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THE 16-VERTEX MODEL OF A FERROELECTRIC WITH PROTON TUNNELING

Thermodynamics of the 1D ferroelectric model with proton tunneling has been reduced to the calculation of the grand partition function of the 1D Ising model of spin $S = 1$ in the transverse field Γ . This allows the behaviour of the initial model to be calculated in the vicinity of its thermodynamic anomaly. In the second order, the dependence of the free energy on temperature, and the tunneling parameter Γ was found.

1. INTRODUCTION

It is known [1, 2] that in ferroelectrics with proton tunneling, quantum effects dramatically change the thermodynamic picture of the phase transitions from proton disordered to proton ordered states. The standard theoretical description of these phenomena [1, 2] uses the Ising model for the spins $S = 1/2$ in the “transverse field Γ ,” where Γ is the energy of the tunnel splitting. This model, however, does not account for two essential features which define the character of phase transitions in ferroelectrics: the geometry of local proton complexes (LC) and the electroneutrality principle for LC configurations, known as Pauling’s “ice rule” [3]. It is desirable in considering the quantum effects in ferroelectrics to have a model that accounts for both these features.

In [4], the simplest model satisfying these conditions was considered. This model is depicted by the one-dimensional sequence of N sites connected in a chain by the bonds. Each pair $((i-1), i)$ of sites is connected by the i -th pair $((i, 1), (i, 2))$ of bonds. The sites with four adjacent bonds form a geometrical vertex. One proton is placed on each bond. A proton on a bond can tunnel between two localized states. On the i -th pair of bonds, these localized states are described by the pair of variables $(\sigma_{i,1}^z, \sigma_{i,2}^z)$, where $\sigma_{i,1}^z$ and $\sigma_{i,2}^z$ are the Pauli matrices. The i -th geometrical vertex has 2^4 proton configurations, each of these we will call a vertex (without the term “geometrical”). The energies of the vertices are described in terms of the spin variables as follows

$$H_0(i) = -v_1[\sigma_{i,1}^z \sigma_{i+1,1}^z + \sigma_{i,2}^z \sigma_{i+1,2}^z] - \frac{1}{2}v_2[\sigma_{i,1}^z \sigma_{i,2}^z + \sigma_{i+1,1}^z \sigma_{i+1,2}^z] - \\ - v_3[\sigma_{i,1}^z \sigma_{i+1,2}^z + \sigma_{i+1,1}^z \sigma_{i,2}^z]. \quad (1)$$

Besides, the term tunnel splitting is associated with each i -th pair of bonds $(i, 1), (i, 2)$

$$v(i) = \left(\frac{\Gamma}{2}\right) [\sigma_{i,1}^z + \sigma_{i,2}^z]. \quad (2)$$

The energy of the vertex (1) can have five values w_i ($i = 1, 2, \dots, 5$). The energies w_1, w_2, w_3 correspond to the six neutral vertices, consisting of all possible positions of two protons on the four bonds of the vertex. The three parameters in model (1) are defined by the three vertex energies

$$v_1 = -\frac{1}{4}(w_1 + w_3), \quad v_2 = \frac{1}{2}(w_1 + w_2), \quad v_3 = -\frac{1}{4}(w_2 + w_3). \quad (3)$$

The energies of other vertices have the following values: eight one-fold charged vertices have the energies $w_4 = 0$, two two-fold charged vertices have the energies $w_5 = -w_1 - \frac{3}{4}(w_2 + w_3)$. In all that follows, we will investigate only the special case $w_1 = w_2 = -w$, $w_3 = -w - \varepsilon_0$, and $0 < \varepsilon_0 < w$. Here the energy w plays the role of the Coulomb gap between the energies of the charged and neutral vertices [4]. The energy ε_0 separates the energies w_3 of the neutral polarized vertices from the energies $w_1 = w_2$ of the neutral nonpolarized vertices.

