K}\left(-\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right) + \widehat{K}\left(\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right) e^{(2\pi i/b)(k_0+k_1+\dots+k_n)} \right. \\ & \quad \left. + \widehat{K}\left(-\left[\frac{k}{b}\right], \left[\frac{k}{b}\right]\right) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} \right). \end{aligned} \quad (43)$$

Another proof of (43). Let $k = k_0 + k_1 b + \dots + k_n b^n$, $k_n \neq 0$. For $(b/2) | k$ we have $k_0 = 0$ or $k_0 = b/2$. Put

$$A = \left\{ 0 + \left[\frac{k}{b}\right] b, \frac{b}{2} + \left[\frac{k}{b}\right] b \right\}.$$

Then

$$\begin{aligned} & \sum_{k \in A} \widehat{K}\left(\left[\frac{k}{b}\right], \left[\frac{k}{b}\right]\right) = 2\widehat{K}\left(\left[\frac{k}{b}\right], \left[\frac{k}{b}\right]\right), \\ & \sum_{k \in A} \widehat{K}\left(-\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right) = 2\widehat{K}\left(-\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right), \\ & \sum_{k \in A} \widehat{K}\left(\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right) e^{(2\pi i/b)(k_0+k_1+\dots+k_n)} \\ & \quad = \widehat{K}\left(\left[\frac{k}{b}\right], -\left[\frac{k}{b}\right]\right) e^{(2\pi i/b)(k_1+\dots+k_n)} (e^{(2\pi i/b)0} + e^{(2\pi i/b)(b/2)}) = 0, \\ & \sum_{k \in A} \widehat{K}\left(-\left[\frac{k}{b}\right], \left[\frac{k}{b}\right]\right) e^{-(2\pi i/b)(k_0+k_1+\dots+k_n)} = 0. \end{aligned}$$

4. Numerical results

4.1. The mean square worst-case error for the kernel $K(x, y) = 1 - \max(x, y)$.

In this Section we apply Theorems 2, 3, 4 and 5 to the Fourier–Walsh expansion of the simple kernel $K(x, y) = 1 - \max(x, y)$.

THEOREM 7. *Let $K(x, y) = \sum_{k, k' \in \mathbb{N}_0} \widehat{K}(k, k') \overline{\text{wal}_k(x)} \text{wal}_{k'}(y)$ be the Fourier–Walsh series of the kernel $K(x, y) = 1 - \max(x, y)$. Then for $k \neq 0$, $k = k_0 + k_1 b + \dots + k_n b^n$, $k_n \neq 0$, we have*

$$\widehat{K}(k, k) = -\frac{1}{2} \frac{1}{b^{2(n+1)}} \left(\frac{1}{3} - \frac{1}{\sin^2(k_n(\pi/b))} \right), \quad (44)$$

¹²This holds for an arbitrary function $\widehat{K}(\mathbf{k}, \mathbf{k})$ directly.

and

$$\widehat{K}(k, -k) = \begin{cases} 0 & \text{if not } 2k_0 = \cdots = 2k_{n-1} = 0 \pmod{b}, \\ \frac{1}{3b^{2(n+1)}} & \text{if } k_n = \frac{b}{2}, 2k_0 = \cdots = 2k_{n-1} = 0 \pmod{b}, \\ \frac{1}{b^{2(n+1)}(e^{(2\pi/b)k_n} - 1)^2} & \text{if } k_n \neq \frac{b}{2}, 2k_0 = \cdots = 2k_{n-1} = 0 \pmod{b}. \end{cases} \quad (45)$$

Proof of (44). Let the kernel $K(x, y)$ be rewritten in the form

$$K(x, y) = 1 - \frac{x + y + |x - y|}{2}. \quad (46)$$

By [5; (12.15)] for $k = k_0 + k_1b + \cdots + k_nb^n$ we have

$$\int_0^1 \int_0^1 |x - y| \text{wal}_k(x) \overline{\text{wal}_k(y)} dx dy = \frac{1}{b^{2(n+1)}} \left(\frac{1}{3} - \frac{1}{\sin^2(k_n(\pi/b))} \right). \quad (47)$$

For $k \neq 0$ we have

$$\int_0^1 \int_0^1 x \text{wal}_k(x) \overline{\text{wal}_k(y)} dx dy = 0$$

then (47) implies (44).

Proof of (45). We have

$$\begin{aligned} \widehat{K}(k, -k) &= \int_0^1 \int_0^1 K(x, y) \text{wal}_k(x) \overline{\text{wal}_{-k}(y)} dx dy \\ &= \int_0^1 \int_0^1 K(x, y) \text{wal}_k(x) \text{wal}_k(y) dx dy = -\frac{1}{2} \int_0^1 \int_0^1 |x - y| \text{wal}_k(x) \text{wal}_k(y) dx dy. \end{aligned} \quad (48)$$

Now, $k = k_0 + k_1b + \cdots + k_nb^n$, $k_n \neq 0$,

$$\begin{aligned} I_i &= \left[\frac{i}{b^{n+1}}, \frac{i+1}{b^{n+1}} \right), \quad i = 0, 1, 2, \dots, b^{n+1} - 1, \\ x &= \frac{x_0}{b} + \frac{x_1}{b^2} + \cdots + \frac{x_n}{b^{n+1}} + \cdots \in I_i, \quad \frac{i}{b^{n+1}} = \frac{x_0}{b} + \frac{x_1}{b^2} + \cdots + \frac{x_n}{b^{n+1}}, \\ y &= \frac{y_0}{b} + \frac{y_1}{b^2} + \cdots + \frac{y_n}{b^{n+1}} + \cdots \in I_j, \quad \frac{j}{b^{n+1}} = \frac{y_0}{b} + \frac{y_1}{b^2} + \cdots + \frac{y_n}{b^{n+1}}. \end{aligned}$$

Using these notations we continue to calculate (48).

$$\begin{aligned} & -\frac{1}{2} \int_0^1 \int_0^1 |x - y| \text{wal}_k(x) \text{wal}_k(y) dx dy \\ &= -\frac{1}{2} \sum_{i,j=0}^{b^{n+1}-1} e^{(2\pi i/b)((x_0+y_0)k_0+(x_1+y_1)k_1+\cdots+(x_n+y_n)k_n)} \iint_{I_i \times I_j} |x - y| dx dy, \end{aligned} \quad (49)$$

where

$$\iint_{I_i \times I_j} |x - y| dx dy = \begin{cases} \frac{1}{3b^{3(n+1)}} & \text{if } i = j, \\ \frac{|i - j|}{b^{3(n+1)}} & \text{if } i \neq j. \end{cases} \quad (50)$$

I) For $i = j$ in (49) we have

$$\begin{aligned}
 & -\frac{1}{2} \frac{1}{3b^{3(n+1)}} \sum_{i=0}^{b^{n+1}-1} e^{(2\pi i/b)(2x_0k_0+2x_1k_1+\dots+2x_nk_n)} \\
 & = -\frac{1}{2} \frac{1}{3b^{3(n+1)}} \left(\sum_{x_0=0}^{b-1} e^{(2\pi i/b)2x_0k_0} \right) \left(\sum_{x_1=0}^{b-1} e^{(2\pi i/b)2x_1k_1} \right) \dots \left(\sum_{x_n=0}^{b-1} e^{(2\pi i/b)2x_nk_n} \right) \\
 & = \begin{cases} -\frac{1}{2} \frac{1}{3b^{2(n+1)}} & \text{if } 2k_0 = 2k_1 = \dots = 2k_n = 0 \pmod{b}, \\ 0, & \text{otherwise.} \end{cases} \tag{51}
 \end{aligned}$$

