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E. Mann, The perturbed sine-Gordon breather equation integrated by Riemann's method, *ТМФ*, 1996, том 107, номер 3, 439–449

DOI: <https://doi.org/10.4213/tmf1168>

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26 апреля 2023 г., 08:10:34

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THE PERTURBED SINE-GORDON BREATHER EQUATION INTEGRATED BY RIEMANN'S METHOD

For the perturbed nonlinear sine-Gordon equation a general first-order perturbation solution in the neighbourhood of the unperturbed breather solution is given. The inhomogeneous linear hyperbolic differential equation is solved by Riemann's method. For the determination of the Riemann function no methods of inverse scattering theory are used. Instead, the Bäcklund transformation and a novel inversion relation are applied. The Riemann function may be expressed in terms of Lommel functions of two variables. It is shown that the so formulated Riemann function has, unlike the discrete part, the correct symmetry. As an example, the asymptotic solution for a low-amplitude breather under a constant perturbation is given, showing that plane waves are radiated to both sides of the breather.

1. INTRODUCTION

In the past forty years much progress has been achieved in the understanding and the solution of nonlinear partial differential equations in physics [1]. One of these, the nonlinear wave equation called sine-Gordon equation, has been thoroughly studied by Seeger et al. already in the fifties [2, 3, 4]. There this equation arises from the so-called line-tension model which describes a crystal dislocation as a stretched string moving in a sinusoidal potential. Later on [5] this equation was also called Enneper equation after the German mathematician who studied this equation in the context of differential geometry of pseudospherical surfaces already in 1870.

While the pure sine-Gordon or Enneper equation is exactly integrable (solutions are, e.g., kinks or breathers), this is no longer the case if there is an additional force or a perturbation term in this equation. This perturbation problem has been handled hitherto almost exclusively by applying inverse scattering methods [1]. In the present treatment no application of the inverse scattering theory is made. Instead, a corresponding transformation is found which may be verified by using the Bäcklund transformation.

In the following a first-order perturbation theory for the perturbed sine-Gordon breather is developed. The inhomogeneous linear hyperbolic differential equation is solved by means of Riemann's method, and the Riemann function is determined with the help of the above-mentioned novel transformation relation. As an example, the asymptotic behaviour of the solution for a constant external force is studied.

2. GENERAL PROBLEM

The perturbed sine-Gordon or Enneper equation

$$u_{tt} - u_{xx} + \sin u = \varepsilon P(x, t), \quad (1)$$

where εP denotes the (small) perturbation, is linearized in the neighbourhood of an unperturbed breather solution u° ,

$$u = u^\circ + v, \quad (2)$$

and leads to the inhomogeneous linear partial differential equation

$$v_{tt} - v_{xx} + (\cos u^\circ)v = \varepsilon P. \quad (3)$$

We consider a breather whose centre is at rest [6, 7, 8]

$$u^\circ = -4 \arctan \left(\frac{1}{\operatorname{sh} \sigma} \frac{\sin \tau}{\operatorname{ch} \xi} \right), \quad \cos u^\circ = 1 - \frac{8 \operatorname{sh}^2 \sigma \operatorname{ch}^2 \xi \sin^2 \tau}{N^2}, \quad (4)$$

with

$$\xi = \frac{x}{\operatorname{ch} \sigma}, \quad \tau = (\operatorname{th} \sigma)t, \quad N = \operatorname{sh}^2 \sigma \operatorname{ch}^2 \xi + \sin^2 \tau. \quad (5)$$

The parameter σ characterizes the amplitude as well as the frequency and the extension of the breather.

In characteristic coordinates

$$p = (x - t)/2, \quad q = (x + t)/2 \quad (6)$$

the perturbed linear equation (3) becomes

$$v_{pq} - (\cos u^\circ)v = -\varepsilon P. \quad (7)$$

In addition, some initial conditions may be given

$$v(x, 0) = f(x), \quad v_t(x, t)|_{t=0} = g(x). \quad (8)$$

This Cauchy problem will be solved by means of Riemann's method [9, 10], which is appropriate to inhomogeneous linear hyperbolic differential equations. The following formulae represent specializations to the present problem with a self-adjoint differential operator. The solution of the Cauchy problem (7), (8) is then given by

$$\begin{aligned} v(p, q) = & \frac{1}{2}(f_A + f_B) + \int_{AB} \frac{ds'}{\sqrt{2}} \left[gR(p', q'; p, q) - \frac{1}{2}f(R_{q'} - R_{p'}) \right] + \\ & + \int_{ABP} R(p', q'; p, q) \varepsilon P(p', q') dp' dq', \end{aligned} \quad (9)$$

FIG. 1

FIG. 2

where $R(p', q'; p, q)$ is the so-called Riemann function. The line integral with the line element ds' extends from A to B on the diagonal $p' = q'$ (see Fig. 1), and the surface integral is to be taken over the triangle ABP.