The Hamiltonian of the considered model is found by the summation of expressions (1), (2) on i :

$$H_N = H_N^0 + V_N = 2w \sum_{i=1}^N [(S_i^z)^2 - (1 + \varepsilon)S_i^z S_i^z - gS_i^x] - Nw, \quad (4)$$

where $\varepsilon = \varepsilon_0/2w$, $g = \Gamma/w$, and

$$S_i^\alpha = \frac{1}{2}(\sigma_{i,1}^\alpha + \sigma_{i,2}^\alpha), \quad i = 1, 2, \dots, N, \quad \alpha = x, z; \quad S_{N+1}^z = 0. \quad (5)$$

The conditions $S_{N+1}^z = 0$ will be called the free end conditions. All Hamiltonians considered below obey these conditions. Operators S_i^α act in 4-dimensional spaces X_i ($i = 1, 2, \dots, N$). Each of X_i breaks into the direct sum of the one-dimensional (X_i^0) and the three-dimensional (X_i^1) subspaces:

$$X_i = X_i^0 + X_i^1. \quad (6)$$

On the subspaces X_i^0 and X_i^1 , the irreducible representations of the Lie algebra of the spin operators with the weights $S = 0$ and $S = 1$ are realized. Model (4) was considered in the classical case of $g = 0$ in [5].

In the infinite Coulomb gap limit, $w \gg \varepsilon_0$, $\varepsilon_0 \sim T$, configurations of the neutral vertices form an autonomous thermodynamic system. This system has the first kind of phase transition at the temperature $T_c = \varepsilon_0/\ln 2$ from the nonpolarized high-temperature state to the polarized low-temperature one. Accounting for the finite w -effects smears this phase transition into the region of the thermodynamic anomaly in the vicinity of T_c [5]. In [4], the tunneling influence on the critical temperature T_c was considered by the use of operator perturbation theory technique [6]. The influence of the tunneling on the form of thermodynamic anomaly lies beyond the scope of methods of [4]. In this paper, we are interested in thermodynamic behavior of model (4) in the vicinity of its thermodynamic anomaly, $T \sim T_c$, for the conditions $\varepsilon < 1$, $g < 1$. Because of a sharp change in thermodynamic

characteristics in this temperature region, it is difficult to use perturbation theory or computer methods for the investigation. Our approach consists of the reduction of model (4) to the 1D Ising problem of spins $S = 1$ in the transverse field. The advantage of this approach is that the reduced model has no thermodynamic anomalies and, therefore, is accessible for investigations by analytic or computer methods. The Hamiltonian $H_N^1 = H_N^{0,1} + V_N^1$ of the reduced model has the form (4), as the initial model, but now the symbols S_i^z, S_i^x mean the operators acting in the X_i^1 subspaces of decomposition (6).

The paper is organized as follows. In the Sec. 2, the above mentioned reduction technique is developed. In Sec. 3, the reduction technique of Sec. 2 and thermodynamic perturbation theory [7] are applied to the investigation of the free energy of 1D ferroelectric model (4). Our conclusions are given in Sec. 4.

2. THE REDUCTION TECHNIQUE

The Hamiltonians H_N and H_N^1 act correspondingly in the spaces

$$\overline{X_N} = \bigotimes_{i=1}^N X_i, \quad \overline{X_N^1} = \bigotimes_{i=1}^N X_i^1. \quad (7)$$

Sequential substitutions of (6) into (7) lead to the decomposition of $\overline{X_N}$ into the direct sum

$$\overline{X_N} = \overline{X_N^1} + \bigoplus_{k=0}^{N-1} \left[\left(\bigotimes_{i=1}^k X_i^1 \right) \otimes X_{k+1}^0 \otimes \left(\bigotimes_{i=k+2}^N X_i \right) \right]. \quad (8)$$

Each summand in (8) is H_N -invariant. Because of the factor X_{k+1}^0 in the k -th summand of (8), the Hamiltonian H_N of this summed reduces to the form

$$H_N \rightarrow H_k^1(1, 2, \dots, k) \oplus H_{N-k-1}(k+2, \dots, N), \quad (9)$$

where operators H_k^1, H_{N-k-1} correspond to the models with free ends, as in (4). As a result of (8) and (9), we get the recurrence formula

$$Q_N = Q_N + \sum_{k=0}^{N-1} Q_k^1 Q_{N-k-1}, \quad (10)$$

where

$$Q_N = \text{Tr exp}(-\beta H_N), \quad Q_N^1 = \text{Tr exp}(-\beta H_N^1) \quad (11)$$

are the partition functions of H_N and H_N^1 . For both of models, characterized by the operators H_N and H_N^1 , the existence of thermodynamic limits at $N \rightarrow \infty$,

