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ON SOME INTEGRABLE GENERALIZATIONS OF THE CONTINUOUS TODA SYSTEM

In the present paper, we obtain some integrable generalizations of the continuous Toda system, generated by a flat connection form taking values in higher grading subspaces of the algebra of the area-preserving diffeomorphism of the torus T^2 , and construct their general solutions. The grading condition which we use here, imposed on the connection, can be realized in terms of some holomorphic distributions on the corresponding homogeneous spaces.

1. INTRODUCTION

In the present paper, we obtain some integrable generalizations of the continuous Toda system generated by a flat connection form taking values in higher grading subspaces of the Lie Poisson bracket algebra $\mathcal{L}$ on a two-dimensional torus T^2 . The main statement is the following. Let $\mathcal{M}$ be a two-dimensional manifold with the local coordinates z_+ and z_- , and G be an exponential mapping of $\mathcal{L}$ defined, e.g., in the sense of the inverse limit of Hilbert¹⁾. Then,

A flat connection in the trivial fiber bundle $\mathcal{M} \times G \rightarrow \mathcal{M}$, with $(1, 0)$ - and $(0, 1)$ -components taking values in the subspaces $\bigoplus_{m=0}^{m_+} \mathcal{G}_m$ and $\bigoplus_{m=0}^{m_-} \mathcal{G}_{-m}$ of the $\mathbf{Z}$ -graded Lie algebra $\mathcal{L} = \bigoplus_m \mathcal{G}_m$, generates an integrable system of partial differential equations

$$\begin{aligned} \partial_+ \partial_- \rho &= \sum_{m=1}^{m_0} m \frac{\partial^2}{\partial \tau^2} (\omega_m^+ \omega_m^- e^{m\rho}), \quad m_0 \equiv \min(m_+, m_-); \\ \partial_{\mp} \omega_m^{\pm} &= \mp i \sum_{n=1}^{m_{\pm}-m} (m+n) (\omega_{m+n}^{\pm})^{\frac{m}{m+n}} \frac{\partial}{\partial \tau} [\omega_n^{\mp} (\omega_{m+n}^{\pm})^{\frac{n}{m+n}} e^{n\rho}], \end{aligned} \quad (1.1)$$

$1 \leq m \leq m_{\pm} - 1$, for the functions ρ and $\omega_m^{\pm}$ depending on three variables $z_{\pm}$ and τ ; $\omega_{m_+}^+ = \omega_{m_-}^- \equiv 1$. The general solution of the Goursat (boundary value) problem for this

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¹⁾Note that an exponential mapping of $\mathcal{L}$ gives a differentiable structure which is weaker than that of a Lie group in the classical sense. Nevertheless, in the same way as for $S_0 \text{ Diff } T^2$, see e.g., [1], it can be defined in the sense of this limit, and just this fact allows us to perform the construction given in the paper.

system is determined by a complete set of arbitrary functions $\Phi_m^+(z_+; \tau)$, $1 \leq m \leq m_+$, and $\Phi_m^-(z_-; \tau)$, $1 \leq m \leq m_-$.

In differential geometry language, the grading condition we use, imposed on the flat connection, can be realized in terms of some holomorphic and antiholomorphic distributions on the homogeneous spaces $F_+ = G/B_-$ and $F_- = G/B_+$, respectively, where $B_\pm$ are exponential mappings of $\bigoplus_{m=0}^\infty \mathcal{G}_{\pm m}$ modifying the reasonings given in [2] for finite-dimensional case. Such a distribution, say on F_+ , is generated by the subspaces of $T^{(1,0)}(F_+)$ with the help of the canonical projection $\pi_+ : G \rightarrow F_+$, as $\pi_{+*}(\mathcal{M}_{+h})$. Here $\mathcal{M}_{+h}$ is obtained by the left translation (with an element h of G , $\pi_+(h) \in F_+$) of the subspace $\mathcal{M}_+ \equiv \bigoplus_{m \leq m_+} \mathcal{G}_{+m}$ invariant under the adjoint action of B_- in $\mathcal{L}$. A distribution on F_- is defined similarly, in terms of the canonical projection $\pi_- : G \rightarrow F_-$, and the subspace $\mathcal{M}_- \equiv \bigoplus_{m \leq m_-} \mathcal{G}_{-m}$.

For the simplest case, where the corresponding connection takes values in the local part $\mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1}$ of $\mathcal{L}$, Eqs. (1.1) are reduced to the well-known continuous Toda system (the heavenly equation, familiar thanks to [3])

$$\partial_+ \partial_- \rho = \frac{\partial^2}{\partial \tau^2} e^\rho, \quad (1.2)$$

for which we briefly discuss the boundary condition on a half-line, $z_+ = z_-$. Quite significant, while still a rather simply represented case, is $m_\pm = 2$, where system (1.1) is reduced to the following equations:

$$\begin{aligned} \partial_+ \partial_- \rho &= \frac{\partial^2}{\partial \tau^2} [2e^{2\rho} + \omega^+ \omega^- e^\rho], \\ \partial_\mp \omega^\pm &= \mp 2i \frac{\partial}{\partial \tau} [\omega^\mp e^\rho]. \end{aligned} \quad (1.3)$$

As a direct by-product of the given consideration, one obtains the integrable equations corresponding to the higher grading subspaces of the centerless Virasoro–Witt algebra.

