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Х. Клейнерт, А. Жук, Эффект Казимира при ненулевой температуре во Вселенной с топологией $S^1 \times S^1 \times S^1$, *ТМФ*, 1996, том 108, номер 3, 482–496

DOI: <https://doi.org/10.4213/tmf1205>

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26 апреля 2023 г., 17:03:29

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THE CASIMIR EFFECT AT
NONZERO TEMPERATURES IN
A UNIVERSE WITH TOPOLOGY $S^1 \times S^1 \times S^1$

We investigate the influence of finite-size effects on the free energy of mass and massless scalar fields at nonzero temperatures in a universe with the topology of a three-dimensional torus. The renormalized free energy and the resulting energy density and pressure are found. Their physical consequences are discussed.

1. INTRODUCTION

The purpose of the present paper is to investigate finite-size and temperature effects in spaces with non-trivial topology. These are either noncompact spaces with boundaries, or compact spaces without boundaries, or spaces with parts of their boundaries glued together. Cylinders are the simplest examples of the latter. For all of these spaces, the spectrum of matter and radiation fluctuations differs from the spectrum of an infinite space. As a result, the partition function contains an additional term equal to the difference between the spectral sum and the spectral integral. In flat space-time, such differences appear in the energy spectrum of electromagnetic waves between two conducting plates and give rise to the attraction known as the Casimir effect [1]. This effect has been discussed extensively in literature (see the reviews in [2–4]).

We shall consider both finite-size and temperature effects calculating, as in [3, 4], the regularized free energy for the field fluctuations. The regularization of the infinite sums is performed by various standard methods described in the literature [5–13]. The Abel–Plana summation formula [5, 6] is especially helpful for these purposes.

In a previous paper [14], the regularization procedure was performed for mass and massless scalar fields with an arbitrary coupling to gravity in the closed Friedmann universe with the space topology of a three-dimensional sphere. It is a compact space without boundaries. In the present paper, we extend this calculation to the universe with the space topology of a three-dimensional torus $S^1 \times S^1 \times S^1$. This manifold is obtained by formal identification (gluing) of the opposite faces of a parallelepiped. It is well-known from the paper by F. Klein [15] that one and the same local structure may correspond to globally different models of the universe. Up to now, we know nothing about the global topology of our universe. There are infinitely many topologically different models of three-dimensional spaces with constant curvature [16, 17]. Some of them may be closed (i.e., finite and boundless)

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and obtained with the help of gluing [17, 18]. All of them are acceptable candidates. One of the observable consequences of gluing would be the existence of multiple images in the celestial sky [18, 19].

In [14], we derived regularized expressions for the energy density and pressure of scalar field fluctuations with arbitrary coupling to gravity in the closed Friedman universe. For the universe with the space topology of a three-dimensional torus, this problem becomes much more complicated because the momentum of the scalar field particle is no longer degenerate as it was in the spherically symmetrical Friedmann universe. Thus, we must regularize the triple sums. To avoid consideration of particle creation during the evolution of this universe, [11, 12, 20], we assume the universe to be static. This is a standard assumption when obtaining pure finite-size effects [21, 22].

At present time, the Casimir effect for the universe with $S^1 \times S^1 \times S^1$ topology was investigated only at zero temperature [3, 4, 23–28] (the vacuum case). In this paper, we shall obtain it for the case of arbitrary temperatures. According to the standard scenario of the hot universe, the entropy of our universe is $S \sim (aT)^3 \sim 10^{87} \gg 1$, where $a \sim 10^{28} \text{cm}$ and $T \sim 2.7^0 \text{K}$ [29]. Thus, it is clear from the very beginning that the finite-size corrections should be negligibly small in this high-temperature limit. The Casimir effect gets swamped here. The finite size of a hot universe has no practical effect. This effect may play an important role in models where the product aT is not too large, for example, in some inflationary models at early stages of their evolution. Finite-size effects should also be significant in multi-dimensional Kaluza–Klein models [30].

2. PARTITION FUNCTION OF A MASSIVE SCALAR FIELD

The action for a scalar field over an arbitrary four-dimensional space-time background is [12, 20]

$$S = \int d^4x |g|^{1/2} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right], \quad (2.1)$$

where the signature of the spacetime metric $g_{\mu\nu}$ is $-+++$. For a massive scalar field, the most general harmonic potential $V(\varphi)$ is

$$V(\varphi) = \frac{\xi R \varphi^2}{2} + \frac{M^2 \varphi^2}{2}, \quad (2.2)$$

where R is the scalar curvature of the metric, M the mass of the scalar field φ , and ξ the coupling constant.