II) For $i \neq j$ in (49) we have

$$\begin{aligned}
 & -\frac{1}{2} \frac{2}{b^{2(n+1)}} \sum_{j=0}^{b^{n+1}-2} \sum_{i=j+1}^{b^{n+1}-1} \left(\frac{i}{b^{n+1}} - \frac{j}{b^{n+1}} \right) e^{(2\pi i/b)((x_0+y_0)k_0+(x_1+y_1)k_1+\dots+(x_n+y_n)k_n)} \\
 & = -\frac{1}{2} \frac{2}{b^{2(n+1)}} \sum_{j=0}^{b^{n+1}-2} \sum_{i=j+1}^{b^{n+1}-1} \left(\frac{x_0-y_0}{b} + \frac{x_1-y_1}{b^2} + \dots + \frac{x_n-y_n}{b^{n+1}} \right) \\
 & \quad \times e^{(2\pi i/b)((x_0+y_0)k_0+(x_1+y_1)k_1+\dots+(x_n+y_n)k_n)}. \tag{52}
 \end{aligned}$$

Now, for fixed $x_0, x_1, \dots, x_{n-1}$ and $y_0, y_1, \dots, y_{n-1}$, we assume that

$$\frac{x_0-y_0}{b} + \frac{x_1-y_1}{b^2} + \dots + \frac{x_{n-1}-y_{n-1}}{b^n} > 0. \tag{53}$$

Similarly, as in [5; (12.11)] the following partial sum of (52) is zero.

$$\begin{aligned}
 & \left| \sum_{x_n=0}^{b-1} \sum_{y_n=0}^{b-1} \left(\frac{x_0-y_0}{b} + \dots + \frac{x_n-y_n}{b^{n+1}} \right) e^{(2\pi i/b)((x_0+y_0)k_0+\dots+(x_n+y_n)k_n)} \right| \\
 & = \left| \sum_{x_n=0}^{b-1} \sum_{y_n=0}^{b-1} \left(\frac{x_0-y_0}{b} + \dots + \frac{x_n-y_n}{b^{n+1}} \right) e^{(2\pi i/b)(x_n+y_n)k_n} \right| \\
 & = \left| \left(\frac{x_0-y_0}{b} + \dots + \frac{x_{n-1}-y_{n-1}}{b^n} \right) \sum_{x_n=0}^{b-1} \sum_{y_n=0}^{b-1} e^{(2\pi i/b)(x_n+y_n)k_n} \right. \\
 & \quad \left. + \sum_{x_n=0}^{b-1} \sum_{y_n=0}^{b-1} \frac{(x_n-y_n)}{b^{n+1}} e^{(2\pi i/b)(x_n+y_n)k_n} \right| = 0, \tag{54}
 \end{aligned}$$

since the final double sums in (54) are equal to zero, bearing in mind $k_n \neq 0$.

III) Note that from (53) follows that

$$\frac{x_0-y_0}{b} + \frac{x_1-y_1}{b^2} + \dots + \frac{x_n-y_n}{b^n} > 0$$

for arbitrary $x_n, y_n = 0, 1, \dots, b-1$. Thus it remains to consider the case

$$\frac{x_0-y_0}{b} + \frac{x_1-y_1}{b^2} + \dots + \frac{x_{n-1}-y_{n-1}}{b^n} = 0, \tag{55}$$

i.e., $x_0 = y_0, x_1 = y_1, \dots, x_{n-1} = y_{n-1}$, but in this case for $j < i$ we need $y_n < x_n$. For such subsum we have

$$\begin{aligned}
& \sum_{j=0}^{b^{n+1}-2} \sum_{i=j+1}^{b^{n+1}-1} \left(\frac{x_0 - y_0}{b} + \dots + \frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)((x_0+y_0)k_0 + \dots + (x_n+y_n)k_n)} \\
&= \sum_{x_0=0}^{b-1} \dots \sum_{x_{n-1}=0}^{b-1} \sum_{y_n=0}^{b-2} \sum_{x_n=y_n+1}^{b-1} \left(\frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)(2x_0k_0 + \dots + 2x_{n-1}k_{n-1} + (x_n+y_n)k_n)} \\
&= \left(\sum_{x_0=0}^{b-1} e^{(2\pi i/b)2x_0k_0} \right) \dots \left(\sum_{x_{n-1}=0}^{b-1} e^{(2\pi i/b)2x_{n-1}k_{n-1}} \right) \\
&\quad \times \sum_{y_n=0}^{b-2} \sum_{x_n=y_n+1}^{b-1} \left(\frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)(x_n+y_n)k_n} \\
&= \begin{cases} b^n \sum_{y_n=0}^{b-2} \sum_{x_n=y_n+1}^{b-1} \left(\frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)(x_n+y_n)k_n} & \text{if } 2k_0 = \dots = 2k_{n-1} = 0 \pmod{b}, \\ 0 & \text{otherwise.} \end{cases} \quad (56)
\end{aligned}$$

Replacing $x_n - y_n = j$, the final double sum in (56) can be expressed as

$$\sum_{y_n=0}^{b-2} e^{(2\pi i/b)2y_nk_n} \sum_{j=1}^{b-1-y_n} \frac{j}{b^{n+1}} e^{(2\pi i/b)jk_n}. \quad (57)$$

Then we use the following summation formula

$$\sum_{j=1}^n j e^{Bj} = \frac{n-1}{1-e^{-B \cdot 1}} - \frac{e^{B(n-1)}-1}{(1-e^{-B \cdot 1})^2} + n \frac{e^{Bn}-1}{1-e^{-B \cdot 1}}, \quad (58)$$

which holds for $n = 0, 1, 2, \dots$ and for an arbitrary nonzero complex number B . Inserting (58) into (57) we find

$$\begin{aligned}
& \sum_{y_n=0}^{b-2} \sum_{x_n=y_n+1}^{b-1} \left(\frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)(x_n+y_n)k_n} \\
&= \frac{1}{b^{n+1}} \frac{e^{-(2\pi i/b)k_n}}{(1-e^{-(2\pi i/b)k_n})^2} \sum_{y_n=0}^{b-2} e^{(2\pi i/b)2y_nk_n} \\
&\quad + \frac{1}{b^{n+1}} \left((b-1) \frac{e^{-(2\pi i/b)k_n}}{1-e^{-(2\pi i/b)k_n}} - \frac{e^{-(4\pi i/b)k_n}}{(1-e^{-(2\pi i/b)k_n})^2} \right) \sum_{y_n=0}^{b-2} e^{(2\pi i/b)y_nk_n} \\
&\quad - \frac{1}{b^{n+1}} \frac{e^{-(2\pi i/b)k_n}}{1-e^{-(2\pi i/b)k_n}} \sum_{y_n=0}^{b-2} y_n e^{(2\pi i/b)y_nk_n}. \quad (59)
\end{aligned}$$