Note that the problem of constructing nonlinear integrable two-dimensional partial differential equations associated with higher $\mathbf{Z}$ -grading subspaces has been already considered in a general and abstract form, that is, for an arbitrary Lie algebra $\mathcal{G}$, see [4]. However, even in the simplest case like A_n , the resulting equations and their general solutions look, in a sense, more complicated than their continuous limit (1.1).

2. SOME INFORMATION ABOUT LIE POISSON BRACKET ALGEBRA

Let us recall briefly some information about the Lie Poisson bracket algebra $\mathcal{L}$ following [5, 6]. This algebra, being considered as a $\mathbf{Z}$ -graded continuum Lie algebra $\mathcal{G}(E; -i\partial/\partial\tau; -i\partial/\partial\tau) = \bigoplus_{m \in \mathbf{Z}} \mathcal{G}_m$, is isomorphic to A_∞ and $S_0 \text{ Diff } T^2$ (infinitesimal area-preserving diffeomorphisms of the torus T^2). Its elements satisfy the commutation relations

$$[X_m(\phi), X_n(\psi)] = iX_{m+n}(n\dot{\phi}\psi - m\phi\dot{\psi}). \quad (2.1)$$

Here $X_m(\phi) = \int d\tau X_m(\tau) \phi(\tau)$ are the elements of the subspaces $\mathcal{G}_m$ parametrised by the functions $\phi(\tau)$ belonging to the algebra E of trigonometrical polynomials on a circle; $\dot{\phi} \equiv \frac{\partial \phi}{\partial \tau}$.

The algebra $\mathcal{G}(E; -i\frac{\partial}{\partial\tau}; -i\frac{\partial}{\partial\tau})$ is of constant growth (in a functional sense) since $\mathcal{G}_n \simeq \mathcal{G}_1 \simeq E$, its Cartan subalgebra $\wp \simeq \mathcal{G}_0$ is infinite-dimensional, and the roots are $n\delta'(\tau)$.

Let $\wp^*$ be an algebra dual to $\wp$, let V be an $\mathcal{L}$ -module and $\lambda \in \wp^*$. Denote by V_λ a set of vectors $v \in V$ satisfying $X_0(\phi)v = \lambda(\phi)v$ for all $\phi \in E$. It can be shown by an appropriate limit procedure (starting from A_r and using the aforementioned isomorphism $\mathcal{G}(E; -i\frac{\partial}{\partial\tau}; -i\frac{\partial}{\partial\tau}) \simeq \mathcal{G}(E; -\frac{\partial^2}{\partial\tau^2}; \text{id}) \simeq A_\infty$) that there exists a nonzero vector $\bar{v} \in V$ such that $\mathcal{G}_m(\bar{v}) = 0$ for $m > 0$ and $U(\mathcal{L})(\bar{v}) = V$. Here $U(\mathcal{L})$ is the universal enveloping algebra for $\mathcal{L}$. By analogy with the usual (“discrete”) case, this $\mathcal{L}$ -module V is called the highest weight module, and $\bar{v}$ the highest weight vector.

A symmetrical bilinear invariant form on the local part of the algebra in question is defined as follows:

$$\text{tr}(X_i(f)X_j(g)) = \delta_{i+j,0}(f,g), \quad i, j = 0, \pm 1;$$

where

$$(f,g) \equiv \int d\tau f(\tau)g(\tau).$$

Thanks to the invariance property of the form,

$$([X_m(f), X_{-m}(g)], X_0(h)) = (X_m(f), [X_{-m}(g), X_0(h)]),$$

the commutation relations (2.1) give

$$-im \left(X_0 \left(\frac{\partial}{\partial\tau}(fg) \right), X_0(h) \right) = im(X_m(f), X_{-m}(g\dot{h}));$$

and so one has

$$(X_m(f), X_{-m}(g\dot{h})) = - \int d\tau \frac{\partial}{\partial\tau}(f(\tau)g(\tau))h(\tau) = \int d\tau fg\dot{h}.$$

Therefore,

$$(X_m(f), X_n(g)) = \delta_{m+n,0}(f,g). \quad (2.2)$$

To this end, denote a continuous version (in the sense of the algebra A_∞ with the elements $X_m^{(A)}$) of the highest weight vectors of the fundamental representations of A_n by $|\tau\rangle$, for which

$$X_0^{(A)}(\phi)|\tau\rangle = \phi(\tau)|\tau\rangle, \quad X_m^{(A)}(\phi)|\tau\rangle = 0 \quad \text{for } m \geq 1, \quad (2.3)$$

so that for the case of $\mathcal{L}$, one has

$$X_0(\phi)|\tau\rangle = \partial_\tau^{-1}\phi(\tau)|\tau\rangle, \quad X_m(\phi)|\tau\rangle = 0 \quad \text{for } m \geq 1. \quad (2.4)$$

3. DERIVATION OF EQUATIONS

Let $\mathcal{M}$ be a two-dimensional manifold with the local coordinates z_+ and z_- in $\mathcal{M}$, and $\mathcal{A}$ be a Lie algebra $\mathcal{G}$ -valued 1-form on $\mathcal{M}$, $\mathcal{A} = \mathcal{A}_+dz_+ + \mathcal{A}_-dz_-$. Any such 1-form generates a connection form of a connection in the trivial fibre bundle. Suppose that $\mathcal{A}$ is flat, so that $\mathcal{A}_\pm$ are mappings from $\mathcal{M}$ to $\mathcal{G}$, satisfying the zero curvature condition

$$\left[\frac{\partial}{\partial z_+} + \mathcal{A}_+, \frac{\partial}{\partial z_-} + \mathcal{A}_- \right] = 0. \quad (3.1)$$