Let the space topology of the universe be $S^1 \times S^1 \times S^1$. This manifold is obtained by a formal identification of the opposite faces of a parallelepiped. Thus, the metric is chosen to be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 dx^2 + b^2 dy^2 + c^2 dz^2, \quad (2.3)$$

where

$$0 \leq x, y, z \leq 1.$$

The metric (2.3) is assumed to be static, and $a, b, c = \text{const}$ throughout our calculations. In this case, the scalar curvature vanishes, $R = 0$. The Euclidean action is obtained from Lorentzian (2.1) by substituting $t \rightarrow -i\tau$. For the static metric, Eq. (2.3) becomes

$$S^e = \frac{abc}{2} \int d^4x \left[(\partial_t \varphi)^2 + \frac{1}{a^2} (\partial_x \varphi)^2 + \frac{1}{b^2} (\partial_y \varphi)^2 + \frac{1}{c^2} (\partial_z \varphi)^2 + M^2 \varphi^2 \right]. \quad (2.4)$$

The free energy of the system with the temperature $k_B T \equiv \frac{1}{\beta}$ is

$$F = -k_B T \ln Z, \quad (2.5)$$

where the partition function Z is defined by the formula

$$Z = \int \mathcal{D}\varphi e^{-S^e}. \quad (2.6)$$

The functional integral is taken over all fields φ that are periodic in imaginary time τ with a period $\hbar\beta$. To calculate the path integral (2.6), we expand the scalar field φ over the eigenfunction basis of the Laplace–Beltrami operator $\Delta_2^{(3)}$ for the metric (2.3)

$$\varphi(x) = \frac{1}{2\sqrt{abc}} \sum_J \left(\varphi_J(\tau) e^{i\vec{k}\vec{x}} + \text{c.c.} \right), \quad (2.7)$$

where $\vec{k} = 2\pi(l, m, n)$, $l, m, n = 0, \pm 1, \pm 2, \dots$, and $J = \{l, m, n\}$ is a cumulative index. The coefficient functions satisfy the periodic boundary conditions:

$$\varphi_J(\tau = 0) = \varphi_J(\tau = \hbar\beta) = 0. \quad (2.8)$$

Substituting (2.7) into (2.4), one finds the Euclidean action

$$S^e = \frac{1}{2} \sum_J \int d\tau \left[|\dot{\varphi}_J|^2 + \omega_J^2 |\varphi_J|^2 \right], \quad (2.9)$$

where the dot denotes differentiation with respect to τ , and the eigenfrequencies ω_J are

$$\omega_J^2 = 4\pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) + M^2. \quad (2.10)$$

The decomposition has reduced the functional integral (2.6) to a product of the simple path integrals of harmonic oscillators. It is then easy to calculate the total free energy [3, 4]

$$F = k_B T \sum_J \ln \left(2 \sinh \frac{\hbar\omega_J}{2k_B T} \right) = k_B T \sum_J \ln \left(1 - e^{-\frac{\hbar\omega_J}{k_B T}} \right) + \frac{1}{2} \sum_J \hbar\omega_J. \quad (2.11)$$

3. THE INFINITE VOLUME LIMIT

The expression for the free energy (2.11) is divergent and we should regularize this triple sum. First of all, we want to derive the regularized formulas for the free energy, energy density, and pressure of the scalar field fluctuations in the infinite volume limit $V = abc \rightarrow \infty$. In this case, (2.11) is

$$F_\infty = k_B T \int_{-\infty}^{+\infty} dJ \ln \left(2 \sinh \frac{\hbar\omega_J}{2k_B T} \right), \quad (3.1)$$

where $\int_{-\infty}^{+\infty} dJ \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dl dm dn$. After a standard regularization ($\int_{-\infty}^{+\infty} dJ \omega_J = 0$), this becomes the relation

$$\begin{aligned} F_{\infty, \text{ren}} &= k_B T \int_{-\infty}^{+\infty} dJ \ln \left[1 - \exp \left(-\frac{\hbar}{k_B T} \sqrt{4\pi^2 \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) + M^2} \right) \right] = \\ &= -\frac{k_B T}{\hbar^3} \frac{abc}{6\pi^2} \int_0^\infty \frac{x^4 dx}{\sqrt{x^2 + \overline{M}^2} \left(\exp \sqrt{x^2 + \overline{M}^2} - 1 \right)}, \end{aligned} \quad (3.2)$$

where

$$\overline{M} = \frac{\hbar M}{k_B T}. \quad (3.3)$$

The internal energy density of the fluctuations is found from the condition

$$\rho = \frac{1}{V} U = \frac{1}{abc} \frac{\partial(\beta F_{\text{ren}})}{\partial \beta}. \quad (3.4)$$

For (3.2) we have

$$\rho_{\infty, \text{ren}} = \frac{1}{2\pi^2} \frac{(k_B T)^4}{\hbar^3} \int_0^\infty dx \frac{x^2 \sqrt{x^2 + \overline{M}^2}}{e^{\sqrt{x^2 + \overline{M}^2}} - 1}. \quad (3.5)$$

We can consider two limits. First, the massless one, $M \rightarrow 0$. In this case,

$$\rho_{\infty, \text{ren}} = \frac{\pi^2}{30} \frac{(k_B T)^4}{\hbar^3}. \quad (3.6)$$

This is the well-known formula for the energy density of a black body scalar field. The second limit is the heavy particle limit (the low-temperature limit) $M \gg k_B T/\hbar$. For this limit,

$$\rho_{\infty, \text{ren}} \approx \frac{3\hbar M^4}{\pi^{3/2} 2^{5/2}} \left(\frac{k_B T}{\hbar M} \right)^{5/2} e^{-\frac{\hbar M}{k_B T}} + \frac{\hbar M^4}{\pi^{3/2} 2^{3/2}} \left(\frac{k_B T}{\hbar M} \right)^{3/2} e^{-\frac{\hbar M}{k_B T}} \equiv \rho_1 + \rho_0, \quad (3.7)$$

where ρ_1 is the kinetic energy density, ρ_0 is the rest mass density, and $\rho_1 \ll \rho_0$ in the low-temperature limit. To find the pressure of the fluctuations, we use the formula

$$P = -\frac{\partial F_{\text{ren}}}{\partial V}. \quad (3.8)$$

For the space topology $S^1 \times S^1 \times S^1$, we have

$$\begin{aligned} P_a &= -\frac{1}{bc} \frac{\partial F_{\text{ren}}}{\partial a}, \\ P_b &= -\frac{1}{ac} \frac{\partial F_{\text{ren}}}{\partial b}, \\ P_c &= -\frac{1}{ab} \frac{\partial F_{\text{ren}}}{\partial c}. \end{aligned} \quad (3.9)$$