Here

$$\sum_{y_n=0}^{b-2} e^{(2\pi i/b)2y_nk_n} = \begin{cases} b-1 & \text{if } k_n = \frac{b}{2}, \\ -e^{-(4\pi i/b)k_n} & \text{if } k_n \neq \frac{b}{2}, \end{cases}$$

$$\sum_{y_n=0}^{b-2} e^{(2\pi i/b)y_n k_n} = -e^{-(2\pi i/b)k_n},$$

$$\sum_{y_n=0}^{b-2} y_n e^{(2\pi i/b)y_n k_n} = \frac{-1}{1 - e^{-(2\pi i/b)k_n}} - \frac{e^{-(6\pi i/b)k_n} - 1}{(1 - e^{-(2\pi i/b)k_n})^2} + (b-1) \frac{e^{-(4\pi i/b)k_n}}{1 - e^{-(2\pi i/b)k_n}}. \quad (60)$$

Inserting expression (60) into (59) we find

$$\sum_{y_n=0}^{b-2} \sum_{x_n=y_n+1}^{b-1} \left(\frac{x_n - y_n}{b^{n+1}} \right) e^{(2\pi i/b)(x_n+y_n)k_n} = \frac{1}{b^{n+1}} \begin{cases} \frac{-be^{-(4\pi i/b)k_n}}{(1 - e^{-(2\pi i/b)k_n})^2} & \text{if } k_n \neq \frac{b}{2}, \\ \frac{(b-2)e^{-(2\pi i/b)k_n}}{1 - e^{-(2\pi i/b)k_n}} + e^{-(2\pi i/b)k_n} & \text{if } k_n = \frac{b}{2}. \end{cases} \quad (61)$$

For $k_n = b/2$ the relevant part of (61) is equal to $-b/2$. Finally, summing up (51), (56) and (61) we find (45).

This completes the proof of Theorem 7.

THEOREM 8. *Let $b \geq 2$, α and β be integer numbers, $\alpha < \beta$, and*

- (i) $K(x, y) = 1 - \max(x, y)$;
- (ii) $N = b^\beta$;
- (iii) $x_0, x_1, \dots, x_{N-1}$ be the van der Corput sequence in base b .¹³

Then

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n \oplus \sigma) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{1}{6b^{2\beta}}, \quad (62)$$

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{1}{6b^{2(\beta-\alpha)}}; \quad (63)$$

for $\Phi(x) = b^\alpha x \bmod 1$ and $\alpha < \beta$;

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{1}{24b^{2(\beta-1)}}, \quad (64)$$

for the tent function $\Phi(x)$, $2 \mid b$.

Proof of (62) By (10), in one-dimensional case, the left hand side of (62) is equal to

$$\sum_{k=1}^{\infty} \widehat{K}(k, k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2. \quad (65)$$

For $N = b^\beta$ and for the van der Corput sequence x_n we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) = \begin{cases} 0 & \text{if } b^\beta \nmid k, \\ 1 & \text{if } b^\beta \mid k. \end{cases} \quad (66)$$

¹³This is the same sequence as

$$\frac{0}{b^\beta}, \quad \frac{1}{b^\beta}, \quad \dots, \quad \frac{b^\beta - 1}{b^\beta}.$$

Inserting (44) into (65) we find

$$\sum_{k=1, b^\beta | k}^{\infty} \widehat{K}(k, k) = -\frac{1}{2} \sum_{n=\beta}^{\infty} \frac{1}{b^{2(n+1)}} b^{n-\beta} \sum_{k_n=1}^{b-1} \left(\frac{1}{3} - \frac{1}{\sin^2(k_n(\pi/b))} \right). \quad (67)$$

Here the term $b^{n-\beta}$ is the number of k , $b^\beta | k$, for which

$$k_\beta = 0, 1, \dots, b-1, \quad k_{\beta+1} = 0, 1, \dots, b-1, \quad \dots, \quad k_{n-1} = 0, 1, \dots, b-1.$$

Since [5; Corollary A23]

$$\sum_{k_n=1}^{b-1} \frac{1}{\sin^2(k_n(\pi/b))} = \frac{b^2-1}{3} \quad (68)$$

we find (62).

Here we add

Proof of (66). Let $N = b^\beta$ and $x_0, x_1, \dots, x_{N-1}$ be the van der Corput sequence,

$$n = n_0 + n_1 b + \dots + n_{\beta-1} b^{\beta-1}, \quad n_i = 0, 1, \dots, b-1, \\ x_n = \frac{n_0}{b} + \frac{n_1}{b} + \dots + \frac{n_{\beta-1}}{b^\beta}, \quad k = k_0 + k_1 b + k_2 b^2 + \dots.$$

We have

$$\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) = \left(\sum_{n_0=0}^{b-1} e^{(2\pi i/b)n_0 k_0} \right) \left(\sum_{n_1=0}^{b-1} e^{(2\pi i/b)n_1 k_1} \right) \dots \left(\sum_{n_{\beta-1}=0}^{b-1} e^{(2\pi i/b)n_{\beta-1} k_{\beta-1}} \right).$$

From this if $k_i \neq 0$ for some $0 \leq i \leq \beta-1$, then $\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) = 0$. If $k_0 = k_1 = \dots = k_{\beta-1} = 0$, i.e. $b^\beta | k$, then $(1/b^\beta) \sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) = 1$.¹⁴

Proof of (63). By one-dimensional case of (13) of Theorem 4 the mean square worst-case error (63) is

$$\sum_{k=1, b^\alpha | k}^{\infty} \widehat{K}\left(\frac{k}{b^\alpha}, \frac{k}{b^\alpha}\right) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2. \quad (69)$$

By (66) the sum (69) is equal to

$$\sum_{k=1, b^\alpha | k, b^\beta | k}^{\infty} \widehat{K}\left(\frac{k}{b^\alpha}, \frac{k}{b^\alpha}\right). \quad (70)$$

Assuming $\alpha < \beta$,

$$k = k_\beta + \dots + k_n b^n, \quad \frac{k}{b^\alpha} = k_\beta b^{\beta-\alpha} + \dots + k_n b^{n-\alpha},$$

then (70) has the form

$$\sum_{k=1, b^\beta | k}^{\infty} \widehat{K}\left(\frac{k}{b^\alpha}, \frac{k}{b^\alpha}\right) = -\frac{1}{2} \sum_{n=\beta}^{\infty} \frac{1}{b^{2(n-\alpha+1)}} b^{n-\beta} \sum_{k_n=1}^{b-1} \left(\frac{1}{3} - \frac{1}{\sin^2(k_n(\pi/b))} \right), \quad (71)$$

which gives (63). Here the exponent in $b^{2(n-\alpha+1)}$ is given by (44) after transformation $k \rightarrow k/b^\alpha$.

¹⁴For an arbitrary N , see (88).