Let the connection $\mathcal{A}$ take values in the algebra $\mathcal{L}$. Choosing the basis $X_m(\phi)$ in $\mathcal{G}(E; -i\frac{\partial}{\partial\tau}; -i\frac{\partial}{\partial\tau})$ and considering the components of the decomposition of $\mathcal{A}_\pm$ over this basis as fields, we can treat (3.1) as a nonlinear system of partial differential equations for the fields. To provide for the nontriviality of such a system, we impose a grading condition on the connection [4], such that the $(1, 0)$ -component $\mathcal{A}_+$ of $\mathcal{A}$ takes values in $\bigoplus_{m \geq 0} \mathcal{G}_{+m}$, and the $(0, 1)$ -component $\mathcal{A}_-$ takes values in $\bigoplus_{m \geq 0} \mathcal{G}_{-m}$. Namely,

$$\mathcal{A}_+ = \sum_{m=0}^{m_+} X_m(f_m^+), \quad \mathcal{A}_- = \sum_{m=0}^{m_-} X_{-m}(f_m^-). \quad (3.2)$$

Then, substituting (3.2) in (3.1), one has

$$\begin{aligned} X_0 \left\{ \partial_+ f_0^- - \partial_- f_0^+ - i \sum_{m=1}^{m_0} m(f_m^+ f_m^-) \right\} &= 0, \\ \sum_{m=1}^{m_+} X_m \left\{ \partial_- f_m^+ + i m f_m^+ \dot{f}_0^- + i \sum_{1 \leq n < m \leq m_+} X_{m-n}(n \dot{f}_m^+ f_n^- + m f_m^+ \dot{f}_n^-) \right\} &= 0, \\ \sum_{m=1}^{m_-} X_{-m} \left\{ \partial_+ f_m^- - i m f_m^- \dot{f}_0^+ - i \sum_{1 \leq n < m \leq m_-} X_{-m+n}(n \dot{f}_m^- f_n^+ + m f_m^- \dot{f}_n^+) \right\} &= 0; \end{aligned} \quad (3.3)$$

or, in terms of the functions $f^\pm(z_+, z_-)$,

$$\begin{aligned} \partial_+ f_0^- - \partial_- f_0^+ &= i \sum_{m=1}^{m_0} m \frac{\partial}{\partial\tau} (f_m^+ f_m^-) = 0; \\ \partial_- f_m^+ + i m f_m^+ \dot{f}_0^- &= -i \sum_{n=1}^{m_+ - m} [n \dot{f}_{m+n}^+ f_n^- + (m+n) f_{m+n}^+ \dot{f}_n^-], \quad 1 \leq m \leq m_+; \\ \partial_+ f_m^- - i m f_m^- \dot{f}_0^+ &= i \sum_{n=1}^{m_- - m} [n \dot{f}_{m+n}^- f_n^+ + (m+n) f_{m+n}^- \dot{f}_n^+], \quad 1 \leq m \leq m_-. \end{aligned} \quad (3.4)$$

It follows from the last two systems with $m = m_+$ and $m = m_-$, respectively, that

$$\dot{f}_0^- = \frac{i}{m_+} \partial_- \ln f_{m_+}^+, \quad \dot{f}_0^+ = -\frac{i}{m_-} \partial_+ \ln f_{m_-}^-, \quad (3.5)$$

and from the other equations of these systems,

$$\begin{aligned} f_m^+ \partial_- \ln [f_m^+ (f_{m_+}^+)^{-\frac{m}{m_+}}] &= \\ = -i \sum_{n=1}^{m_+ - m} f_{m+n}^+ f_n^- \frac{\partial}{\partial\tau} \ln [(f_{m+n}^+)^n (f_n^-)^{m+n}], \quad 1 \leq m \leq m_+ - 1; \end{aligned} \quad (3.6)$$

$$\begin{aligned} f_m^- \partial_+ \ln [f_m^- (f_{m_-}^-)^{-\frac{m}{m_-}}] &= \\ = i \sum_{n=1}^{m_- - m} f_{m+n}^- f_n^+ \frac{\partial}{\partial\tau} \ln [(f_{m+n}^-)^n (f_n^+)^{m+n}], \quad 1 \leq m \leq m_- - 1. \end{aligned} \quad (3.7)$$

Substituting (3.5) in the first equation (3.4) differentiated over τ , one has

$$\partial_+ \partial_- \ln[(f_{m+}^+)^{1/m_+} (f_{m-}^-)^{1/m_-}] = \sum_{m=1}^{m_0} m \frac{\partial^2}{\partial \tau^2} f_m^+ f_m^-. \quad (3.8)$$

Then, with the notations

$$f_m^\pm (f_{m\pm}^\pm)^{-\frac{m}{m_\pm}} \equiv \omega_m^\pm, \quad 1 \leq m \leq m_\pm - 1; \quad (f_{m+}^+)^{1/m_+} (f_{m-}^-)^{1/m_-} \equiv e^\rho, \quad (3.9)$$

our equations are finally written as (1.1). In the simplest case, where $m_\pm = 1$, this system is reduced to Eq. (1.2), whose general solution is determined [7] by two arbitrary functions $\Phi^+(z_+, \tau)$ and $\Phi^-(z_-, \tau)$; for $m_\pm = 2$ – to equations (1.3).