If $a = b = c$, then

$$P_a = P_b = P_c \equiv P = -\frac{1}{3a^2} \frac{\partial F_{\text{ren}}(a = b = c)}{\partial a}. \quad (3.10)$$

Substitution of Eq. (3.2) into (3.9) gives, for any a , b and c ,

$$P_a = P_b = P_c \equiv P_{\infty, \text{ren}} = \frac{(k_B T)^4}{\hbar^3} \frac{1}{6\pi^2} \int_0^\infty \frac{x^4 dx}{\sqrt{x^2 + \overline{M}^2} \left(\exp \sqrt{x^2 + \overline{M}^2} - 1 \right)}. \quad (3.11)$$

There is no dependence on direction. It is easy to see that in the case of radiation ($M = 0$), we have the usual equation of state

$$P_{\infty, \text{ren}} = \frac{1}{3} \rho_{\infty, \text{ren}}. \quad (3.12)$$

In the low-temperature limit $M \gg k_B T / \hbar$, pressure is described by the relation $P_{\infty, \text{ren}} \approx 2/3 \rho_1$, which coincides with the equation of state for the Fermi and Bose gases of elementary particles [31].

4. FINITE-SIZE EFFECTS ON FIELD FLUCTUATIONS

The expression for the free energy of the scalar field is given by Eq. (2.11). To isolate the finite effects, we add and subtract the infinite volume limit (3.1) and write

$$F = (F - F_\infty) + F_\infty \equiv F_{\text{fs}} + F_\infty. \quad (4.1)$$

The renormalized free energy is

$$F_{\text{ren}} = F_{\text{fs}} + F_{\infty, \text{ren}}, \quad (4.2)$$

where $F_{\infty, \text{ren}}$ is defined by Eq. (3.2), and the finite-size effects are contained in the sum-minus-integral expression

$$\begin{aligned} F_{\text{fs}} &= k_B T \left[\sum_J \ln \left(2 \sinh \frac{\hbar \omega_J}{2k_B T} \right) - \int_{-\infty}^{+\infty} dJ \ln \left(2 \sinh \frac{\hbar \omega_J}{2k_B T} \right) \right] = \\ &= k_B T \left\{ \sum_J \left[\ln \frac{\hbar \omega_J}{k_B T} + \sum_{p=1}^{\infty} \ln \left(1 + \frac{1}{p^2} \frac{\hbar^2 \omega_J^2}{(2\pi k_B T)^2} \right) \right] - \right. \\ &\quad \left. - \int_{-\infty}^{+\infty} dJ \ln \left(2 \sinh \frac{\hbar \omega_J}{2k_B T} \right) \right\}. \end{aligned} \quad (4.3)$$

To carry out the regularization procedure, it is convenient to use the Abel–Plana summation formula [11; 32 (formula 1.9 (11))]

$$\sum_{n=0}^{\infty} f(n) = \frac{f(0)}{2} + \int_0^\infty f(n) dn + i \int_0^\infty \frac{f(i\nu) - f(-i\nu)}{[\exp(2\pi\nu) - 1]} d\nu, \quad (4.4)$$

which is correct if $f(\nu)$ is regular for $\Re \nu \geq 0$ (on the imaginary axis, $f(\nu)$ may have poles and branching points, which are passed on the right during integration, i.e., with $\Re \nu > 0$). For example, for a typical sum, we have

$$\begin{aligned} \text{reg} \sum_{l=0}^{\infty} \ln \left(\frac{l^2}{a^2} + M_0^2 \right) &\equiv \sum_{l=0}^{\infty} \ln \left(\frac{l^2}{a^2} + M_0^2 \right) - \int_0^{\infty} dl \ln \left(\frac{l^2}{a^2} + M_0^2 \right) = \\ &= \ln M_0 - 2\pi \int_{M_0 a}^{\infty} \frac{d\nu}{e^{2\pi\nu} - 1}, \end{aligned} \quad (4.5)$$

where the last integral in (4.5) is convergent, and

$$M_0 \equiv \frac{M}{2\pi}. \quad (4.6)$$

For the triple sums in (4.3), we should apply the Abel–Plana summation formula (4.4) three times. After long but not difficult calculations, we get