$$\lim \left(\frac{1}{N} \ln Q_N \right) = -\beta f, \quad \lim \left(\frac{1}{N} \ln Q_N^1 \right) = -\beta f^1, \quad (12)$$

has been proved [8]. Here f and f^1 are the free energies per site. It follows from (12) that these free energies are related to the convergence radii, R and R^1 , of the grand partition functions

$$G(z) = \sum_{N=0}^{\infty} z^N Q_N, \quad G^1(z) = \sum_{N=0}^{\infty} z^N Q_N^1 \quad (13)$$

by the Hadamard formulas [9]

$$R = \exp(\beta f), \quad R^1 = \exp(\beta f^1). \quad (14)$$

Since $Q_N^1 \leq Q_N$, we have $R \leq R^1$. So, after rewriting recurrence (10) in terms of the grand partition functions (13), at $|z| < R$, we get

$$G(z) = \frac{G^1(z)}{1 - zG^1(z)}. \quad (15)$$

In the Appendix, we proved that the following strict inequality holds:

$$R < R^1. \quad (16)$$

Since the function $G^1(z)$ is analytical in the circle $\{z : |z| < R^1\}$, formula (15) gives the analytical continuation of $G(z)$ to the R^1 circle. Therefore, $G(z)$ can have only those poles corresponding to the zeros of the denominator in the r.h.s. of (15) in this R^1 circle. It is simple to prove, using (16), that in the real interval $0 < x < 1$, the monotonously decreasing function $1 - xG^1(x)$ has a zero and this real zero is the nearest to the origin. Consequently, the only real root of the equation

$$RG^1(R) = 1 \quad (17)$$

is the convergence radius for the grand partition function $G(z)$. So, using formulas (14) and (13), we have reduced the initial problem to a calculation of the grand partition function of the 1D Ising problem of spins $S = 1$ in the transverse field (see formula (4) for H_N^1). It should be noted that the existence of the root of Eq. (17) is a direct consequence of the strict inequality (16).

It is instructive to consider how the reduction technique works in the limiting case $g = 0$, when both initial and reduced problems are solved by the transfer-matrix method [10]. The resulting formulas will be used in the following section. The partition function $Q_N^1(0)$, accounting for the free end conditions, is expressed through the transfer-matrix U as follows:

$$Q_N^1(0) = \sum_s (U^N)_{s,0}, \quad U_{s,s'} = \exp[-\beta s^2 - \beta(2 + \varepsilon)ss']. \quad (18)$$

The three parameters

$$c_1 = \exp(\beta\varepsilon), \quad c_2 = \exp(-\beta), \quad c_3 = \exp(-\beta(2 + \varepsilon)) \quad (19)$$

entering the transfer-matrix U have different scales : $c_1 > c_2 > c_3$. In all that follows, we will neglect the c_3 -parameter. Using (18) and (19), we get

$$Q_N^1(0) = d_1 l_1^N + d_2 l_2^N, \quad N = 1, 2, \dots, \quad (20)$$

where l_1 and l_2 are two of the three transfer-matrix U eigenvalues:

$$l_{1,2} = \frac{1}{2}(c_1 + 1) \pm \frac{1}{2}[c_1 - 1)^2 + 8c_2]^{1/2}. \quad (21)$$

The weights d_1, d_2 in (20) are:

$$d_i = 2c_2 l_i [2l_2 + (l_i - 1)^2]^{-1}, \quad i = 1, 2. \quad (22)$$

The eigenvectors corresponding to l_1, l_2 are

$$\bar{l}_i = (a_i)^{1/2} \left[\frac{l_i - 1}{2\sqrt{c_2}}, 1, \frac{l_i - 1}{2\sqrt{c_2}} \right], \quad a_i = \left(\frac{d_i}{l_i} \right), \quad i = 1, 2. \quad (23)$$

It is evident from (20) that the free energy $f^1(0) = -\beta^{-1} l_1$ is an analytical function of the temperature. Substituting (20) into (13), we find $G^1(z)$ as a rational function with two simple poles, l_1 and l_2 . Substituting $G^1(z)$ into (15), we get $G(z)$ in the form of a rational function with two simple poles, L_1 and L_2 . Then, by expanding $G(z)$ into a series, we get

$$Q_N(0) = D_1 L_1^N + D_2 L_2^N, \quad (24)$$

where

$$L_{1,2} = \frac{1}{2}(c_1 + 2) \pm \frac{1}{2}[(c_1 - 2)^2 + 16c_2]^{1/2}. \quad (25)$$