The corresponding connection components in terms of the functions ρ and $\omega_m^\pm$ are rewritten as

$$\mathcal{A}_\pm = X_0 \left(\mp \frac{i}{m_\mp} \partial_\pm \partial_\tau^{-1} \ln f_{m_\mp}^\mp \right) + \sum_{m=1}^{m_\pm-1} X_{\pm m} (\omega_m^\pm (f_{m\pm}^\pm)^{\frac{m}{m_\pm}}) + X_{\pm m_\pm} (f_{m_\pm}^\pm); \quad (3.10)$$

or, in an appropriate gauge, as

$$\mathcal{A}_\pm = X_0 \left(\mp \frac{i}{2} \partial_\pm \partial_\tau^{-1} \rho \right) + \sum_{m=1}^{m_\pm-1} X_{\pm m} (\omega_m^\pm e^{\frac{m}{2}\rho}) + X_{\pm m_\pm} (e^{\frac{m_\pm}{2}\rho}). \quad (3.11)$$

For the case when all the functions depend on τ and $z_+ - z_-$ only, putting $-i\partial_+ + i\partial_- = \frac{1}{2} \frac{\partial}{\partial r}$ and choosing a gauge with $f_{m_\pm}^\pm = e^\rho$, for e.g., $m_\pm = 2$, we have the Lax operator

$$L \equiv \mathcal{A}_+ + \mathcal{A}_- = X_0 (\partial_r \partial_\tau^{-1} \rho) + X_1 (\omega^+ e^{\rho/2}) + X_{-1} (\omega^- e^{\rho/2}) + X_2 (e^\rho) + X_{-2} (e^\rho). \quad (3.12)$$

Then one can calculate, using (2.2), all conserved quantities as $\mathcal{I}_p \equiv \frac{1}{p} \text{tr } L^p$, for which $\frac{\partial}{\partial r} \mathcal{I}_p = 0$, e.g.

$$\mathcal{I}_2 = \int d\tau \left[\frac{1}{2} \left(\frac{\partial y}{\partial r} \right)^2 + e^{\frac{\partial y}{\partial \tau}} \left(e^{\frac{\partial y}{\partial \tau}} + \omega^+ \omega^- \right) \right], \quad (3.13)$$

where $y = \partial_\tau^{-1} \rho$.

Finishing the Section, let us note that Eqs. (1.1), remaining integrable, admit a natural reduction to a system associated with the higher grading subspaces of the centerless Virasoro–Witt algebra $\mathcal{W}$. Indeed, it is clear that with a choice $\phi(\tau) \simeq \tau$, the elements $X_m(\phi)$ of the Lie Poisson bracket algebra $\mathcal{L}$, given by the commutation relations (2.1), generate the algebra $\mathcal{W}$ with the elements L_m satisfying the relations

$$[L_m, L_n] = (m - n) L_{m+n}. \quad (3.14)$$

Then Eqs. (1.1) take the form

$$\begin{aligned}\partial_+ \partial_- \rho &= 2 \sum_{m=1}^{m_0} m \omega_m^+ \omega_m^- e^{m\rho}; \\ \partial_{\mp} \omega_m^{\pm} &= \mp i \sum_{n=1}^{m_{\pm}-m} (m+2n) \omega_n^{\mp} \omega_{m+n}^{\pm} e^{n\rho}, \quad 1 \leq m \leq m_{\pm} - 1,\end{aligned}\quad (3.15)$$

which for the special case $m_+ = m_- = 2$, coincides with those in [8].

4. CONSTRUCTION OF THE GENERAL SOLUTION

Let us construct the general solution to system (1.1) using an appropriate modification of the method given in [4]. The connection $\mathcal{A}$ is represented in the gradient form,

$$\mathcal{A}_{\pm} = g^{-1} \partial_{\pm} g, \quad (4.1)$$

with $g \in G$, and we take for $\mathcal{A}_{\pm}$ the Gauss type decomposition of g ,²⁾

$$g = M_- N_+ g_{0-} \quad \text{and} \quad g = M_+ N_- g_{0+},$$

respectively. The grading conditions (3.2) provide the holomorphic property of $M_{\pm}$, namely that the functions $M_{\pm}(z_{\pm}) \in G_{\pm}$ satisfy the initial value problem

$$\partial_{\pm} M_{\pm}(z_{\pm}) = M_{\pm}(z_{\pm}) L_{\pm}(z_{\pm}), \quad (4.2)$$

where

$$L_{\pm}(z_{\pm}) = \sum_{m=1}^{m_{\pm}} X_{\pm m}(\Phi_m^{\pm}(z_{\pm})) \quad (4.3)$$

with the arbitrary functions $\Phi_m^{\pm}(z_{\pm}; \tau)$ determining the general solution to our system. Then one has

$$\mathcal{A}_{\pm} = g_{0\mp}^{-1} (N_{\pm}^{-1} \partial_{\pm} N_{\pm}) g_{0\mp} + g_{0\mp}^{-1} \partial_{\pm} g_{0\mp}. \quad (4.4)$$

Differentiating the identity

$$M_+^{-1} M_- = N_- g_0 N_+^{-1} \quad (4.5)$$

($g_0 \equiv g_{0+} g_{0-}^{-1}$), over $z_{\pm}$, and using Eqs. (4.2), one can become convinced that the elements $N_{\pm}^{-1} \partial_{\pm} N_{\pm}$ take values in the subspaces $\bigoplus_{m=1}^{m_{\pm}} \mathcal{G}_{\pm m}$. Let us rewrite decomposition (4.4) as

$$\mathcal{A}_{\pm} = g_{0\pm}^{-1} \tilde{L}_{\pm} g_{0\pm} + g_{0\mp}^{-1} \partial_{\pm} g_{0\mp}, \quad (4.6)$$

where

$$\tilde{L}_{\pm} \equiv g_0^{\pm 1} (N_{\pm}^{-1} \partial_{\pm} N_{\pm}) g_0^{\mp 1} = \sum_{m=1}^{m_{\pm}} X_{\pm m}(F_m^{\pm}),$$