$$\begin{aligned} F_{\text{fs}} = & \frac{4\pi}{3} k_B T \sum_{p=0}^{\infty} \left\{ - \int_{M_p a}^{\infty} d\nu I(\nu) + \right. \\ & + \int_0^{\infty} dl \left[\int_{[b^2(\frac{l^2}{a^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) + \int_{[c^2(\frac{l^2}{a^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) \right] + \\ & + 2 \sum_{l=0}^{\infty} \left[\int_{[b^2(\frac{l^2}{a^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) + \int_{[c^2(\frac{l^2}{a^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) \right] - \\ & - 4 \int_0^{\infty} \int_0^{\infty} dn dm \int_{[a^2(\frac{m^2}{b^2} + \frac{n^2}{c^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) - \\ & - 2 \int_0^{\infty} dn \left(\sum_{l=0}^{\infty} \int_{[b^2(\frac{l^2}{a^2} + \frac{n^2}{c^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) + \right. \\ & \quad \left. + \sum_{m=0}^{\infty} \int_{[a^2(\frac{m^2}{b^2} + \frac{n^2}{c^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) \right) - \\ & \left. - 4 \sum_{l,m=0}^{\infty} \int_{[c^2(\frac{l^2}{a^2} + \frac{m^2}{b^2} + M_p^2)]^{1/2}}^{\infty} d\nu I(\nu) + \text{C.P.} \right\}, \end{aligned} \quad (4.7)$$

where

$$M_p^2 = M_0^2 + \left(\frac{pk_B T}{\hbar} \right)^2, \quad (4.8)$$

$$I(\nu) = [\exp(2\pi\nu) - 1]^{-1}, \quad (4.9)$$

and C.P. means terms obtained by cyclic permutations $\{a, l; b, m; c, n\}$, for example, $b^2 \left(\frac{n^2}{c^2} + M_p^2 \right) \rightarrow c^2 \left(\frac{l^2}{a^2} + M_p^2 \right) \rightarrow a^2 \left(\frac{m^2}{b^2} + M_p^2 \right)$. The prime over the first sum indicates that the term with $p = 0$ should be counted with the weight $1/2$.

The total renormalized free energy is given by Eq. (4.2) where F_{fs} is defined by (4.7) and $F_{\infty, \text{ren}}$ is equal to (3.2). In this paper, we assume that our universe is approximately equal in size in all three directions. Therefore, we usually assume $a = b = c$.

After this assumption, we obtain

$$\begin{aligned}
 F_{\text{fs}} \stackrel{a=b=c}{=} & 4\pi k_B T \sum_{p=0}^{\infty} \left\{ - \int_{M_p a}^{\infty} d\nu I(\nu) + 2 \int_0^{\infty} dl \int_{\sqrt{l^2 + M_p^2 a^2}}^{\infty} d\nu I(\nu) + \right. \\
 & + 4 \sum_{l=0}^{\infty} \int_{\sqrt{l^2 + M_p^2 a^2}}^{\infty} d\nu I(\nu) - 4 \int_0^{\infty} \int_0^{\infty} dl dm \int_{\sqrt{l^2 + m^2 + M_p^2 a^2}}^{\infty} d\nu I(\nu) - \\
 & \left. - 4 \int_0^{\infty} dl \sum_{m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_p^2 a^2}}^{\infty} d\nu I(\nu) - 4 \sum_{l,m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_p^2 a^2}}^{\infty} d\nu I(\nu) \right\} \equiv \\
 \equiv & 4\pi k_B T \sum_{p=0}^{\infty} I_1(p). \tag{4.10}
 \end{aligned}$$

The internal energy density of the fluctuations is defined by Eq. (3.4), where F_{ren} is found from Eq. (4.2). It gives

$$\rho = \rho_{\text{fs}} + \rho_{\infty, \text{ren}}, \tag{4.11}$$

where $\rho_{\infty, \text{ren}}$ is defined by formula (3.5) and the finite-size energy density is

$$\rho_{\text{fs}} = \frac{4\pi}{\hbar} \frac{\Theta^2}{a^4} \sum_{p=1}^{\infty} p^2 I_2(p), \tag{4.12}$$

where ($a = b = c$)

$$\begin{aligned}
 I_2(p) = & -\mathcal{I}(p; 0, 0) + 2 \int_0^{\infty} dl \mathcal{I}(p; l, 0) + \\
 & + 4 \sum_{l=0}^{\infty} \mathcal{I}(p; l, 0) - 4 \int_0^{\infty} \int_0^{\infty} dl dm \mathcal{I}(p; l, m) - \\
 & - 4 \int_0^{\infty} dl \sum_{m=0}^{\infty} \mathcal{I}(p; l, m) - 4 \sum_{l,m=0}^{\infty} \mathcal{I}(p; l, m), \tag{4.13}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{I}(p; l, m) = & \left[\sqrt{p^2 + (l^2 + m^2) \frac{\hbar^2}{\Theta^2} + \left(\frac{M_0 \hbar}{k_B T} \right)^2} \times \right. \\
 & \left. \times \left(\exp 2\pi \frac{\Theta}{\hbar} \sqrt{p^2 + (l^2 + m^2) \frac{\hbar^2}{\Theta^2} + \left(\frac{M_0 \hbar}{k_B T} \right)^2} - 1 \right) \right]^{-1}. \tag{4.14}
 \end{aligned}$$

Here, we introduced the dimensionless quantity

$$\Theta = k_B T \cdot a. \tag{4.15}$$

To find the pressure of the fluctuations, we use (3.10),

$$P = P_{\text{fs}} + P_{\infty}, \tag{4.16}$$

where $P_{\infty, \text{ren}}$ is defined by (3.11), and the pressure caused by the finite-size effects is ($a = b = c$)

$$P_{\text{fs}} = \frac{1}{3} \rho_{\text{fs}} + \frac{4\pi}{3\hbar} \left(\frac{M_0 \hbar}{k_B T} \right)^2 \frac{\Theta^2}{a^4} \sum_{p=0}^{\infty} I_2(p). \quad (4.17)$$

From (3.12) and (4.17), it is easy to see that in the massless limit, we have the standard radiation state equation

$$P \stackrel{M=0}{=} \frac{1}{3} \rho. \quad (4.18)$$

In this limit, if the number of particles during the evolution of the universe is conserved, i.e., $\rho \cdot a^4 = \text{const}$, from (4.11)–(4.13) we have

$$k_B T \cdot a = \Theta \stackrel{M=0}{=} \text{const}. \quad (4.19)$$

This coincides with the usual relation between the temperature of the radiation and the scale factor in the Friedmann universe [29].