While the use of characteristic coordinates is in general advantageous, sometimes the formulation in (x, t) -coordinates is of interest (see Fig. 2)

$$v(x, t) = \frac{1}{2}(f_A + f_B) + \frac{1}{2} \int_{x-t}^{x+t} dx' \left[g(x') \tilde{R}(x', 0; x, t) - f(x') \tilde{R}_{t'}(x', t'; x, t) \Big|_{t'=0} \right] + \frac{1}{2} \int_{ABP} \tilde{R}(x', t'; x, t) \varepsilon P(x', t') dx' dt'. \quad (10)$$

Here, of course, $\tilde{R}(x', t'; x, t) = R(p', q'; p, q)$ has been introduced (in the following we shall use the same symbol for such functions with different arguments).

The Riemann function R is to be determined from the conditions

$$R_{p'q'} - (\cos u^\circ) R = 0, \quad (11)$$

$$R(p', q'; p, q) = 1 \quad \text{on } AP(p' = p) \quad \text{and} \quad \text{on } PB(q' = q). \quad (12)$$

This means that $R(p', q'; p, q)$ has to satisfy the homogeneous part of our starting equation (7), with coordinates p, q replaced by p', q' . Very generally, the Riemann function for a self-adjoint problem should have the symmetry property

$$R(p', q'; p, q) = R(p, q; p', q'). \quad (13)$$

3. DETERMINATION OF THE RIEMANN FUNCTION

The boundary value problem (11), (12) divides into two parts, the finding of solutions of the homogeneous equation and the adjustment to the boundary conditions.

3.1. General homogeneous solution. For finding solutions of the homogeneous equation

$$\varphi_{pq} - (\cos u^\circ) \varphi = 0, \quad (14)$$

we remark that (14) represents the linearized version of the homogeneous nonlinear equation (1) (with $\varepsilon = 0$) in the neighbourhood of the breather solution u° . The most general solution of the Enneper equation is obtained by continued application of the Bäcklund transformation [4–8]. Starting from a solution $u_\circ$ of the Enneper equation, another solution u_i is determined by

$$\frac{1}{2} \frac{\partial}{\partial p} (u_i - u_\circ) = \beta_i \sin \frac{u_i + u_\circ}{2}, \quad \frac{1}{2} \frac{\partial}{\partial q} (u_i + u_\circ) = \beta_i^{-1} \sin \frac{u_i - u_\circ}{2}, \quad (15)$$

where β_i is a parameter. Each such a transformation produces a generalized kink (soliton). The most general two-kink solution with the initial solution $u_\circ \equiv 0$ contains four parameters. For a special choice of these parameters the breather solution (4) results. Therefore, a variation of the general two-kink solution with respect to the four parameters in the neighbourhood of the breather values yields four independent solutions, the so-called discrete solutions, of the equation (14). For $u_\circ$ nonzero but small, $u_\circ$ obeys $u_{\circ,tt} - u_{\circ,xx} + u_\circ = 0$, the Klein–Gordon equation, with the general solution

$$u_\circ = A e^{i(kx+\omega t)} + B e^{i(kx-\omega t)}, \quad \omega = +(1+k^2)^{1/2}, \quad (16)$$

where the wavenumber k is a parameter. Starting from this solution, a first-order treatment of the general two-kink solution with respect to A and B , and taking the breather values for the four discrete parameters, yields another (continuous) solution of (14).