²⁾It seems believable that such a decomposition for G can be considered as an appropriate continuous limit of the Gauss decomposition for A_r .

with some functions $F_m^\pm(z_+, z_-)$. In the same way as for the case of a simple Lie algebra, here the functions $\tilde{L}_\pm$ are equal to $L_\pm$ only when $m_+ = m_- = 1$. However, thanks to identity (4.5), the elements $N_\pm$ and g_0 are determined by the elements $M_\pm$, and hence the functions $F_m^\pm$ entering $\tilde{L}_\pm$ can be expressed in terms of the functions $\Phi_m^\pm(z_\pm)$ for arbitrary values of $m_\pm$; moreover, it is clear that $F_{m_\pm}^\pm = \Phi_{m_\pm}^\pm$. A construction of the general solution to system (1.1) can be done in an explicit way. Before we give it, let us discuss briefly the solution to Eqs. (4.2).

The functions $M_\pm(z_\pm) \in G_\pm$ satisfying the initial value problem (4.2) are given by the corresponding multiplicative integrals

$$M_\pm = \sum_{n=0}^{\infty} \int_{Z_\pm^n(z_\pm)} dy_\pm L_\pm(y_\pm^n) L_\pm(y_\pm^{n-1}) \cdots L_\pm(y_\pm^1), \quad (4.7)$$

or, equivalently,

$$M_\pm^{-1} = \sum_{n=0}^{\infty} (-1)^n \int_{Z_\pm^n(z_\pm)} dy_\pm L_\pm(y_\pm^1) L_\pm(y_\pm^2) \cdots L_\pm(y_\pm^n), \quad (4.8)$$

where $Z_\pm^n(z_\pm) = \{y_\pm \in \mathbf{R}^n : a_\pm \leq y_\pm^n \leq y_\pm^{n-1} \leq \cdots \leq y_\pm^1 \leq z_\pm\}$, $a_\pm$ are constants determining the problem, and $M_\pm(a_\pm) = 1$. Note that there is also a noncommutative version of the well-known exponential formula, see [9, 4]:

$$M_\pm = \exp \sum_{n=1}^{\infty} \sum_{\omega} \frac{(-1)^{\epsilon(\omega)}}{n^2 C_{\epsilon(\omega)}^{n-1}} \int_{Z_\pm^n} dy_\pm [\cdots [L_\pm(y_\pm^{\omega(n)}), L_\pm(y_\pm^{\omega(n-1)})] \cdots] L_\pm(y_\pm^{\omega(1)}). \quad (4.9)$$

Here ω is a permutation of the set $\{1, 2, \dots, n\}$, $\epsilon(\omega)$ is the number of errors in the consecutive order of terms in $\{\omega(1), \omega(2), \dots, \omega(n)\}$, and $C_{\epsilon(\omega)}^{n-1}$ are the binomial coefficients.

The algebra $\mathcal{G}(E; -i\partial/\partial\tau; -i\partial/\partial\tau)$, which we have introduced as a $\mathbf{Z}$ -graded algebra, can be represented as a $\mathbf{Z}^2$ -graded algebra with one-dimensional components, e.g., with the elements $X_{\mathbf{m}}$ satisfying the commutation relations [10]

$$[X_{\mathbf{m}}, X_{\mathbf{n}}] = (\mathbf{m} \times \mathbf{n}) X_{\mathbf{m}+\mathbf{n}}, \quad (4.10)$$

where $\mathbf{m} = (m_1, m_2)$ and $\mathbf{n} = (n_1, n_2)$ are two-dimensional integer vectors; $(\mathbf{m} \times \mathbf{n}) \equiv m_1 n_2 - m_2 n_1$. Accounting for this fact, the multiple commutators of the functions

$$L(y^{(k)}) \equiv \sum_{m_1^{(k)}=1}^{m_\pm} X_{m_1^{(k)}}(\Phi_{m_1^{(k)}}^{(k)})$$

in the exponential (4.9) are given by the formula

$$\begin{aligned} & [\cdots [L(y^{(n)}), L(y^{(n-1)})] \cdots] L(y^{(1)}) = \\ & = \sum_{m_2^{(1)}, \dots, m_2^{(n)}} \Phi_{\mathbf{m}^{(n)}}^{(n)} \prod_{l=1}^{n-1} \left(\left(\sum_{j=l+1}^n \mathbf{m}^{(j)} \right) \times \mathbf{m}^{(l)} \right) \Phi_{\mathbf{m}^{(l)}}^{(l)} X_{\sum_{j=1}^n \mathbf{m}^{(j)}}, \end{aligned} \quad (4.11)$$

where

$$\Phi_{m_1^{(k)}}^{(k)}(z; \tau) = \sum_{m_2^{(k)}} \Phi_{\mathbf{m}^{(k)}}^{(k)}(z) e^{im_2^{(k)}\tau}.$$