5. THE HIGH-TEMPERATURE LIMIT

Expressions (4.10), (4.12), and (4.16) are valid in the high-temperature approximation $k_B T \gg \hbar/a$. Let us obtain the energy density in this limit.

The massless case: $M = 0$. It is clear that in this case, the term with $p = 1$ gives the leading contribution to ρ_{fs} . If we take into account that all momenta l, m from zero to $\Theta/\hbar$ give approximately equal contributions, with the weight $\exp(-2\pi\Theta/\hbar)$, we come to the estimate

$$\rho \approx -\frac{48\pi}{\hbar^3} \frac{\Theta^4}{a^4} e^{-2\pi\frac{\Theta}{\hbar}} + \frac{\pi^2}{30\hbar^3} \frac{\Theta^4}{a^4}. \quad (5.1)$$

The limit of light particles: $\frac{M\hbar}{k_B T} = \frac{Ma}{\Theta/\hbar} \leq 1$. In this case, we get the estimate

$$\rho \approx \frac{4\pi}{\hbar} \frac{\Theta^2}{a^4} I_2(p=1) + \rho_{\infty, \text{ren}}. \quad (5.2)$$

For $M \rightarrow 0$, we reproduce the result (5.1).

The limit of heavy particles: $\frac{M\hbar}{k_B T} = \frac{Ma}{\Theta/\hbar} \gg 1$. In this limit, all terms $I_2(p)$ with p ranging from 1 to $p_c = [M_0 \hbar / k_B T] \gg 1$, p_c being the largest integer in $M_0 \hbar / k_B T$, give approximately equal contributions to the sum over p . Thus,

$$\rho_{\text{fs}} \approx \frac{4\pi}{\hbar} \frac{\Theta^2}{a^4} I_2(p=1) \sum_{p=1}^{p_c} p^2 \approx \frac{4\pi}{3\hbar} \frac{\Theta^2}{a^4} I_2(p=0) p_c^3, \quad (5.3)$$

where we took into account that for $M\hbar/k_B T \gg 1$: $I_2(p=1) \approx I_2(p=0)$. For the total energy density in this limit, we get the estimate

$$\rho \approx \frac{4\pi}{3\hbar} \frac{\Theta^2}{a^4} I_2(p=0) p_c^3 + \rho_0, \quad (5.4)$$

where ρ_0 is defined by (3.7).

For the pressure of the fluctuations (4.15), (4.16), similar estimates can be obtained.

6. FINITE-SIZE VACUUM EFFECTS

In this section, we will derive the formulas for pure vacuum fluctuations. This implies the limit transition $T \rightarrow 0$ in the equation for the free energy (2.11). The expression for the free energy of vacuum fluctuations reads

$$F_{\text{vf}} = \frac{\hbar}{2} \sum_J \omega_J - \frac{\hbar}{2} \int_{-\infty}^{+\infty} dJ \omega_J, \quad (6.1)$$

where the frequency ω_J is defined by Eq. (2.10). After application of the Abel–Plana summation formula (4.4) to the triple sum in (6.1), for arbitrary scale factors a, b , and c , we get the expression for the free energy of the vacuum fluctuations,

$$F_{\text{vf}} = \frac{2}{k_B T} F_{\text{fs}} \quad (p = 0). \quad (6.2)$$

This means that in expression (4.7) for F_{fs} , only the term with $p = 0$ should be left, and all integrals of the type

$$\int_{a\sqrt{\frac{m^2}{b^2} + \frac{n^2}{c^2} + M_0^2}}^{\infty} d\nu I(\nu)$$

should be replaced by the integrals

$$\int_{a\sqrt{\frac{m^2}{b^2} + \frac{n^2}{c^2} + M_0^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{m}{b}, \frac{n}{c}, M_0\right),$$

where

$$I\left(\frac{\nu}{a}; \frac{m}{b}, \frac{n}{c}, M_0\right) = \sqrt{\frac{\nu^2}{a^2} - \frac{m^2}{b^2} - \frac{n^2}{c^2} - M_0^2} (e^{2\pi\nu} - 1)^{-1}. \quad (6.3)$$

For example, in the symmetrical case, $a = b = c$, we get

$$\begin{aligned} F_{\text{vf}} = 4\pi\hbar \bigg\{ & - \int_{M_0 a}^{\infty} d\nu I\left(\frac{\nu}{a}; 0, 0, M_0\right) + \\ & + 2 \int_0^{\infty} dl \int_{\sqrt{l^2 + M_0^2 a^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{l}{a}, 0, M_0\right) + \\ & + 4 \sum_{l=0}^{\infty} \int_{\sqrt{l^2 + M_0^2 a^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{l}{a}, 0, M_0\right) - \\ & - 4 \int_0^{\infty} \int_0^{\infty} dl dm \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{l}{a}, \frac{m}{a}, M_0\right) - \\ & - 4 \int_0^{\infty} dl \sum_{m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{l}{a}, \frac{m}{a}, M_0\right) - \\ & - 4 \sum_{l, m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{l}{a}, \frac{m}{a}, M_0\right) \bigg\}. \quad (6.4) \end{aligned}$$