REMARK 3. The formula (63) follows also from the second expression (14) in Theorem 4 using

$$\frac{1}{b^\beta} \sum_{n=0}^{b^\beta-1} \text{wal}_k(\Phi(x_n)) = \begin{cases} 0 & \text{if } b^{\beta-\alpha} \nmid k, \\ 1 & \text{if } b^{\beta-\alpha} \mid k. \end{cases} \quad (72)$$

Proof of (64). We start with

$$\sum_{k=k_\beta b^\beta + \dots + k_n b^n}^{\infty} \widehat{K}\left(\frac{k}{b}, \frac{k}{b}\right) = \sum_{k=k_\beta b^\beta + \dots + k_n b^n}^{\infty} \widehat{K}\left(-\frac{k}{b}, -\frac{k}{b}\right) = \frac{1}{6b^{2(\beta-1)}} \quad (73)$$

and

$$\begin{aligned} \sum_{k=k_\beta b^\beta + \dots + k_n b^n}^{\infty} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)(k_\beta + \dots + k_n)} &= \sum_{k=k_\beta b^\beta + \dots + k_n b^n}^{\infty} \widehat{K}\left(-\frac{k}{b}, \frac{k}{b}\right) e^{-(2\pi i/b)(k_\beta + \dots + k_n)} \\ &= -\frac{1}{12b^{2(\beta-1)}}. \end{aligned} \quad (74)$$

Inserting (73) and (74) into (25) then by Theorem 5 the mean square worst-case error is

$$= \frac{1}{4} \left(\frac{1}{6b^{2(\beta-1)}} + \frac{1}{6b^{2(\beta-1)}} - \frac{1}{12b^{2(\beta-1)}} - \frac{1}{12b^{2(\beta-1)}} \right) = \frac{1}{24b^{2(\beta-1)}},$$

which gives (64).

Proof of (74). By (45), if $2k_\beta = \dots = 2k_{n-1} = 0 \pmod{b}$, then the value of $\widehat{K}(k/b, -k/b)$ depends only on k_n , thus

$$\begin{aligned} &\sum_{k=k_\beta b^\beta + \dots + k_n b^n}^{\infty} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)(k_\beta + \dots + k_n)} \\ &= \sum_{n=\beta}^{\infty} \sum_{k_\beta=0, b/2, \dots, k_{n-1}=0, b/2} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)(k_\beta + \dots + k_n)} \\ &= \left(\sum_{k_\beta=0, b/2} e^{(2\pi i/b)k_\beta} \right) \dots \left(\sum_{k_{n-1}=0, b/2} e^{(2\pi i/b)k_{n-1}} \right) \sum_{k_n=1}^{b-1} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)k_n} \end{aligned} \quad (75)$$

and every sum in parenthesis is equal to zero. But for $n = \beta$ the set of such parenthesis is empty and by using (45) we have that (75) is equal

$$\sum_{k_n=1}^{b-1} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)k_n} = -\frac{1}{3b^{2\beta}} + \sum_{k_n=1, k_n \neq b/2}^{b-1} \frac{e^{(2\pi i/b)k_n}}{(e^{(2\pi i/b)k_n} - 1)^2}. \quad (76)$$

Note that we apply (45) for $k/b = k_\beta b^\beta + \dots + k_n b^{n-1}$ and for $n = \beta$ we have $1/b^{2(n-1+1)} = 1/b^{2\beta}$. Moreover, by elementary operations (all the time $k_n \neq 0$)

$$\frac{e^{(2\pi i/b)k_n}}{(e^{(2\pi i/b)k_n} - 1)^2} = -\frac{1}{4 \sin^2((\pi/b)k_n)}.$$

Using (68) we find

$$\sum_{k_n=1}^{b-1} \frac{e^{(2\pi i/b)k_n}}{(e^{(2\pi i/b)k_n} - 1)^2} = -\frac{1}{12}(b^2 - 1). \quad (77)$$

For the sum in (76) we need to omit in (77) the term for $k_n = b/2$, which is $-1/(-1-1)^2 = -1/4$.
Summary,

$$\sum_{\substack{k=1 \\ k=k_\beta b^\beta + \dots + k_n b^n}}^{\infty} \widehat{K}\left(\frac{k}{b}, -\frac{k}{b}\right) e^{(2\pi i/b)(k_\beta + \dots + k_n)} = -\frac{1}{3b^{2\beta}} - \frac{1}{4b^{2\beta}} \frac{b^2 - 1}{3} + \frac{1}{4b^{2\beta}} = -\frac{1}{12b^{2(\beta-1)}}. \quad (78)$$

Final notes. By definition of Fourier–Walsh coefficients we have

$$\begin{aligned} \widehat{K}(k, -k) &= \int_0^1 \int_0^1 K(x, y) \text{wal}_k(x) \overline{\text{wal}_{-k}(y)} dx dy = \int_0^1 \int_0^1 K(x, y) \text{wal}_k(x) \text{wal}_k(y) dx dy, \\ \widehat{K}(-k, k) &= \int_0^1 \int_0^1 K(x, y) \text{wal}_{-k}(x) \overline{\text{wal}_k(y)} dx dy = \int_0^1 \int_0^1 K(x, y) \overline{\text{wal}_k(x)} \text{wal}_k(y) dx dy. \end{aligned}$$

Since the kernel $K(x, y)$ is real we have $\widehat{K}(-k, k) = \overline{\widehat{K}(k, -k)}$. Then by (45) we have

$$\widehat{K}(-k, k) = \frac{1}{b^{2(n+1)}(e^{-(2\pi i/b)k_n} - 1)^2}$$

if $k_n \neq b/2$. Since

$$\frac{e^{-(2\pi i/b)k_n}}{(e^{-(2\pi i/b)k_n} - 1)^2} = \frac{e^{(2\pi i/b)k_n}}{(e^{(2\pi i/b)k_n} - 1)^2}$$

we have (74).

The Theorem 8 follows.

REMARK 4. For the weighted Sobolev space with the reproducing kernel given by

$$K(x, y) = 1 + \gamma \left(\frac{1}{2} B_2(\{x - y\}) + \left(x - \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \right),$$

Dick and Pillichshammer [11; Theorem 10] proved that for every $\varepsilon > 0$ there exists a digital net such that the mean square worst-case error is bounded by $O(1/N^{2-\varepsilon})$. But in (62) the ε does not occur.

4.2. The mean square worst-case error for the weighted Sobolev space of order 2.

THEOREM 9. Let b, α and β be integer numbers, $\alpha < \beta$, and

(i) $K(x, y)$ be a kernel of the form

$$K(x, y) = 1 + \gamma B_1(x) B_1(y) + \frac{\gamma^2}{4} B_2(x) B_2(y) - \frac{\gamma^2}{24} B_4(|x - y|);$$

(ii) $b = 2, N = 2^\beta$;

(iii) $x_0, x_1, \dots, x_{N-1}$ be the van der Corput sequence in base $b = 2$.

Then

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n \oplus \sigma) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{\gamma}{12 \cdot 2^{2\beta}} + \frac{\gamma^2}{360 \cdot 2^{4\beta}}; \quad (79)$$

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{\gamma}{12 \cdot 2^{2(\beta-\alpha)}} + \frac{\gamma^2}{360 \cdot 2^{4(\beta-\alpha)}} \quad (80)$$

for $\Phi(x) = 2^\alpha x \bmod 1$ and $\alpha < \beta$;

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{\gamma^2}{30 \cdot 2^{4\beta}} \quad (81)$$

$$\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma = \frac{\gamma^2}{30 \cdot 2^{4\beta}} \quad (82)$$

for the tent transformation $\Phi(x) = \Phi_0(x) = 1 - |2x - 1|$.