The most general solution of the homogeneous equation (14) is then given by a sum or integral of these solutions [7, 8]

$$\begin{aligned} \varphi(x, t) &= \sum_{\mu=1}^4 A_\mu \varphi_\mu(x, t) + \int_{-\infty}^{+\infty} dk [A(k) \phi(x, t; k, \omega) + B(k) \phi(x, t; k, -\omega)], \\ \varphi_1 &= \frac{1}{N} \operatorname{sh} \xi \sin \tau, \quad \varphi_3 = \frac{1}{N} [\tau \operatorname{sh} \xi \sin \tau - \operatorname{sh}^2 \sigma \xi \operatorname{ch} \xi \cos \tau], \\ \varphi_2 &= \frac{1}{N} \operatorname{ch} \xi \cos \tau, \quad \varphi_4 = \frac{1}{N} [\operatorname{ch}^2 \sigma \operatorname{ch} \xi \sin \tau - \tau \operatorname{ch} \xi \cos \tau - \operatorname{sh}^2 \sigma \xi \operatorname{sh} \xi \sin \tau], \\ \phi(x, t; k, \omega) &= e^{i(kx+\omega t)} [1 + \psi(x, t; k, \omega)], \\ \psi(x, t; k, \omega) &= \frac{2(\sin^2 \tau - \operatorname{ch}^2 \xi) + i(k \operatorname{ch} \sigma \operatorname{sh} 2\xi + \omega \operatorname{cth} \sigma \sin 2\tau)}{(1 + \operatorname{ch}^2 \sigma k^2) N / \operatorname{sh}^2 \sigma}. \end{aligned} \quad (17)$$

Using characteristic coordinates (6) and a new integration variable s , with

$$s = k + \omega, \quad \frac{1}{s} = \omega - k, \quad k = \frac{1}{2} \left(s - \frac{1}{s} \right), \quad \omega = \frac{1}{2} \left(s + \frac{1}{s} \right), \quad (18)$$

the two functions $A(k)$ and $B(k)$ may be comprised into one function $C(s)$:

$$\varphi(p, q) = \sum_{\mu=1}^4 A_\mu \varphi_\mu(p, q) + \int_{-\infty}^{+\infty} ds C(s) \phi(p, q; s). \quad (19)$$

3.2. Inversion relation and consideration of boundary conditions. In order to determine the coefficients in the general solution (19) from some initial or boundary conditions like (12), the form (19) is not well suited without knowledge of any orthogonality relations for the φ_μ and ϕ functions. In the inverse scattering method these have been established [11, 12], but, as stated in the introduction, we intend to show that it is not necessary to use that method. Rather, we shall present an inversion relation which has been derived in a heuristic way and may be verified by repeated use of the Bäcklund transformation. This relation, which seemingly has not been stated by other authors, is [7]

$$\begin{aligned} \varphi_{xx} + \varphi_{tt} + B_1(x, t)\varphi_x + B_2(x, t)\varphi_t + B_3(x, t)\varphi = \\ = -2 \int_{-\infty}^{+\infty} dk \left(\frac{1}{\text{ch}^2 \sigma} + k^2 \right) \left[A(k)e^{i(kx+\omega t)} + B(k)e^{i(kx-\omega t)} \right], \end{aligned} \quad (20)$$

with

$$\begin{aligned} B_1 = \frac{2}{N} \frac{\text{sh}^2 \sigma}{\text{ch} \sigma} \text{sh} \left(2 \frac{x}{\text{ch} \sigma} \right), \quad B_2 = \frac{2}{N} \text{th} \sigma \sin(2 \text{th} \sigma t), \\ B_3 = 1 + \frac{2}{\text{ch}^2 \sigma} - \frac{4 \sin^2(\text{th} \sigma t)}{N}. \end{aligned} \quad (21)$$

Expressed in the variables p, q (6) and s (18) this becomes

$$\varphi_{pp} + \varphi_{qq} + C_1 \varphi_p + C_2 \varphi_q + C_3 \varphi = - \int_{-\infty}^{+\infty} ds \bar{C}(s) e^{i(sq-s^{-1}p)}, \quad (22)$$

$$\begin{aligned} C_1(p, q) = B_1 - B_2, \quad C_2(p, q) = B_1 + B_2, \quad C_3(p, q) = 2B_3, \\ \bar{C}(s) = \left[\frac{4}{\text{ch}^2 \sigma} + \left(s - \frac{1}{s} \right)^2 \right] C(s). \end{aligned} \quad (23)$$

These relations may be interpreted as “inversions” of the expressions (17) and (19) for the general solution φ in so far as expressions for φ in terms of complicated functions have been inverted into equations with much simpler factors at the coefficients and a more complicated dependence on φ . Moreover, it is quite remarkable that the discrete terms φ_μ do no longer appear in these relations.