The free energy of the vacuum fluctuations does not depend on the temperature. Therefore, the energy density of the vacuum fluctuations is

$$\rho_{\text{vf}} = \frac{1}{abc} \frac{\partial(\beta F_{\text{vf}})}{\partial \beta} = \frac{1}{abc} F_{\text{vf}}, \quad (6.5)$$

where F_{vf} is defined either by Eq. (6.2), or, in the symmetrical case $a = b = c$, by formula (6.4).

Let us consider the symmetrical case $a = b = c$ in more detail. It is not difficult to see that the first term in Eq. (6.4) gives the leading contribution. For example, in the massless limit $M_0 = 0$, we have a rough estimate

$$\rho_{\text{vf}} \approx -\frac{4\pi\hbar}{a^3} \int_0^\infty d\nu I\left(\frac{\nu}{a}, 0, 0, 0\right) = -\frac{\pi}{6} \frac{\hbar}{a^4} = -0,5233 \frac{\hbar}{a^4}. \quad (6.6)$$

Counting all terms in (6.4), one gets [5, 6, 23–27]

$$\rho_{\text{vf}} \approx -0,8375 \frac{\hbar}{a^4}. \quad (6.7)$$

To get the estimate in the heavy particle limit $M_0 a \gg 1$, it is convenient to apply the Watson lemma, which is:

$$\int_0^\infty dt e^{-zt} \varphi(t) \underset{z \rightarrow \infty}{\sim} \sum_k a_k \Gamma(\gamma_k + 1) z^{-(\gamma_k + 1)}, \quad (6.8)$$

if

$$\varphi(t) \underset{t \rightarrow 0}{\sim} \sum_k a_k t^{\gamma_k} \quad (\gamma_k > -1). \quad (6.9)$$

For example, we have the following estimate for the typical integral:

$$\int_{M_0 a}^\infty d\nu I\left(\frac{\nu}{a}, 0, 0, M_0\right) \underset{M_0 a \gg 1}{\approx} \frac{(M_0 a)^{1/2}}{4\pi a} e^{-2\pi M_0 a}. \quad (6.10)$$

Applying the Watson lemma to the whole (6.4) expression, in the limit $M_0 a \gg 1$, we get the estimate for the energy density of the vacuum fluctuations:

$$\rho_{\text{vf}} \approx -\frac{12\hbar}{a^4} (M_0 a)^{5/2} e^{-2\pi M_0 a}. \quad (6.11)$$

For the pressure of the vacuum fluctuations in the symmetrical case, we have

$$\begin{aligned}
 P_{\text{vf}} &= -\frac{1}{3a^2} \frac{\partial F_{\text{vf}}(a=b=c)}{\partial a} = \\
 &= \frac{1}{3} \rho_{\text{vf}} + \frac{4\pi\hbar}{3} \frac{(M_0 a)^2}{a^4} \left\{ - \int_{M_0 a}^{\infty} d\nu \left[\sqrt{\nu^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} + \right. \\
 &\quad + 2 \int_0^{\infty} dl \int_{\sqrt{l^2 + M_0^2 a^2}}^{\infty} d\nu \left[\sqrt{\nu^2 - l^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} + \\
 &\quad + 4 \sum_{l=0}^{\infty} \int_{\sqrt{l^2 + M_0^2 a^2}}^{\infty} d\nu \left[\sqrt{\nu^2 - l^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} - \\
 &\quad - 4 \int_0^{\infty} \int_0^{\infty} dl dm \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu \left[\sqrt{\nu^2 - l^2 - m^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} - \\
 &\quad - 4 \int_0^{\infty} dl \sum_{m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu \left[\sqrt{\nu^2 - l^2 - m^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} - \\
 &\quad \left. - 4 \sum_{l,m=0}^{\infty} \int_{\sqrt{l^2 + m^2 + M_0^2 a^2}}^{\infty} d\nu \left[\sqrt{\nu^2 - l^2 - m^2 - M_0^2 a^2} (e^{2\pi\nu} - 1) \right]^{-1} \right\}. \quad (6.12)
 \end{aligned}$$

In the massless case $M = 0$, we get the usual state equation for radiation,

$$P_{\text{vf}} \stackrel{M=0}{=} \frac{1}{3} \rho_{\text{vf}}. \quad (6.13)$$

In the limit of heavy particles $M_0 a \gg 1$, applying the Watson lemma to (6.12), we get the estimate for the pressure of vacuum fluctuations:

$$P_{\text{vf}} \approx -\frac{8\pi\hbar}{a^4} (M_0 a)^{7/2} e^{-2\pi M_0 a}. \quad (6.14)$$

7. AN ALTERNATIVE FORM

Formula (4.10) does not work in the low-temperature limit $\Theta/\hbar \ll 1$. To consider this case, we may use the method [7–9, 14]. According to the Poisson summation formula, the following identity holds:

$$\sum_{p=-\infty}^{+\infty} \sigma(p) = 2\pi \sum_{k=-\infty}^{+\infty} c(2\pi k), \quad (7.1)$$

if $\sigma(p)$ and $c(k)$ are connected by the Fourier transform

$$c(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma(x) e^{-i\alpha x} dx. \quad (7.2)$$