Proof. In [1; p. 447] we have, for $k \geq 1$,¹⁵

$$\widehat{K}(k, k) = \begin{cases} \frac{\gamma}{4 \cdot 2^{2(n+1)}} + \frac{\gamma^2}{120 \cdot 2^{4(n+1)}} & \text{if } k = 2^n, \\ \frac{\gamma^2}{12 \cdot 2^{2(n+1+j+1)}} + \frac{\gamma^2}{20 \cdot 2^{4(n+1)}} & \text{if } k = k_0 + k_1 \cdot 2 + \dots + 2^j + 2^n, \ 0 \leq j < n, \end{cases} \quad (83)$$

and for the tent transformation $\Phi_0(x)$

$$\widehat{K}_1(k, k) = \begin{cases} \frac{\gamma}{4 \cdot 2^{2n}} + \frac{\gamma^2}{120 \cdot 2^{4n}} & \text{if } k = 1 + 2^n, \\ \frac{\gamma^2}{12 \cdot 2^{2(n+j)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} & \text{if } k = k_0 + k_1 \cdot 2 + \dots + 2^j + 2^n, \\ & 1 \leq j < n, \ 2 \mid (k_0 + k_1 + \dots + k_{j-1}), \\ 0 & \text{otherwise.} \end{cases} \quad (84)$$

Proof of (79). Again, for the van der Corput sequence x_n , $n = 0, 1, \dots, N-1$, $N = 2^\beta$, we have (see (66))

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2 = \begin{cases} 0 & \text{if } 2^\beta \nmid k, \\ 1 & \text{if } 2^\beta \mid k, \end{cases}$$

and by (10), in one-dimensional case, the left hand side of (79) is equal to

$$\begin{aligned} & \sum_{k=1}^{\infty} \widehat{K}(k, k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2 \\ &= \sum_{\substack{k=1 \\ 2^\beta \mid k}}^{\infty} \widehat{K}(k, k) = \sum_{n=\beta}^{\infty} \widehat{K}(2^n, 2^n) + \sum_{n=\beta+1}^{\infty} \sum_{\substack{k=k_\beta 2^\beta + \dots + 2^j + 2^n \\ \beta \leq j < n}} \widehat{K}(k, k) \\ &= \sum_{n=\beta}^{\infty} \left(\frac{\gamma}{4 \cdot 2^{2(n+1)}} + \frac{\gamma^2}{120 \cdot 2^{4(n+1)}} \right) + \sum_{n=\beta+1}^{\infty} \sum_{j=\beta}^{n-1} 2^{j-1-\beta+1} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+1+j+1)}} + \frac{\gamma^2}{20 \cdot 2^{4(n+1)}} \right), \end{aligned} \quad (85)$$

which gives (79).

Proof of (82). By the tent transformation $\Phi_0(x)$ and by (12), the error (82) has the form

$$\sum_{k=1}^{\infty} \widehat{K}_1(k, k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2 = \sum_{\substack{k=1 \\ 2^\beta \mid k}}^{\infty} \widehat{K}_1(k, k) \quad (86)$$

¹⁵In this case $\widehat{K}(k, -k) = \widehat{K}(k, k)$ and (83) and (84) related by Theorem 5.

$$\begin{aligned}
&= \widehat{K}_1(2^\beta, 2^\beta) + \widehat{K}_1(2^{\beta+1}, 2^{\beta+1}) + \widehat{K}_1(2^\beta + 2^{\beta+1}, 2^\beta + 2^{\beta+1}) + \sum_{n=\beta+2}^{\infty} \widehat{K}_1(2^\beta + 2^n, 2^\beta + 2^n) \\
&\quad + \sum_{n=\beta+2}^{\infty} \sum_{j=\beta}^{n-1} \widehat{K}_1(k_\beta 2^\beta + \dots + 2^j + 2^n, k_\beta 2^\beta + \dots + 2^j + 2^n) \\
&= 0 + 0 + \left(\frac{\gamma^2}{12 \cdot 2^{2(\beta+1+\beta)}} + \frac{\gamma^2}{20 \cdot 2^{4(\beta+1)}} \right) + \sum_{n=\beta+2}^{\infty} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+\beta)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right) \\
&\quad + \sum_{n=\beta+2}^{\infty} \sum_{j=\beta+1}^{n-1} 2^{j-\beta-1} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+j)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right), \tag{87}
\end{aligned}$$

which gives (82).

Proof of (80). We use (69) in the form

$$\sum_{k=1, 2^\alpha | k}^{\infty} \widehat{K}\left(\frac{k}{2^\alpha}, \frac{k}{2^\alpha}\right) \left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2 = \sum_{k=1, 2^\beta | k}^{\infty} \widehat{K}\left(\frac{k}{2^\alpha}, \frac{k}{2^\alpha}\right) = \sum_{k=1, 2^{\beta-\alpha} | k}^{\infty} \widehat{K}(k, k).$$

Thus, Theorem 9 is proved.

REMARK 5. Thus we must take into the sum (85) also the terms $\widehat{K}(2^n, 2^n)$, $\beta \leq n$, in which the sum has the order $O(N^{-2})$. In (86) by (84) we do not take into account the terms $\widehat{K}_1(1+2^n, 1+2^n)$. This gives that the error (82) is better than the error (79) without special properties of the tent function $\Phi_0(x)$. The sums (87) in the both cases are of the order $O(N^{-4})$.

Finally, we also show that the result $O(N^{-4})$ in (82) also depend on the number $N = 2^\beta$ of terms $x_0, x_1, \dots, x_{N-1}$. Naturally, N can be selected arbitrary too, e.g.

$$N = N_0 + N_1 b + N_2 b^2 + \dots + N_{\beta-1} b^{\beta-1} + N_\beta b^\beta, \quad N_\beta \neq 0.$$

Then, for an arbitrary $k = k_0 + k_1 b + \dots$, and the van der Corput sequence $x_0, x_1, \dots, x_{N-1}$ we have

$$\begin{aligned}
&\sum_{n=0}^{N-1} \text{wal}_k(x_n) \\
&= \left(\sum_{n_0=0}^{b-1} e^{(2\pi i/b) k_0 n_0} \right) \dots \left(\sum_{n_{\beta-1}=0}^{b-1} e^{(2\pi i/b) k_{\beta-1} n_{\beta-1}} \right) \left(\sum_{n_\beta=0}^{N_\beta-1} e^{(2\pi i/b) k_\beta n_\beta} \right) \\
&\quad + \left(\sum_{n_0=0}^{b-1} e^{(2\pi i/b) k_0 n_0} \right) \dots \left(\sum_{n_{\beta-1}=0}^{b-1} e^{(2\pi i/b) k_{\beta-2} n_{\beta-2}} \right) \left(\sum_{n_\beta=0}^{N_{\beta-1}-1} e^{(2\pi i/b) k_{\beta-1} n_{\beta-1}} \right) e^{(2\pi i/b) k_\beta n_\beta} \\
&\quad + \dots + \left(\sum_{n_0=0}^{b-1} e^{(2\pi i/b) k_0 n_0} \right) \left(\sum_{n_1=0}^{N_1-1} e^{(2\pi i/b) k_1 n_1} \right) e^{(2\pi i/b) (k_2 N_2 + \dots + k_\beta N_\beta)} \\
&\quad + \left(\sum_{n_0=0}^{b-1} e^{(2\pi i/b) k_0 n_0} \right) e^{(2\pi i/b) (k_1 N_1 + k_2 N_2 + \dots + k_\beta N_\beta)}, \tag{88}
\end{aligned}$$

where $n = n_0 + n_1 b + \dots$ and $(\sum_{n_\beta=0}^{N_\beta-1} e^{(2\pi i/b)k_j n_j}) = 0$ if $N_j = 0$. Putting $N = 2^\beta + 1$ in (88) we find