The weights D_1, D_2 in (24) are the positive constants like those of (22). From (24) we get the free energy of the H_N^0 model in the form $F_N^0 = -N\beta^{-1} \ln L_1$. The dependence of L_1 on c_1 (remember that $c_1 \gg c_2$) sharply changes in the vicinity of $c_1 = 2$, i.e., in the vicinity of the temperature $T_c = \varepsilon_0 / \ln 2$. This is the known thermodynamic anomaly [5]. At $c_2 \sim 0$, this anomaly concentrates at the point T_c of the first kind of phase transition.

3. THERMODYNAMIC PERTURBATION THEORY

In the first nonzero approach, the correction to $Q_N^1(0)$ has the form

$$\Delta Q_N^1 = g^2 \sum_{k=1}^N \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \operatorname{Tr} \left\{ \exp[(\tau_1 - \tau_2 - \beta) H_N^0] S_k^x \exp[(\tau_1 - \tau_2) H_N^0] S_k^x \right\}. \quad (26)$$

The trace in (26) is calculated by using eigenvectors (23), eigenvalues (21) and weights (22) of the three-dimensional transfer-matrix (18). After simple but cumbersome algebra, we get

$$\Delta Q_N^1 = \sum_{k=1}^N \Delta Q_N^1(k), \quad \Delta Q_N^1(k) = \beta g^2 \sum_{i,j=1,2} l_i^{k-1} l_j^{N-k} f_{i,j}, \quad (27)$$

where

$$f_{i,j} = a_i a_j \{I_1 + (l_i - l_j - 2)I_2 + (l_i - 1)(l_j - 1)I_3\}, \quad (28)$$

and where

$$I_1 = 1 - c_2, \quad I_2 = (c_1 - 1)/2\varepsilon, \quad I_3 = \frac{1}{2}(1 - c_2) + \frac{1}{4c_2} \left(\frac{c_1^2 - c_2}{1 + 2\varepsilon} + \frac{c_2}{3 + 2\varepsilon} \right). \quad (29)$$

After the summation in (27), we get

$$\Delta Q_N^1 = \beta g^2 N (f_{1,1} l_1^{N-1} + f_{2,2} l_2^{N-1}) + 2\beta g^2 f_{1,2} \left(\frac{l_1^N - l_2^N}{l_1 - l_2} \right). \quad (30)$$

Now, from (20) and (28), we find the $N \rightarrow \infty$ asymptotic behavior:

$$\frac{\Delta Q_N^1}{Q_N^1(0)} \sim N \left(\beta g^2 \frac{f_{1,1}}{d_1 l_1} \right) + \frac{2\beta g^2 f_{1,2}}{d_1 (l_1 - l_2)} + O \left(\frac{l_1}{l_2} \right)^N. \quad (31)$$

In correspondence with the linked clusters theorem [7], the first term of (31) is nothing but the correction to the free energy $Nf^1(g)$. So, the behavior of $Q_N^1(g)$ at the thermodynamic limit has to have the form

$$Q_N^1(g) = Q_N^1(0) \exp \left(N \beta g^2 \frac{f_{1,1}}{d_1 l_1} + \frac{2\beta g^2}{d_1 (l_1 - l_2)} \right). \quad (32)$$

The following expression (33) has a structure like one for $Q_N^1(0)$ of Eq. (20). Expression (33) interpolates between the formulas (30) and (32):

$$Q_N^1(g) = d_1^* (l_1^*)^N + d_2^* (l_2^*)^N, \quad (33)$$

$$d_{1,2}^* = d_{1,2} \pm 2\beta g^2 \frac{f_{1,2}}{l_1 - l_2}, \quad l_i^* = l_i \exp \left(\beta g^2 \frac{f_{i,2}}{d_i l_i} \right), \quad i = 1, 2. \quad (34)$$

Calculating $G_1(z)$ by the substitution of (33) into (13), we can write Eq. (17) in the form

$$\frac{d_1^* R}{1 - l_1^* R} + \frac{d_2^* R}{1 - l_2^* R} = 1. \quad (35)$$

The real root of (35) nearest to zero defines, as can be seen from (14), the free energy f of the ferroelectric model

$$\exp(-\beta f) = \frac{1}{2} \left\{ (l_1^* + l_2^* + 1) + [(l_1^* + d_1^* - l_2^* - d_2^*)^2 + 4d_1^* d_2^*]^{1/2} \right\}. \quad (36)$$