In our case,

$$\sigma(p) = I_1(p), \quad (7.3)$$

where $I_1(p)$ is defined in (4.10). Then,

$$c(\alpha) = \frac{1}{\pi} \int_0^{\infty} dx \cos \alpha x I_1(x), \quad (7.4)$$

and we can rewrite Eq. (4.10) as

$$F_{\text{fs}} = 4\pi^2 k_B T \left\{ c(0) + 2 \sum_{k=1}^{\infty} c(2\pi k) \right\}. \quad (7.5)$$

It is easy to see that $c(0)$ is proportional to the free energy of the vacuum fluctuations (6.4):

$$4\pi^2 k_B T c(0) = F_{\text{vf}}. \quad (7.6)$$

For the $c(2\pi k)$ terms, we have

$$\begin{aligned} c(2\pi k) = & -c(2\pi k; M_0 a) + 2 \int_0^{\infty} dl c\left(2\pi k; \sqrt{l^2 + M_0^2 a^2}\right) + \\ & + 4 \sum_{l=0}^{\infty} c\left(2\pi k; \sqrt{l^2 + M_0^2 a^2}\right) - \\ & - 4 \int_0^{\infty} \int_0^{\infty} dl dm c\left(2\pi k; \sqrt{l^2 + m^2 + M_0^2 a^2}\right) - \\ & - 4 \int_0^{\infty} dl \sum_{m=0}^{\infty} c\left(2\pi k; \sqrt{l^2 + m^2 + M_0^2 a^2}\right) - \\ & - 4 \sum_{l,m=0}^{\infty} c\left(2\pi k; \sqrt{l^2 + m^2 + M_0^2 a^2}\right), \end{aligned} \quad (7.7)$$

where

$$c(\alpha; D) = \frac{1}{2\pi^2 \alpha} \int_{2\pi D}^{\infty} \frac{\sin \left[\frac{\alpha \hbar}{2\pi \Theta} \sqrt{y^2 - 4\pi^2 D^2} \right]}{e^y - 1} dy. \quad (7.8)$$

Equation (7.5) takes the form

$$F_{\text{fs}} = F_{\text{vf}} + 8\pi^2 k_B T \sum_{k=1}^{\infty} c(2\pi k). \quad (7.9)$$

Thus, an alternative expression for the total renormalized free energy is $F_{\text{ren}} = F_{\text{fs}} + F_{\infty, \text{ren}}$, where F_{fs} is defined by (7.7)–(7.9) and $F_{\infty, \text{ren}}$ is given by (3.2).

This alternative expression is the most convenient one to use in the massless case where the terms $c(2\pi k)$, as a rule, can be calculated exactly [14]. Unfortunately, for the universe with the space topology of a three-dimensional torus, we cannot do this even for $M = 0$, and expression (7.9) may be useful for numerical calculations in the low-temperature limit $\Theta/\hbar \ll 1$. If it is necessary to obtain analytic estimates in the low-temperature limit, one should try to find another representation of the same expression.

8. THE LOW-TEMPERATURE LIMIT

It is easy to obtain the low-temperature limit for F_{fs} directly from the formula (4.3), which can be written in the form

$$F_{\text{fs}} = k_B T \sum_J \ln \left[1 - \exp \left(-\frac{\hbar \omega_J}{k_B T} \right) \right] + F_{\text{vf}} - F_{\infty, \text{ren}}. \quad (8.1)$$

Adding $F_{\infty, \text{ren}}$ to this expression, we have

$$F_{\text{ren}} = k_B T \sum_J \ln \left[1 - \exp \left(-\frac{\hbar \omega_J}{k_B T} \right) \right] + F_{\text{vf}}. \quad (8.2)$$

Substituting (8.2) into (3.4), we get

$$\rho = \frac{\hbar}{abc} \sum_J \frac{\omega_J}{\exp \left(\frac{\hbar \omega_J}{k_B T} \right) - 1} + \rho_{\text{vf}}, \quad (8.3)$$

or, in the symmetrical case $a = b = c$,

$$\rho = \frac{2\pi\hbar}{a^4} \sum_J \frac{(l^2 + m^2 + n^2 + M_0^2 a^2)^{1/2}}{\exp \left[\frac{2\pi}{\Theta/\hbar} (l^2 + m^2 + n^2 + M_0^2 a^2)^{1/2} \right] - 1} + \rho_{\text{vf}}. \quad (8.4)$$

For the pressure, we have from (8.2) and (3.10),

$$P = P_{\text{vf}} + \frac{2\pi\hbar}{3a^4} \times \sum_J'' \frac{l^2 + m^2 + n^2}{(l^2 + m^2 + n^2 + M_0^2 a^2)^{1/2} \left\{ \exp \left[\frac{2\pi}{\Theta/\hbar} (l^2 + m^2 + n^2 + M_0^2 a^2)^{1/2} \right] - 1 \right\}}, \quad (8.5)$$

where the double prime denotes that the zero mode $l = m = n = 0$ gives no contribution to the pressure. In the massless case $M = 0$, we have the usual state equation for radiation $P = \frac{1}{3}\rho$ if we drop the zero mode contribution to the energy density, as well.