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2 = \begin{cases} \frac{1}{(2^\beta + 1)^2} & \text{if } 2^\beta \nmid k, \\ 1 & \text{if } 2^\beta \mid k, \quad k_\beta = 0, \\ \left(\frac{2^\beta - 1}{2^\beta + 1} \right)^2 & \text{if } 2^\beta \mid k, \quad k_\beta = 1. \end{cases}$$

For such $N = 2^\beta + 1$, summing $\sum_{k=1}^{\infty} \widehat{K}_1(k, k) \left| (1/N) \sum_{n=0}^{N-1} \text{wal}_k(x_n) \right|^2$ over $\widehat{K}_1(k, k)$ in (84) we see that the mean square worst-case error has only the order $O(N^{-2})$. In the following theorem we compute mean square worst-case error exactly dividing the sum $\sum_{n=0}^{b^\beta} \text{wal}_k(x_n)$ into two parts $\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n)$ and $\text{wal}_k(x_{b^\beta})$.

THEOREM 10. *Let*

(i) $K(x, y)$ be a kernel of the form

$$K(x, y) = 1 + \gamma B_1(x)B_1(y) + \frac{\gamma^2}{4} B_2(x)B_2(y) - \frac{\gamma^2}{24} B_4(|x - y|);$$

(ii) $b = 2, N = 2^\beta + 1$;

(iii) $x_0, x_1, \dots, x_{N-1}$ be the van der Corput sequence in base $b = 2$.

Then for the tent transformation $\Phi_0(x) = 1 - |2x - 1|$ we have

$$\begin{aligned} & \int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi_0(x_n \oplus \sigma)) - \int_0^1 f(x) dx \right|^2 d\sigma \\ &= \left(\frac{2^\beta}{2^\beta + 1} \right) \frac{\gamma^2}{30 \cdot 2^{4\beta}} + \left(\frac{\gamma}{12} + \frac{\gamma^2}{360} \right) \frac{1}{(2^\beta + 1)^2} + 2 \frac{2^\beta}{(2^\beta + 1)^2} \frac{\gamma^2}{2^{4\beta}} \frac{7}{240}. \end{aligned} \quad (89)$$

Proof. The following part holds for an arbitrary base b and an arbitrary k . Divide

$$\sum_{n=0}^{b^\beta} \text{wal}_k(x_n) = \sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) + \text{wal}_k(x_{b^\beta})$$

then we have

$$\begin{aligned} \left| \sum_{n=0}^{b^\beta} \text{wal}_k(x_n) \right|^2 &= \left| \sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) \right|^2 + \text{wal}_k(x_{b^\beta}) \overline{\text{wal}_k(x_{b^\beta})} \\ &\quad + \left(\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) \right) \overline{\text{wal}_k(x_{b^\beta})} + \overline{\left(\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) \right)} \text{wal}_k(x_{b^\beta}). \end{aligned} \quad (90)$$

As we have proved

$$\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) = \begin{cases} b^\beta & \text{if } b^\beta \mid k, \\ 0 & \text{otherwise,} \end{cases}$$

and for $b^\beta | k$ we have $\text{wal}_k(x_{b^\beta}) = e^{(2\pi i/b)k_\beta}$. Inserting (90) in (12) we find

$$\begin{aligned} & \sum_{k=1}^{\infty} \widehat{K}_1(k, k) \frac{1}{(b^\beta + 1)^2} \left| \sum_{n=0}^{b^\beta} \text{wal}_k(x_n) \right|^2 \\ &= \frac{1}{(b^\beta + 1)^2} \sum_{k=1}^{\infty} \widehat{K}_1(k, k) \left| \sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) \right|^2 \end{aligned} \quad (91)$$

$$+ \frac{1}{(b^\beta + 1)^2} \sum_{k=1}^{\infty} \widehat{K}_1(k, k) \quad (92)$$

$$+ \frac{1}{(b^\beta + 1)^2} \sum_{k=1}^{\infty} \widehat{K}_1(k, k) \left(\sum_{n=0}^{b^\beta-1} \text{wal}_k(x_n) \right) 2 \cos \left(\frac{2\pi i}{b} k_\beta \right). \quad (93)$$

Now, by (82) the term (91) is equal to (here $b = 2$)

$$\left(\frac{b^\beta}{b^\beta + 1} \right)^2 \frac{\gamma^2}{30 \cdot 2^{4\beta}}.$$

By Theorem 6 for the term (92) we have

$$\sum_{k=1}^{\infty} \widehat{K}_1(k, k) = \int_0^1 K(x, x) dx - \int_0^1 \int_0^1 K(x, y) dx dy,$$

and for Sobolev kernel we obtain

$$\frac{\gamma}{12} + \frac{\gamma^2}{360}.$$

The final term (93) we express in the form

$$2 \frac{b^\beta}{(b^\beta + 1)^2} \left(\sum_{\substack{k=1 \\ k \in A}}^{\infty} \widehat{K}_1(k, k) - \sum_{\substack{k=1 \\ k \in B}}^{\infty} \widehat{K}_1(k, k) \right), \quad (94)$$

where A is the set of all k satisfying

$$\begin{aligned} k &= k_0 + k_1 \cdot 2 + \cdots + 2^j + 2^n, \quad 1 \leq j < n, \\ 2 &\mid (k_0 + k_1 + \cdots + k_{j-1}), \quad 2^\beta \mid k, \quad k_\beta = 0, \end{aligned}$$

and B is the set of all k satisfying

$$\begin{aligned} k &= k_0 + k_1 \cdot 2 + \cdots + 2^j + 2^n, \quad 1 \leq j < n, \\ 2 &\mid (k_0 + k_1 + \cdots + k_{j-1}), \quad 2^\beta \mid k, \quad k_\beta = 1. \end{aligned}$$

We find

$$\sum_{\substack{k=1 \\ k \in A}}^{\infty} \widehat{K}_1(k, k) = \sum_{n=\beta+2}^{\infty} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+\beta+1)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right) + \sum_{n=\beta+3}^{\infty} \sum_{j=\beta+2}^{n-1} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+j)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right) \frac{2^{j-1-\beta}}{2}$$

and

$$\sum_{\substack{k=1 \\ k \in B}}^{\infty} \widehat{K}_1(k, k) = \sum_{n=\beta+1}^{\infty} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+\beta)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right) + \sum_{n=\beta+3}^{\infty} \sum_{j=\beta+2}^{n-1} \left(\frac{\gamma^2}{12 \cdot 2^{2(n+j)}} + \frac{\gamma^2}{20 \cdot 2^{4n}} \right) \frac{2^{j-1-\beta}}{2}$$

which gives (94) in the form

$$2 \frac{b^\beta}{(b^\beta + 1)^2} \frac{\gamma^2}{2^{4\beta}} \frac{7}{240}.$$

5. Concluding remarks

This section contains several remarks inspired by comments of referees.