Now let us give an explicit solution to our system (1.1). Equating expressions (3.11) and (4.6), we have

$$g_{0\mp}^{-1} \partial_{\pm} g_{0\mp} = X_0 \left(\mp \frac{i}{2} \partial_{\pm} \partial_{\tau}^{-1} \rho \right), \quad (4.12)$$

therefore,

$$g_{0\mp} = g_0^{\mp}(z_{\mp}) e^{\mp \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)},$$

and hence,

$$g_0 = g_0^+(z_+) e^{iX_0(\partial_{\tau}^{-1} \rho)} (g_0^-(z_-))^{-1}.$$

Here $g_0^{\pm} \equiv g_0^{\pm}(z_{\pm}) \in G_0$ are arbitrary functions of their arguments, and are expressed in terms of $\Phi_m^{\pm}$. Moreover,

$$\begin{aligned} g_{0\pm}^{-1} \tilde{L}_{\pm} g_{0\pm} &= e^{\mp \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)} (g_0^{\pm})^{-1} \sum_{m=1}^{m_{\pm}} X_{\pm m} (F_m^{\pm}) g_0^{\pm} e^{\pm \frac{i}{2} X_0(\partial_{\tau}^{-1} \rho)} = \\ &= \sum_{m=1}^{m_{\pm}-1} X_{\pm m} (\omega_m^{\pm} e^{\frac{m}{2} \rho}) + X_{\pm m_{\pm}} (e^{\frac{m_{\pm}}{2} \rho}). \end{aligned} \quad (4.13)$$

Rewriting identity (4.5) in the form

$$(g_0^+)^{-1} M_+^{-1} M_- g_0^- = [(g_0^+)^{-1} N_- g_0^+] e^{iX_0(\partial_{\tau}^{-1} \rho)} [(g_0^-)^{-1} N_+^{-1} g_0^-],$$

one has

$$\langle \tau | (g_0^+)^{-1} M_+^{-1} M_- g_0^- | \tau \rangle = \langle \tau | e^{iX_0(\partial_{\tau}^{-1} \rho)} | \tau \rangle, \quad (4.14)$$

where the brackets are taken between the basis vector $|\tau\rangle$ and its dual, $\langle\tau|$, annihilated by the subspaces $\mathcal{G}_+$ and $\mathcal{G}_-$, respectively. This matrix element realises a continuous version of the tau-function, depending on the necessary number of arbitrary functions $\Phi_m^{\pm}(z_{\pm}, \tau)$ which determine the general solution to system (1.1). It can be rewritten as a series over the nested integrals of the products of these functions in the same way as it was done for the case of the continuous Toda system (1.2) in terms of the basis (2.4) with the help of the commutation relations (2.1), see e.g., [6], or with the discrete basis (4.10) and formulas (4.11).

Now, since

$$e^{X_0(\Phi)} X_m(\Psi) e^{-X_0(\Phi)} = X_m(e^{im\partial_{\tau}\Phi}\Psi), \quad (4.15)$$

in accordance with (2.1), one has from (4.13) that

$$(g_0^{\pm})^{-1} \sum_{m=1}^{m_{\pm}} X_{\pm m} (F_m^{\pm} e^{\frac{m}{2} \rho}) g_0^{\pm} = \sum_{m=1}^{m_{\pm}-1} X_{\pm m} (\omega_m^{\pm} e^{\frac{m}{2} \rho}) + X_{\pm m_{\pm}} (e^{\frac{m_{\pm}}{2} \rho}),$$

and hence,

$$\sum_{m=1}^{m_{\pm}} X_{\pm m}(F_m^{\pm}) = \sum_{m=1}^{m_{\pm}-1} g_0^{\pm} X_{\pm m}(\omega_m^{\pm})(g_0^{\pm})^{-1} + g_0^{\pm} X_{\pm m_{\pm}}(g_0^{\pm})^{-1}. \quad (4.16)$$

Thanks to (4.16), with the parametrization $g_0^{\pm} = e^{X_0(\partial_{\tau}^{-1}\nu_{\pm})}$, one has $F_{m_{\pm}}^{\pm} = \Phi_{m_{\pm}}^{\pm} = e^{\pm im_{\pm}\nu_{\pm}}$, so that

$$g_0^{\pm}(z_{\pm}) = e^{\mp \frac{i}{m_{\pm}} X_0(\partial_{\tau}^{-1} \ln \Phi_{m_{\pm}}^{\pm}(z_{\pm}))}. \quad (4.17)$$

Now, accounting for (2.4) and (4.17), expression (4.14) implies that

$$e^{\rho} = (\Phi_{m_+}^+)^{\frac{1}{m_+}} (\Phi_{m_-}^-)^{\frac{1}{m_-}} e^{-i\partial_{\tau}^2 \ln \langle \tau | M_+^{-1} M_- | \tau \rangle}, \quad (4.18)$$

where $M_{\pm}(z_{\pm})$ are determined from the initial value problem (4.2) by the infinite series (4.7), (4.8), or exponentials (4.9). Here the continuous analogue of the tau-function $\langle \tau | M_+^{-1} M_- | \tau \rangle$ is represented by the series

$$\sum_{m,n=0}^{\infty} (-1)^n \int_{Z_+^n(z_+)} dy_+ \int_{Z_-^m(z_-)} dy_- \langle \tau | L_+(y_+^1) \cdots L_+(y_+^n) L_-(y_-^m) L_-(y_-^1) | \tau \rangle, \quad (4.19)$$

where nonzero contributions come only from the terms with the same number of the elements L_+ and L_- in their products in (4.19), i.e.,

$$\begin{aligned} \langle \tau | M_+^{-1} M_- | \tau \rangle &= 1 + \sum_{n \geq 1} (-1)^n \int_{Z_+^n(z_+)} dy_+ \int_{Z_-^n(z_-)} dy_- \times \\ &\times \langle \tau | L_+(y_+^1) \cdots L_+(y_+^n) L_-(y_-^n) L_-(y_-^1) | \tau \rangle. \end{aligned} \quad (4.20)$$

The rule for explicitly calculating the matrix elements in series (4.20) consists in successive displacements of all L_+ and L_- to the extreme right and, respectively, left positions at which they annihilate the highest vectors $|\tau\rangle$ and $\langle\tau|$. In this, one uses the decomposition of $L_{\pm}$ as series (4.3), and relations (2.4) and (2.1).