The massless case: $M = 0$. It is clear that for the low-temperature limit $\Theta/\hbar \ll 1$ in the massless case, the zero mode and the terms with $l^2 + m^2 + n^2 = 1$ give the leading contributions to (8.4):

$$\rho \approx \frac{\hbar}{a^4} \cdot \frac{\Theta}{\hbar} + \frac{12\pi\hbar}{a^4} e^{-\frac{2\pi\hbar}{\Theta}} + \rho_{\text{vf}}(M = 0), \quad (8.6)$$

where the first term results from the zero mode and $\rho_{\text{vf}}(M = 0)$ is defined by (6.7). For the low-temperature limit, $|\rho_{\text{vf}}(M = 0)| \gg \frac{\hbar}{a^4} \frac{\Theta}{\hbar}$. Thus, in this case, $\rho < 0$.

The limit of ultra-light-particles: $Ma < \Theta/\hbar \ll 1$. In this limit, again, the zero mode and terms with $l^2 + m^2 + n^2 = 1$ make the leading contributions to (8.4)

$$\rho \approx \frac{\hbar}{a^4} \frac{\Theta}{\hbar} \left(1 - \frac{1}{2} \frac{Ma}{\Theta/\hbar} \right) + \frac{12\pi\hbar}{a^4} e^{-\frac{2\pi\hbar}{\Theta}} + \rho_{\text{vf}}, \quad (8.7)$$

where $\rho_{\text{vf}} \approx \rho_{\text{vf}}(M = 0)$. If $M \rightarrow 0$, we restore formula (8.6).

The limit of light particles: $\Theta/\hbar < Ma < 2\pi$. It is clear that in this case, the term with $l = m = n = 0$ gives the main contribution in (8.4), and we obtain the estimate

$$\rho \approx \frac{\hbar}{a^4} Ma e^{-\frac{Ma\hbar}{\Theta}} + \rho_{\text{vf}}. \quad (8.8)$$

The limit of heavy particles: $Ma \gg 1$. In this limit, all terms in sum (8.4) with $0 \leq l, m, n < n_c = [M_0 a]$, where n_c is the largest integer in $M_0 a$, give approximately equal contributions, with the weight $\sim M \exp[-Ma\hbar/\Theta]$. Thus, we have the estimate

$$\rho \approx 8n_c^3 \frac{\hbar}{a^4} (Ma) e^{-\frac{Ma\hbar}{\Theta}} + \rho_{\text{vf}}, \quad (8.9)$$

where the approximation for the energy density of the vacuum fluctuations ρ_{vf} is given by Eq. (6.11).

It is easy to see that in all four cases $\rho < 0$.

9. CONCLUSION

We have already stressed in the Introduction that according to the standard scenario [29], the hot universe is in the high-temperature limit. If we assume that in the present state of the universe the value $a \sim 12, 48 \cdot 10^{27}$ cm, and the temperature of the microwave radiation $T \sim 2.7^\circ$ K, then the dimensionless parameter $T \cdot a / \hbar = \Theta / \hbar \sim 1.5 \cdot 10^{29} \gg 1$. This parameter does not vary during the evolution of the universe since $T \sim 1/a$ for radiation. This estimate was derived for a closed universe with the space topology of a three-dimensional sphere. For the universe with the space topology of a three-dimensional torus, the scale factor a is less than for the closed Friedmann universe [19]. However, the difference is not essential, and we again get $\Theta / \hbar \gg 1$. It is also useful to obtain an estimate for another dimensionless parameter, Ma . For example, if we take the mass of the electron $M_e \sim 10^{11}$ cm $^{-1}$, then for the present state of the universe, we get $M_e a \sim 10^{39} \gg 1$. Moreover, $M_e a \gg \Theta / \hbar$. Thus, at the present state of the evolution of the universe, all massive particles that are not ultra-light [33] are in the heavy particle limit. The dimensionless parameter Ma varies during the evolution of the universe, and the particles pass gradually to the light-particle limit when $a \rightarrow 0$.

These estimates demonstrate that at the present stage of the evolution of the universe, the finite-size corrections should be exponentially small and have no practical effect. The method developed here for calculating the temperature and finite-size effects should be useful for applications in the models of cold universes or for other objects with non trivial topology where these dimensionless parameters are not so drastically large.

We would also like to note that the formulas obtained here can be generalized to universes with the space topology $R^1 \times S^1 \times S^1$ and $R^1 \times R^1 \times S^1$ with the help of the limiting transitions in Eq. (4.7) where, respectively, one or two scale factors become infinitely large.

Acknowledgements. One of the authors (A.Z.) wishes to thank the Institute of Theoretical Physics at the Free University of Berlin for the hospitality during the preparation of this paper. A.Z. was partly supported by Deutsche Forschungsgemeinschaft under grant №436 UKR 17-30-93.

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Поступила в редакцию
 6.IV.1995 г.

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ЭФФЕКТ КАЗИМИРА ПРИ НЕНУЛЕВОЙ ТЕМПЕРАТУРЕ ВО ВСЕЛЕННОЙ С ТОПОЛОГИЕЙ $S^1 \times S^1 \times S^1$

Исследуется эффект Казимира для массивных и безмассовых скалярных полей при ненулевой температуре во Вселенной с топологией трехмерного тора. Найдены перенормированная свободная энергия и соответствующие плотность энергии и давление. Обсуждаются физические следствия полученных результатов.