In expression (36) we have $d_1^* \sim c_2 \ll 1$. Therefore, the temperature of the thermodynamic anomaly $T_c(g)$ is defined by the condition

$$l_1^* + d_1^* = l_2^* + d_2^*. \quad (37)$$

Dependence of the free energy $f = F/N$ on temperature T and $g = \Gamma/w$ for $g = 0$; 0.1; 0.2; 0.25 and $\epsilon = 0.05$

After the substitution the obvious expressions for l_i^* , d_i^* from the formulas (34), (21), (22), (28) into the formula (37), we obtain the obvious form of (37):

$$\exp(\beta\epsilon) = [1 + \exp(\beta g^2)] \exp\left(-\frac{\beta g^2}{2(1 + \epsilon)}\right). \quad (38)$$

After changing $(1 + \epsilon) \rightarrow 1$ on the r.h.s of (38), we get Eq. (29) of [4]. This equation defines the dependence of $T_c(g)$ on $\Gamma = 2wg$. In Figure, the temperature dependence of the free energy of the 1D ferroelectric is depicted for several values of g : $g = 0$; 0.1; 0.2; 0.25 and for $\epsilon = 0.05$. Since the thermodynamic anomalies are very narrow, they are seen in Figure as cusps on the curves.

4. CONCLUSIONS

We reduced the thermodynamics of the 1D ferroelectric model (4) with proton tunneling to the 1D Ising model of spins $S = 1$ in a transverse field. The advantage of this reduction consists of possibility to calculate the thermodynamic behavior of the initial model in the vicinity of its thermodynamic anomaly. To evaluate the influence of the tunneling on the free energy, we applied the standard thermodynamic perturbation theory. As the result, we got the dependence (36) of the free energy of the ferroelectric model f on the temperature and the tunneling frequency. This dependence provides a good description of the thermodynamic behavior in the vicinity of the thermodynamic anomaly inherent to the considered ferroelectric model. This anomaly mimics the first kind of phase transition to the low temperature ordered states in high dimensional ferroelectrics. The dependence of the center,

T_c , of the thermodynamic anomaly on the tunneling frequency is described by the Eq. (38). This equation coincides with that of [4].

The proposed reduction method opens the possibility to investigate the thermodynamics of the considered one-dimensional ferroelectric model by computer methods. We also evaluated the applicability region of the perturbation theory. The analysis showed that the parameter βg^2 , in d_1^* , d_2^* and l_1^* with the factor $c_2 = \exp(-\beta)$ securing the form $\beta \exp(-\beta)$, is finite over the entire range of temperatures. But in l_2^* , the factor βg^2 was not compensated by the factor c_2 . This is the reason why the perturbation theory applicability is reduced to the region $\beta g^2 < 1$; $g^2 < \varepsilon$. It is interesting to note that the condition $\beta g^2 < 1$ arose as a consequence of accounting for contributions from finite N , since the terms $(l_2^*)^N$ are lost in the thermodynamic limit.

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APPENDIX

Inequality (16) is the central point in arriving at the reduction equation (17). Therefore, it needs to be checked. This checking consists of three different steps.

1. We solve the recurrence (10):

$$Q_N = Q_N^1 + B_N^1 + \sum_{l_1+l_2=N}^* Q_{l_1}^1 B_{l_2}^1, \quad (\text{A.1})$$

where

$$B_1^1 = 1, \quad B_m^1 = \sum_{k+n=m}^* A_k^1(n), \quad m \geq 2, \quad A_k^1(n) = \sum_{l_1+l_2+\dots+l_s=n}^* C_k^n Q_{l_1}^1 Q_{l_2}^1 \dots Q_{l_s}^1.$$

In (A.1), the summations with the stars mean $l_1 \geq 1, l_2 \geq 1$, and so on.

2. From the convexities of the logarithms of the traces of the operator exponents, [11], the sequence of inequalities follows

$$x^n Q_N^1 \leq Q_{l_1}^1 Q_{l_2}^1 \dots Q_{l_{n+1}}^1 \leq x^{-n} Q_N^1, \quad (\text{A.2})$$

where $x = \exp(-2(1+\varepsilon)\beta)$, $l_1 \geq 1, \dots, l_{n+1} \geq 1$, $l_1 + \dots + l_{n+1} = N$.