Moreover, from (4.16) we obtain

$$X_{\pm m}(F_m^{\pm}) = g_0^{\pm} X_{\pm m}(\omega_m^{\pm})(g_0^{\pm})^{-1} = X_{\pm m}(\omega_m^{\pm} e^{\frac{m}{m_{\pm}} \ln \Phi_{m_{\pm}}^{\pm}}), \quad 1 \leq m \leq m_{\pm} - 1,$$

and hence,

$$\omega_m^{\pm} = F_m^{\pm} (\Phi_{m_{\pm}}^{\pm})^{-\frac{m}{m_{\pm}}}, \quad 1 \leq m \leq m_{\pm} - 1. \quad (4.21)$$

So, formulas (4.18) and (4.21) define the general solution to system (1.1).

The functions $F_m^{\pm}$, $1 \leq m \leq m_{\pm} - 1$, entering solution (4.21) and in turn, $\tilde{L}_{\pm}$, are determined in terms of the matrix elements of the known element $M_+^{-1} M_-$ taken between some but not necessarily the highest vectors of the representation space. Consider, for example, the matrix element with the highest bra-vector $\langle\tau|$ and a ket-vector $|\tau\rangle^{(1)}$ which is annihilated by the action of the subspaces $\mathcal{G}_m$, $m > 1$. Then

$$\langle \tau | g_0^{-1} M_+^{-1} M_- | \tau \rangle^{(1)} = \langle \tau | (g_0^{-1} N_- g_0) N_+^{-1} | \tau \rangle^{(1)} = \langle \tau | N_+^{-1} | \tau \rangle^{(1)},$$

and differentiating this equality over z_+ , one has a sequence of equalities

$$\begin{aligned}\partial_+ \langle \tau | g_0^{-1} M_+^{-1} M_- | \tau \rangle^{(1)} &= - \langle \tau | g_0^{-1} \tilde{L}_+ g_0 N_+^{-1} | \tau \rangle^{(1)} = \\ &= - \langle \tau | g_0^{-1} X_1(F_1^+) g_0 N_+^{-1} | \tau \rangle^{(1)} = - \langle \tau | g_0^{-1} X_1(F_1^+) g_0 | \tau \rangle^{(1)},\end{aligned}$$

which determines the function F_1^+ . Now, knowing this function, and in order to find F_2^+ , we consider the matrix element with a ket-vector $|\tau\rangle^{(2)}$ which is annihilated by the action of the subspaces $\mathcal{G}_m$, $m > 2$, etc. A similar procedure allows to determine the functions F_m^- .

It seems interesting to investigate the problem of the integrability of system (1.1) on a half-line, $r = 0$. Let us study it with the simplest example of the heavenly equation (1.2) which shows that this problem respects the boundary condition

$$\partial_r x = 0 \quad \text{at} \quad r = 0; \quad \rho \equiv \partial_\tau^2 x. \quad (4.22)$$

To become convinced of it, one can equate the corresponding $W_m^\pm$ -elements, $\partial_\mp W_m^\pm = 0$, at $z_+ = z_-$. Then it follows from the expression

$$W_2^\pm = \int d\tau \left[\partial_\pm^2 x + \frac{1}{2} (\partial_\pm \partial_\tau x)^2 \right],$$

that

$$\partial_\tau x = \lambda \cdot e^{\rho/2} \quad \text{at} \quad r = 0, \quad (4.23)$$

where λ is an arbitrary constant, yet formally, a similar condition takes place for the case of the Liouville equation [11]. Then, already from the $W^\pm$ -elements of the 3rd order,

$$W_3^\pm = \int d\tau \left[\tau \partial_\pm^3 x + (\partial_\pm x) \cdot (\partial_\pm^2 \partial_\tau x) + \tau (\partial_\pm \partial_\tau x) \cdot (\partial_\pm^2 \partial_\tau x) - \frac{1}{3} (\partial_\pm \partial_\tau x)^3 \right],$$

one sees that this constant λ should be equal to zero. Recall that for the affine Toda field theory associated with the series $A_n^{(1)}$, and, as can also be easily shown for the corresponding finite case A_n , there are two possibilities for the constants λ_i entering the boundary conditions like (4.23), or its affine deformation, written for the fields x_i , $1 \leq i \leq r$, namely $|\lambda_i| = 2$, or all $\lambda_i = 0$ [12]³⁾. It is interesting to note that, in general, non-Abelian Toda systems do not admit such a rigid restriction on these constants. In particular, for the case of B_2 , the boundary conditions contain one arbitrary constant, as can be seen by equating the corresponding $W^\pm$ -elements, as all three of them are of the 2nd order [13].