In determining the coefficients in the homogeneous solution from the boundary conditions (12), it is typographically more convenient to consider the Riemann function $R(p, q; p', q')$. Then R satisfies the homogeneous equation (14), and the appropriate figure is Fig. 1 with p, q and p', q' interchanged. The fixed point is now $P=(p', q')$ and the running variables are p, q . We insert the conditions (12) into the relation (22) and try to determine the coefficient $\bar{C}(s)$.

First we consider the boundary $p = p'$. Here $\varphi = 1$ for $q \leq q'$, and we assume $\varphi = 0$ for $q > q'$ (which is in agreement with the behaviour of the final Riemann function). Then $\varphi_q = -\delta(q - q')$, $\varphi_{qq} = -\delta_q(q - q')$. We imagine φ continued to $q \rightarrow -\infty$, multiply equation (22) by $\exp(-is'q)$, integrate over q from $-\infty$ to $+\infty$, and replace s' by s to obtain

$$\begin{aligned} -2\pi \bar{C}(s) e^{-is^{-1}p'} &= \int_{-\infty}^{+\infty} dq (\varphi_{pp} + \varphi_{qq} + C_1 \varphi_p + C_2 \varphi_q + C_3 \varphi) e^{-isq} = \\ &= -(is + C_2) e^{-isq'} + \int_{-\infty}^{+\infty} dq (\varphi_{pp} + C_1 \varphi_p + C_3 \varphi) e^{-isq}. \end{aligned} \quad (24)$$

For reasons of convergence, $s = \text{Re } s + i\varepsilon$ is assumed. With $\varphi_{pq} = (\cos u^\circ)\varphi \equiv c(p, q)\varphi$ and $\varphi_{ppq} = (c\varphi)_p$ an integration by parts gives

$$\begin{aligned} -2\pi\bar{C}(s)e^{-is^{-1}p'} &= -(is + C_2 - \frac{i}{s}C_3)e^{-isq'} - \\ &\quad - \frac{i}{s} \int_{-\infty}^{+\infty} dq [(c + C_{1,q})\varphi_p + (c_p + C_1c + C_{3,q})\varphi]e^{-isq}. \end{aligned} \quad (25)$$

Because of the general relationships

$$c + C_{1,q} = 1, \quad c_p + C_1c + C_{3,q} = C_1, \quad (26)$$

the second integration by parts leads to

$$2\pi\bar{C}(s)e^{i(sq' - s^{-1}p')} = \frac{1}{is}C_3(p', q') + C_2(p', q') - \frac{1}{s^2}C_1(p', q') + \frac{i}{s}\left(s^2 + \frac{1}{s^2}\right). \quad (27)$$

The boundary condition on $q = q'$ gives the same result.

With (21) and (23) the coefficient $C(s)$ now becomes, using ψ of (17),

$$C(s) = \frac{-1}{2\pi is}e^{-i(sq' - s^{-1}p')}(1 + \psi^*(p', q'; s)), \quad (28)$$

so that the integral term of φ in (19) can be written as

$$\begin{aligned} \int_{-\infty}^{+\infty} ds C(s)\phi &= \frac{-1}{2\pi i} \int_{-\infty}^{+\infty} \frac{ds}{s} e^{is(q-q') + is^{-1}(p'-p)}(1 + \psi)(1 + \psi'^*) = \\ &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{ds}{s} e^{is(q'-q) + is^{-1}(p-p')}(1 + \psi^*)(1 + \psi'), \end{aligned} \quad (29)$$

where $\psi' \equiv \psi(p', q'; s)$. The last expression results by taking the complex conjugate. This is permitted since the left-hand side like the Riemann function is real. For definiteness, s should then have a small negative imaginary part, $s = \text{Re } s - i\varepsilon$ for $p - p' > 0$. In case of $\psi \equiv 0$, just the Bessel function J_0 results:

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{ds}{s} e^{is(q'-q) + is^{-1}(p-p')} = J_0\left(2[(p-p')(q'-q)]^{1/2}\right). \quad (30)$$

Now the determination of the coefficients A_μ in (19) is left. They are defined by, for example, again requiring $\varphi = 1$ for $p = p'$ and q variable. The part (30) of (29) just gives 1. Since $\psi \sim s$ for small s , but $\sim 1/s$ for large s , the integral (29) may easily be evaluated by contour integration round the poles of ψ . The result is

$$\sum_{\mu=1}^4 A_\mu \varphi_\mu(p, q) = 2 \text{sh } \sigma [\varphi'_3 \varphi_1 + \varphi'_4 \varphi_2 - \varphi'_1 \varphi_3 - \varphi'_2 \varphi_4]. \quad (31)$$

We observe that both the coefficients $C(s)$ and A_μ are expressed through the functions ϕ and φ_μ of the general homogeneous solution (19).