5.1. Classical method. The classical rectangular method of integration, for dimension $s = 1$ and

$$x_n = \frac{n + 1/2}{N}, \quad n = 0, 1, \dots, N-1,$$

gives

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) - \int_0^1 f(x) dx \right|^2 \leq \frac{1}{12N^2} \int_0^1 (f'(x))^2 dx, \quad (95)$$

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) - \int_0^1 f(x) dx \right|^2 \leq \frac{1}{320N^4} \int_0^1 (f''(x))^2 dx. \quad (96)$$

It can be compared with (62) and (82) so that for van der Corput sequence $x_0, x_1, \dots, x_{N-1}$ in base b , $N = b^\beta$, there exists $\sigma_1 \in [0, 1]$ such that for every $f \in H_1$, $\|f\| \leq 1$, we have

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(x_n \oplus \sigma_1) - \int_0^1 f(x) dx \right|^2 \leq \frac{1}{6N^2}$$

and there exists $\sigma_2 \in [0, 1]$ such that for every $f \in H_2$, $\|f\| \leq 1$, we have

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi_0(x_n \oplus \sigma_2)) - \int_0^1 f(x) dx \right|^2 \leq \frac{\gamma^2}{30 \cdot N^4}$$

for the tent transformation $\Phi_0(x) = 1 - |2x - 1|$ and $b = 2$. Here for $f \in H_1$ there exists $\int_0^1 (f'(x))^2 dx$ and for $f \in H_2$ there exists $\int_0^1 (f''(x))^2 dx$.

If we construct a product rule for s dimensions based on an N -point rule in one dimension, it requires N^s points to achieve the same degree of accuracy (see Antia [12; p. 243]). Identifying N^s as N , then from (95) we obtain $O(1/N^{2/s})$ and from (96) we obtain $O(1/N^{4/s})$. So, to get the errors $O(N^{-2+\varepsilon})$ for H_1 , $s > 1$, and $O(N^{-4+\varepsilon})$ for H_2 , $s > 1$, one has to consider complicated sequences x_n , e.g. lattices or digital nets (see [1], [2]).

Note that sequences x_n , $n = 1, 2, \dots$, like van der Corput sequence have the advantage over sequences like $x_n = (n + 1/2)/N$: to estimate the integral using $N' > N$ points, we do not need to re-calculate $f(x_n)$ for $n < N$. This and the following limit relation (97) is the basis of the quasi-Monte Carlo method (QMC).

5.2. quasi-Monte Carlo method. Define the s -dimensional step d.f. as

$$F_N(\mathbf{x}) = \frac{1}{N} \# \{0 \leq n \leq N : \mathbf{x}_n \in [0, \mathbf{x}]\}.$$

If $f: [0, 1]^s \rightarrow \mathbb{R}$ is continuous, then the Riemann–Stieltjes integral gives

$$\frac{1}{N} \sum_{n=1}^N f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x}) dF_N(\mathbf{x}).$$

Then Second Helly theorem (see [7; Theorem 4.1.0.11]) shows that the weak limit $F_N(\mathbf{x}) \rightarrow \mathbf{x}$ implies

$$\int_{[0,1]^s} f(\mathbf{x}) dF_N(\mathbf{x}) \rightarrow \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \quad (97)$$

for every continuous $f: [0, 1]^s \rightarrow \mathbb{R}$. This is called Weyl's limit relation [7; Theorem 1.11.1.1]. For the rate of convergence of (97) it can be used.

Koksma–Hlawka's inequality [7; Theorem 1.11.3.1]: Let $f: [0, 1]^s \rightarrow \mathbb{R}$ be of the bounded variation $V(f)$ in the sense of Hardy and Krause. Then for any sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in $[0, 1]^s$ we have

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right| \leq V(f) D_N^*(\mathbf{x}_n), \quad (98)$$

where the star discrepancy $D_N^*(\mathbf{x}_n)$ is defined by

$$D_N^*(\mathbf{x}_n) = \sup_{\mathbf{x} \in [0,1]^s} |F_N(\mathbf{x}) - \mathbf{x}|.$$

The limit $D_N^*(\mathbf{x}_n) \rightarrow 0$ characterizes uniform distribution of the sequence $\mathbf{x}_n$, $n = 0, 1, \dots$, in $[0, 1]^s$.

5.3. L^2 discrepancy. a) The star-discrepancy D_N^* in (98) can be replaced by L^2 discrepancy $D_N^{(2)}(\mathbf{x}_n) = \int_{[0,1]^s} (F_N(\mathbf{x}) - \mathbf{x})^2 d\mathbf{x}$ expressed as [7; Item 1.11.4]

$$D_N^{(2)} = \frac{1}{3^s} + \frac{1}{N^2} \sum_{m,n=1}^N \prod_{j=1}^s \left(1 - \max(x_{m,j}, x_{n,j}) \right) - \frac{1}{2^{s-1}N} \sum_{n=1}^N \prod_{j=1}^s (1 - x_{n,j}^2),$$

where $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$. Let $\mathbf{x}_n^{(1)}$, $n = 0, 1, \dots, N-1$ minimize $D_N^{(2)}(\mathbf{x}_n)$. Let $\Phi(\mathbf{x})$ be a u.d.p. map and $\Phi(\mathbf{x}_n^{(2)}) = \mathbf{x}_n^{(1)}$, $n = 0, 1, \dots, N-1$. Then

$$D_N^{(2)}(\Phi(\mathbf{x}_n^{(2)})) \leq D_N^{(2)}(\mathbf{x}_n^{(2)}), \quad D_N^{(2)}(\mathbf{x}_n^{(1)}) \leq D_N^{(2)}(\Phi(\mathbf{x}_n^{(1)})).$$

Thus any u.d.p. map $\Phi(\mathbf{x})$ does not improve all sequences $\mathbf{x}_n$. For $s = 1$ we have [7; Theorem 1.9.0.4]

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) dx \right| \leq \sqrt{D_N^{(2)}} \sqrt{\int_0^1 (f'(x))^2 dx}$$

and $D_N^{(2)} = (1/N^2) \sum_{m,n=1}^N F_0(x_m, x_n)$, where

$$F_0(x, y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}.$$

b) Hickernell [10] suggested, for Hilbert space integration, look at the square worst-case quasi-Monte Carlo error (6) as analogy of L^2 discrepancy:

$$D_N^K = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} - \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} K(\mathbf{x}_n, \mathbf{y}) d\mathbf{y} + \frac{1}{N^2} \sum_{n,m=0}^{N-1} K(\mathbf{x}_m, \mathbf{x}_n)$$

and he found that for every $f \in H$ and $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ in $[0, 1]^s$ we have

$$\left| \frac{1}{N} \sum_{n=1}^N f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right| \leq \sqrt{D_N^K} V_K(f),$$

where $V_K(f) = \|f(\mathbf{x}) - \mathbf{1}(f(\mathbf{x}) \cdot \mathbf{1})/(\mathbf{1} \cdot \mathbf{1})\|$, see [10; p. 117]. The limit $D_N^K \rightarrow 0$ characterized uniformly distributed sequences in $[0, 1]^s$, too.