5. ALGEBRAIC STRUCTURE OF THE SOLUTION

The matrix element (4.20) determining the general solution to system (1.1) realizes a continuous version of the well-known object in the theory of integrable systems – the tau-function. At the same time, this matrix element is closely related to the so-called Shapovalov form defined here on the Lie algebra $\mathcal{L}$, which can clarify the algebraic structure of the solution. Such a relationship is quite common and takes place for a wide class of nonlinear

³⁾The same as for the Liouville case, the sine (or sinh)-Gordon model is less restricted in this sense; see [12] for relevant references.

integrable systems, including, in particular, Abelian and non-Abelian Toda systems associated with the simple Lie algebras $\mathcal{G}$. Let us discuss this briefly.

Recall, see e.g. [14], that the standard Shapovalov form defines the linear mapping $U(\mathcal{G}) \otimes_{\mathbb{C}} U(\mathcal{G}) \mapsto U(\wp)$ and is realized as a bilinear form $(x^\vee y)_0$ for any two elements $x, y \in U(\mathcal{G})$. Here $x \rightarrow x^\vee$ is the Chevalley involution for $x^\vee = x'$, and the Hermitean Chevalley involution for $x^\vee = x^*$; the subscript 0 means the projection of $U(\mathcal{G})$ on $U(\wp)$ which is parallel to $\mathcal{G}_- U(\mathcal{G}) + U(\mathcal{G}) \mathcal{G}_+$. The given definition is naturally extended for the case of the algebra $U'(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\wp)} R(\wp^*)$, where $R(\wp^*)$ is the algebra of the rational functions over $\wp^*$. It is very important to note that the form $(x^\vee y)_0$ is degenerated on the left ideal $U'(\mathcal{G}) \mathcal{G}_+$, and is not degenerated on the subalgebra $U(\mathcal{G}_-)$, and hence, is not degenerated on the space $U'(\mathcal{G})/U'(\mathcal{G}) \mathcal{G}_+$ which is a rational span of the corresponding Verma module. In fact, those forms which are most relevant for describing the general solution to the Toda type equations [4] are related with an appropriate holomorphic extension of the algebra $U(\mathcal{G})$, namely $U_h(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\wp)} h(\wp^*)$ where $h(\wp^*)$ is the algebra of holomorphic functions over $\wp^*$. Note that a special class of such extensions was introduced in [15], see also [16]. In our case, we deal with a holomorphic extension of the algebra $U(\mathcal{L})$, and the role of the holomorphic and antiholomorphic functions is played by the functions $\Phi_m^+(z)$ and $\Phi_m^-(\bar{z})$, respectively, under a relevant reality condition imposed on the solution to Eqs. (1.1) similar to those in [2] for the Toda system.

A basis in $U(\mathcal{L})$ is constructed with the help of the monomials $\hat{X}(\{M\}) = X_{m_n} \cdots X_{m_1}$ in terms of the basis elements $X_m = X_m(\Phi_m)$ of $\mathcal{L}$, see (4.19). In particular, let $\hat{X}_\pm(\{M\})$, $\hat{X}_-(\{M\}) = \hat{X}_+^\vee(\{M\})$, be such a basis in $U(\mathcal{G}_\pm)$ with the weight $\mu(\{M\})$. Then the elements $\hat{X}_+(\{M\}) \hat{X}_-(\{N\})$ generate a basis of $U'(\mathcal{L})$ over $R(\wp^*)$, and this procedure gives a vector space $F_\mu(\mathcal{L})$ for any weight $\mu \in \wp^*$ of all formal series $\sum_{\{M, N\}} c_{\{M, N\}} \hat{X}_+(\{M\}) \times \hat{X}_-(\{N\})$ with $c_{\{M, N\}} \in R(\wp^*)$, where the sum runs over all monomials of the weight $\mu = \mu(\{M\}) - \mu(\{N\})$. The subspaces $F_\mu(\mathcal{L})$ are, in turn, the subspaces of the algebra $F(\mathcal{L})$ graded by the weights μ . To calculate explicitly the corresponding Shapovalov form or its Hermitean version, $(x^\vee y)$, one needs to transform the elements $\hat{X}_+(\{M\}) \hat{X}_-(\{N\})$ entering (4.20) to the series of the monomials $\hat{X}_- \hat{X}_0 \hat{X}_+$, with $\hat{X}_0 \in U(\wp)$, using the commutation relations (2.1) of the algebra $\mathcal{L}$. This is just the rule that we mentioned below formula (4.20). Now one can be convinced that the matrix element (4.20) is a Shapovalov type form $(x^\vee y)_0$ for two special elements $x, y \in U_h(\mathcal{G}_-)$ of the Lie algebra $\mathcal{L}$ in question.

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ИНТЕГРИРУЕМЫЕ ОБОБЩЕНИЯ НЕПРЕРЫВНОЙ ТОДОВСКОЙ СИСТЕМЫ

Получены интегрируемые обобщения непрерывной тодовской системы, порождаемые плоской связностью, принимающей значения в высших градуировочных подпространствах алгебры, сохраняющих площадь диффеоморфизмов на двумерном торе, и построены их общие решения. Используемое градуировочное условие, накладываемое на связность, может быть реализовано в терминах некоторых голоморфных распределений на соответствующих однородных пространствах.

Note Added in Proof. In September 1995, when the paper was completed and appeared as a preprint LPTENS-95/17, V. E. Zakharov informed me that Eq. (1.2) had been obtained in his talk "Integrable systems in multidimensional spaces", Lecture Notes in Physics, Springer–Verlag, vol. 153 (1981), 190.