3.3. The Riemann function. The homogeneous solution (19) with coefficients adjusted to the boundary conditions (12) represents just the desired Riemann function. Returning to the initially chosen running coordinates p', q' , for which Fig. 1 is appropriate, the Riemann function now is given by

$$R(p', q'; p, q) = R_d + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{ds}{s} e^{is(q-q') + is^{-1}(p'-p)} (1 + \psi)(1 + \psi'^*), \quad (32)$$

where

$$R_d = -2 \operatorname{sh} \sigma [\varphi'_3 \varphi_1 + \varphi'_4 \varphi_2 - \varphi'_1 \varphi_3 - \varphi'_2 \varphi_4] \quad (33)$$

denotes the “discrete” part of R . The inhomogeneous solution (9) or (10) with this form of R agrees with the solution found in [7] by means of Duhamel’s method.

The perturbation term in the inhomogeneous solution (9) constructed with the discrete part R_d represents, as in [7], the linear expansion of the so-called adiabatic approximation [13] and thus arises in a natural way [8]. From a mathematical point of view, however, this approximate solution is not satisfactory since R_d , unlike the true Riemann function, is not symmetric in interchanging p, q with p', q' , as may be seen from (33). The sum of both terms in (32) should, of course, have the correct symmetry.

An appropriate representation is obtained by the following considerations. Since $\psi(p, q; s)$ possesses four poles s_ν with $|s_\nu| = 1$,

$$s_{1,2} = \pm \operatorname{th} \sigma + \frac{i}{\operatorname{ch} \sigma}, \quad s_{3,4} = \pm \operatorname{th} \sigma - \frac{i}{\operatorname{ch} \sigma}, \quad (34)$$

the terms beyond 1 of the last product in (32) may be written as

$$\psi(p, q; s) + \psi^*(p', q'; s) + \psi\psi'^* = \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s - s_\nu}, \quad (35)$$

where β_ν and γ_ν are functions of p, q, p', q' . Thus (32) becomes

$$R(p', q'; p, q) = R_d + J_0(y) + \frac{1}{2\pi i} \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \times \\ \times \int_{-\infty}^{+\infty} \frac{ds}{s(s - s_\nu)} e^{is(q-q') + is^{-1}(p'-p)}, \quad (36)$$

where $J_0(y)$ denotes the Bessel function defined in (30), with $y = 2(q - q')^{1/2}(p' - p)^{1/2}$. For $q - q' > 0$ and $p' - p > 0$ (cf. Fig. 1), the path of integration in (36) just below the real axis can be closed in the upper half-plane, and the integral is a sum of contour integrals round the poles $s = 0$ and $s = s_1, s_2$

$$R(p', q'; p, q) = R_d + J_0(y) + \frac{1}{2\pi i} \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \times \\ \times \oint \frac{ds}{s(s - s_\nu)} e^{is(q-q') + is^{-1}(p'-p)} + \\ + \sum_{\nu=1,2} \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} e^{is_\nu(q-q') + is_\nu^{-1}(p'-p)}. \quad (37)$$

As can be shown, the last term in (37) just compensates R_d , so that the discrete terms and with them the above mentioned discrepancy have disappeared. Other useful relationships are

$$\sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} = 0, \quad \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} e^{is_\nu(q-q') + is_\nu^{-1}(p'-p)} = 0. \quad (38)$$

By a decomposition into partial fractions and applying the first relation of (38) the Riemann function (37) may also be written as

$$R(p', q'; p, q) = J_0(y) + \frac{1}{2\pi i} \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} \oint \frac{ds}{s - s_\nu} e^{is(q-q') + is^{-1}(p'-p)}, \quad (39)$$

where the contour integral is taken round $s = 0$ with $|s| < 1$ or, because of the second relation in (38), $|s| \geq 1$.