c) Let W be a space of functions $f: [0, 1]^s \rightarrow \mathbb{R}$ equipped with a measure of the Wiener type df and $\mathbf{x}_n$, $n = 0, 1, \dots, N-1$, be a sequence in $[0, 1]^s$. To characterize a rate of convergence

$$\frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) \rightarrow \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x}$$

it can be used the integral

$$\int_W \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 df.$$

For $s = 1$, Temirgaliev [13] proved that

$$\int_W \left| \frac{1}{N} \sum_{n=0}^{N-1} f\left(\frac{n}{N}\right) - \int_{[0,1]^s} f(x) dx \right|^2 df = \left(\frac{\kappa}{12} + \frac{m^2}{16} \right) \frac{1}{N^2} + O\left(\frac{1}{N^3}\right).$$

Here he used: $F(x)$ is an infinitely divisible distribution function, $m = \int_{-\infty}^{+\infty} u dF(u)$, $\kappa = \int_{-\infty}^{+\infty} u^2 dF(u) - m^2$, W is a family of real valued functions $f(x)$ defined by Kolmogorov's theorem on the existence of stochastic processes and df is a measure on W designed by $F(x)$. Methods of construction of df Temirgaliev discusses in [14], too.

5.4. Multivariate integration. In this example we shall see, again as in Theorem 10, that the tent map does not improve the following multidimensional sequence (99). Let $\mathbf{x}_n$ be an s -dimensional sequence,

$$\mathbf{x}_n = \left(\frac{n_1}{b_1^{\beta_1}}, \frac{n_2}{b_2^{\beta_2}}, \dots, \frac{n_s}{b_s^{\beta_s}} \right), \quad (99)$$

where $n_j = 0, 1, \dots, b_j^{\beta_j} - 1$, $j = 1, 2, \dots, s$. Let $K(\mathbf{x}, \mathbf{y})$ be a kernel of the type

$$K(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^s K_j(x_j, y_j),$$

where $\mathbf{x} = (x_1, x_2, \dots, x_s)$ and $\mathbf{y} = (y_1, y_2, \dots, y_s)$ and let $\Phi(\mathbf{x})$ be a u.d.p. map of the type

$$\Phi(\mathbf{x}) = (\Phi_1(x_1), \Phi_2(x_2), \dots, \Phi_s(x_s)),$$

where $\Phi_j(x)$ are one-dimensional u.d.p. maps. To short we denote $N_j = b_j^{\beta_j}$. Then the number of terms of (99) is $N = N_1 N_2 \dots N_s$ and simplified $\mathbf{x}_n = (n_1/N_1, n_2/N_2, \dots, n_s/N_s)$. By definition in Section 2, for fixed $\mathbf{k} = (k_1, k_2, \dots, k_s)$, we have

$$\text{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^s \text{wal}_{k_j} \left(\frac{n_j}{N_j} \right),$$

where the Walsh function $\text{wal}_{k_j}(x)$ is defined with respect to base b_j . Then

$$\frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) = \prod_{j=1}^s \left(\frac{1}{N_j} \sum_{n_j=0}^{N_j-1} \text{wal}_{k_j} \left(\frac{n_j}{N_j} \right) \right)$$

and by means of one-dimensional result (66) we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \text{wal}_{\mathbf{k}}(\mathbf{x}_n) = \begin{cases} 0 & \text{if } \mathbf{b}^{\beta} \nmid \mathbf{k}, \\ 1 & \text{if } \mathbf{b}^{\beta} \mid \mathbf{k}. \end{cases} \quad (100)$$

Also

$$\begin{aligned}\widehat{K}_1(\mathbf{k}, \mathbf{k}) &= \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) \text{wal}_{\mathbf{k}}(\mathbf{x}) \overline{\text{wal}_{\mathbf{k}}(\mathbf{y})} d\mathbf{x} d\mathbf{y} \\ &= \prod_{j=1}^s \int_0^1 \int_0^1 K_j(\Phi_j(x_j), \Phi_j(y_j)) \text{wal}_{k_j}(x_j) \overline{\text{wal}_{k_j}(y_j)} dx_j dy_j = \prod_{j=1}^s K_{1,j}(k_j, k_j).\end{aligned}$$

Applying (100) in Theorem 3 we find

$$\begin{aligned}& \int_{[0,1]^s} \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\sigma} \\ &= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^s \\ \mathbf{k} \neq \mathbf{0}, \mathbf{b}^\beta | \mathbf{k}}} \widehat{K}_1(\mathbf{k}, \mathbf{k}) = \prod_{j=1}^s \left(\sum_{\substack{k_j=0 \\ b_j^{\beta_j} | k_j}}^{\infty} \widehat{K}_{1,j}(k_j, k_j) \right) - \widehat{K}_1(\mathbf{0}, \mathbf{0}) \\ &= \prod_{j=1}^s \left(\sum_{\substack{k_j=1 \\ b_j^{\beta_j} | k_j}}^{\infty} \widehat{K}_{1,j}(k_j, k_j) + \widehat{K}_j(0, 0) \right) - \prod_{j=1}^s \widehat{K}_j(0, 0) \\ &= \prod_{j=1}^s \left(\int_0^1 \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \frac{1}{N_j} \sum_{n_j=0}^{N_j-1} f\left(\Phi_j\left(\frac{n_j}{N_j} \oplus \sigma\right)\right) - \int_0^1 f(x) dx \right|^2 d\sigma + \int_0^1 \int_0^1 K_j(x, y) dx dy \right) \\ &\quad - \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \tag{101}\end{aligned}$$

because

$$\widehat{K}_1(\mathbf{0}, \mathbf{0}) = \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y})) d\mathbf{x} d\mathbf{y} = \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = \widehat{K}(\mathbf{0}, \mathbf{0}).$$

Now, putting

$$\begin{aligned}b_1 &= b_2 = \dots = b_s = 2, & \beta_1 &= \dots = \beta_s = \beta, & N &= (2^\beta)^s, \\ K_j(x, y) &= K(x, y) = 1 + \gamma B_1(x) B_1(y) + \frac{\gamma^2}{4} B_2(x) B_2(y) - \frac{\gamma^2}{24} B_4(|x - y|),\end{aligned}$$

and using the s -dimensional tent u.d.p. map

$$\Phi(x) = \Phi_0(\mathbf{x}) = (1 - |2x_1 - 1|, 1 - |2x_2 - 1|, \dots, 1 - |2x_s - 1|),$$

and inserting (82) to (101), also using $\int_0^1 \int_0^1 K(x, y) dx dy = 1$, we find the s -dimensional mean square worst-case error (101) as

$$\left(\frac{\gamma^2}{30 \cdot 2^{4\beta}} + 1 \right)^s - 1.$$

Thus, for the sequence (99) and applying the tent u.d.p. map $\Phi_0(\mathbf{x})$, we obtain the error $O(1/N^{4/s})$ what is contrary to the [1], where in [1; Theorem 5] is proved that for every $\varepsilon > 0$ there exists a digital net such that the mean square worst-case error is bounded by $O(1/N^{4-\varepsilon})$.

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