We also need the inequalities

$$Q_{N-k}^1 \leq y^k Q_N^1, \quad 1 \leq k \leq N, \quad x = y Q_1^1, \quad (\text{A.3})$$

derived from (A.2) by the substitution of $l_1 = l_2 = \dots = l_n = 1$.

3. We make a lower estimations of B_N^1 , bearing in mind that we need to get the final estimation only in terms of Q_N^1 . Using (A.2) for the lower estimation, we get

$$A_k^1(n) \geq \sum_{s=1} C_k^s C_{n-1}^{s-1} x^{s-1} Q_n^1 \quad (\text{A.4})$$

with the upper limit in the sum of (A.4) equaling $\min(k, n)$. The next step consists of the substitution of (A.4) into B_N^1 in (A.1) and making the lower estimation, using (A.3). The cumbersome result of this estimation is simplified by the following lower estimations, consisting of the increase in some degrees of x and changing the summation limits. At last, we obtain

$$B_N^1 \geq (xy) \sum_{k=0}^{\frac{N-2}{2}} C_{N-k-2}^k (xy)^k Q_N^1. \quad (\text{A.5})$$

The final lower estimation on the r.h.s. of (A.5) is obtained by using the inequality

$$C_{2M-k}^k \geq C_M^k.$$

So, as the consequence of (A.1), we get:

$$Q_N > B_N > (xy)(1+xy)^{\frac{N}{2}-1} Q_N^1.$$

Using (12) and (14), and taking the thermodynamic limit, we get the final estimation

$$(1+xy)^{\frac{1}{2}} R \leq R^1,$$

which follows immediately from the inequality $R < R^1$.

To prove (A.2), we use the convexity of the free energies $F(t)$, [11],

$$d^2 F/dt^2 < 0, \quad dF/dt \equiv \langle V \rangle_t = \text{Tr}[V \exp(-\beta H_t)] / \text{Tr}[\exp(-\beta H_t)], \quad (\text{A.6})$$

defined on the one-parametric Hamiltonian family

$$H_t = H_0 + tV.$$

For each $k = 1, 2, \dots, N-1$, we can write the Hamiltonian H_N^1 of (4) in the form (all energies and temperatures written in the $2w$ -scale):

$$H_N^1 = H_N^1(k) - 2(1+\varepsilon)S_k^z S_{k+1}^z,$$

where

$$H_N^1(k) = H_k^1(1, 2, \dots, k) \oplus H_{N-k}^1(k+1, \dots, N).$$

It follows from this decomposition that

$$\text{Tr} \exp(-\beta H_N^1(k)) = Q_k^1 Q_{N-k}^1, \quad \text{Tr} \exp(-\beta H_N^1) = Q_N^1 \quad \text{Tr} \exp(-\beta H_N^1) = Q_N^1.$$

Now, using (A.6) for the family with

$$H_0 = H_N^1(k), \quad V = -2(1+\varepsilon)S_k^z S_{k+1}^z,$$

integrating (A.6) on $0 \leq t \leq 1$, and using the evaluation $|\langle S_k^z S_{k+1}^z \rangle_t| \leq 1$, we find

$$F(0) - 2(1 + \varepsilon) \leq F(1) \leq F(0) + 2(1 + \varepsilon),$$

or equivalently:

$$xQ_N^1 \leq Q_k^1 Q_{N-k}^1 \leq x^{-1}Q_N^1, \quad k = 1, 2, \dots, (N-1).$$

Inequalities (A.2) immediately follow from this.

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В. Н. Лихачев, В. А. Онищук, А. Розенгрэн 16-ВЕРШИННАЯ МОДЕЛЬ ФЕРРОЭЛЕКТРИКА С ТУННЕЛИРОВАНИЕМ ПРОТОНОВ

Термодинамика одномерной модели ферроэлектрика с туннелированием протонов сведена к вычислению большой статистической суммы одномерной модели Изинга для спинов $S = 1$ в поперечном поле Γ . Этот прием позволяет исследовать термодинамические свойства исходной модели в окрестности ее аномального поведения. Во втором порядке теории возмущений найдена зависимость свободной энергии от температуры и параметра туннельного расщепления Γ .