By introducing in (36) the identities

$$\frac{1}{s - s_\nu} = \pm \frac{1}{i} \int_0^\infty e^{\pm i(s-s_\nu)\bar{q}} d\bar{q}, \quad \text{with } + \text{ for } \nu = 3, 4, \text{ and } - \text{ for } \nu = 1, 2, \quad (40)$$

and noting that for $\nu = 3, 4$

$$\int_0^\infty e^{-is_\nu(q-q')x} J_0(y\sqrt{x}) dx = \frac{1}{is_\nu(q-q')} e^{is_\nu^{-1}(p'-p)}, \quad (41)$$

one may also obtain

$$R = J_0(y) + i(q - q') \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \int_0^1 dx J_0(y\sqrt{x}) e^{is_\nu(q-q')(1-x)}. \quad (42)$$

This integral is just the Bessel transform of Lommel functions of two variables [14]

$$R = J_0(y) + i \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} [U_1(w_\nu, y) + iU_2(w_\nu, y)] \quad (43)$$

with $w_\nu = 2(q - q')s_\nu$. Definitions for the Lommel functions are

$$\begin{aligned} U_n(w, z) &= \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z} \right)^{n+2m} J_{n+2m}(z) = \cos \left(\frac{w}{2} + \frac{z^2}{2w} - n\frac{\pi}{2} \right) + \\ &+ (-1)^n \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{w} \right)^{2-n+2m} J_{2-n+2m}(z). \end{aligned} \quad (44)$$

Because of (38) then also holds, with $v_\nu = 2(p' - p)/s_\nu$,

$$R = J_0(y) - i \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} [U_1(v_\nu, y) + iU_2(v_\nu, y)]. \quad (45)$$

Noticing that

$$\frac{\partial^n}{\partial p^n} J_0(y) = \left(\frac{q - q'}{p' - p} \right)^{n/2} J_n, \quad \frac{\partial^n}{\partial q^n} J_0(y) = (-1)^n \left(\frac{p' - p}{q - q'} \right)^{n/2} J_n,$$

other useful forms may be derived

$$R = J_0(y) + \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} \frac{1}{1 - i s_\nu \partial / \partial p} J_0(y), \quad (46)$$

$$R = J_0(y) - \sum_{\nu=1}^4 \left(\beta_\nu + \gamma_\nu \frac{\partial}{\partial s_\nu} \right) \frac{1}{s_\nu} \frac{1}{1 + i s_\nu^{-1} \partial / \partial q} J_0(y). \quad (47)$$

By expanding the above relations the symmetry relation (13) is easily verified. Furthermore, by adding (43) and (45) or (46) and (47) it is rather easy to show that $R(x', t'; x, t)$ is invariant in exchanging x, x' for $-x, -x'$ and/or t, t' for $-t, -t'$.

4. APPLICATION

As an example we consider the asymptotic behaviour ($x \gg \text{ch } \sigma$) of the solution of equation (3) for a constant perturbation $\varepsilon P = S$. As initial conditions (8) we choose $f(x) = S$, corresponding to the rest position $u = \arcsin S \approx S$ according to (1), and $g(x) = 0$, which means that the initial velocity is that of the unperturbed breather. The functions β_ν, γ_ν defined in (35) reduce for $x \gg \text{ch } \sigma$ to $\beta_\nu = \gamma_\nu = 0$ for $\nu = 3, 4$ and

$$\begin{aligned} \beta_1/s_1 &= \frac{-1}{N'} \left[-i \text{ch } \sigma \text{th } \sigma + (1 - i \text{ch } \sigma \text{th } \sigma) e^{-2\xi'} + e^{2i\tau'} \right], \\ \beta_2/s_2 &= (\beta_1/s_1)^*, \quad \gamma_1/s_1^2 = \frac{-1}{N'} \left(e^{-2\xi'} + e^{2i\tau'} \right), \quad \gamma_2/s_2^2 = (\gamma_1/s_1^2)^*, \end{aligned} \quad (48)$$

with the abbreviations (5).

The term $J_0(y)$ of the Riemann function R contributes to the surface integral in (9) or (10) strictly

$$\begin{aligned} S \int_{\text{ABP}} J_0(y) dp' dq' &= \\ &= \frac{1}{2} S \int_{\text{ABP}} J_0 \left([(t - t')^2 - (x - x')^2]^{1/2} \right) dx' dt' = S(1 - \cos t). \end{aligned} \quad (49)$$

For the term $\bar{R} = R - J_0$ we take the form (47). Since the functions β_ν, γ_ν vanish at the integration limit $q' = q = (x + t)/2 \gg 1$, integrations over $\beta_\nu J_0$ and $\gamma_\nu J_0$ can be carried out before applying the operator with $\partial / \partial q$. Explicit analytic results are obtainable only in limiting cases. We consider the low-amplitude breather defined by $\text{sh } \sigma \gg 1$. With the approximation $N' \approx \text{sh}^2 \sigma \text{ch}^2 \xi'$, from (48) only the terms $\sim \exp(2i\tau')$ contribute.

Assuming $x, t, t - x \gg 1$, the integrations with respect to t' and x' may be performed and the q -operation carried out; for example one obtains

$$\begin{aligned} \frac{1}{1 + is_1^{-1} \partial / \partial q} \exp \left\{ 2i \operatorname{th} \sigma t - i [(2 \operatorname{th} \sigma)^2 - 1]^{1/2} x \right\} = \\ = \frac{1}{1 - s_1^{-1} (2 \operatorname{th} \sigma - [(2 \operatorname{th} \sigma)^2 - 1]^{1/2})} \exp \{ \}. \end{aligned}$$

The final result is

$$\begin{aligned} \frac{1}{2} \int_{\text{ABP}} \bar{R} dx' dt' = \pi \operatorname{cth}^2 \sigma \exp \left\{ -\frac{\pi}{2} \operatorname{ch} \sigma [(2 \operatorname{th} \sigma)^2 - 1]^{1/2} \right\} \times \\ \times \sin([(2 \operatorname{th} \sigma)^2 - 1]^{1/2} x - 2 \operatorname{th} \sigma t). \end{aligned} \quad (50)$$

The line integral in (9) or (10) defined with the Riemann function R is not simply calculated. Rather, it is much simpler to solve directly the initial-value problem for the homogeneous equation (14) via the inversion equation (20) and its time derivative [7]. For $f(x) = S$, $g(x) = 0$ follows $A(k) = B(k) = -(S/2)\delta(k)$, from (17) $A_\mu = 0$, and therefore

$$v_{\text{hom}}(x, t) = -S \left[\cos t \left(1 + \frac{2(\sin^2 \tau - \operatorname{ch}^2 \xi)}{\operatorname{ch}^2 \xi + \sin^2 \tau / \operatorname{sh}^2 \sigma} \right) - \sin t \frac{\operatorname{cth} \sigma \sin 2\tau}{\operatorname{ch}^2 \xi + \sin^2 \tau / \operatorname{sh}^2 \sigma} \right], \quad (51)$$

valid for all x, t . Thus the asymptotic solution for $x \gg \operatorname{ch} \sigma$, $t > x$, and for $\sigma \gg 1$ becomes by adding (49), (50), and (51)

$$v(x, t) \Rightarrow S + S\pi \exp(-\operatorname{ch} \sigma \pi \sqrt{3}/2) \sin(\sqrt{3}x - 2t). \quad (52)$$

If we had considered an expansion of $1/N'$ in terms of $\sin^2 \tau' / \operatorname{sh}^2 \sigma \operatorname{ch}^2 \xi'$, each term would have contributed an expression like the second in (52), with different numerical factors of the same order. Therefore, only the functional form of (52) is of importance. The result (52), which agrees with the former result in [7], shows that at a distance x larger than the extension $\operatorname{ch} \sigma$ of the unperturbed breather after times $t > x$ a plane wave with wavenumber $\sqrt{3}$ and frequency 2 is running out, corresponding to an energy radiation [7], which, however, because of the exponential factor is only weak. The emission power derived from (52) is [7]

$$W = 2\sqrt{3}\pi^2 S^2 \exp(-\pi\sqrt{3} \operatorname{ch} \sigma), \quad (53)$$

in accordance with the statements by Malomed [15, 1].

Acknowledgement

The author wishes to thank Professor A. Seeger for his stimulating interest and Professor S. Yu. Slavyanov for his kind support.

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Поступила в редакцию
11.I.1996 г.