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A. Kagan, M. Konikov, The structure of the UMVUEs from categorical data, *Теория вероятн. и ее примен.*, 2005, том 50, выпуск 3, 597–604

DOI: <https://doi.org/10.4213/tvp100>

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Поступила в редакцию
1.VI.2005

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THE STRUCTURE OF THE UMVUEs FROM CATEGORICAL DATA

Пусть наблюдаемая случайная величина X принимает конечное число значений с вероятностями $p_1(\theta), \dots, p_N(\theta)$, зависящими от абстрактного параметра $\theta \in \Theta$. Доказывается, что статистика является несмещенной оценкой с равномерно минимальной дисперсией (UMVUE) тогда и только тогда, когда она измерима относительно подалгебры конечной алгебры, порожденной X . В общем случае эта подалгебра меньше, чем минимальная достаточная подалгебра для θ , и описывается в явном виде. Она связана со специальным разбиением конечного множества элементов абстрактного линейного пространства.

Ключевые слова и фразы: оценка, линейное пространство, подалгебра, достаточность.

1. Introduction. Let $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ be a family of probability distributions of a random element X taking values in a measurable space $(\mathcal{X}, \mathcal{A})$. The structure of uniformly minimum variance unbiased estimators (UMVUEs) of parameter functions has interested statisticians since long ago. Due to the Rao–Blackwell theorem, a UMVUE is a function of the minimal sufficient statistic when the latter exists, in particular, for

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dominated families $\mathcal{P}$. If the minimal sufficient statistic is complete, the class of UMVUEs coincides with the class of statistics with finite second moment that are functions of the minimal sufficient statistic (see, e.g., [6]).

Note in passing that when it exists, the minimal sufficient statistic is unique and a complete sufficient statistic is minimal.

We call a parameter function $\gamma(\theta)$, $\gamma: \Theta \rightarrow \mathbf{R}$, *estimable*, if there exists a statistic $g(X)$, $g: \mathcal{X} \rightarrow \mathbf{R}$, such that

$$\mathbf{E}_\theta g(X) = \gamma(\theta), \quad \mathbf{D}_\theta g(X) < \infty, \quad \theta \in \Theta.$$

In [1] it is proved that if every estimable function possesses a UMVUE, there exists a minimal sufficient statistic for θ . In a general case, some estimable parameter functions possess UMVUEs while the others do not.

Let $\chi(X)$ be an unbiased estimator of zero with finite variance, i.e.,

$$\mathbf{E}_\theta \chi(X) = 0, \quad \mathbf{D}_\theta \chi(X) < \infty, \quad \theta \in \Theta.$$

The following well-known lemma (see, e.g., [5, Lemma 7.2.1]) gives a necessary and sufficient condition for a statistic $g(X)$ to be a UMVUE.

Lemma 1. *A statistic $g(X)$ with finite second moment is a UMVUE if and only if it is uncorrelated with any unbiased estimator of zero $\chi(X)$, i.e.,*

$$\text{cov}(g, \chi) = \mathbf{E}_\theta \{g(X) \chi(X)\} = 0, \quad \theta \in \Theta. \quad (1)$$

Plainly, if g is a UMVUE, it is the UMVUE of the parameter function

$$\gamma(\theta) = \mathbf{E}_\theta g(X).$$

An important observation was made in [8] (for a related result see [5, Theorem 7.2.1]). Suppose that $g(X)$ is a UMVUE. By Lemma 1, (1) holds for any unbiased estimator of zero $\chi(X)$ so that $g\chi$ is also an unbiased estimator of zero and, thus,

$$\text{cov}(g^2, \chi) = \mathbf{E}_\theta \{(g(X))^2 \chi(X)\} = 0, \quad \theta \in \Theta.$$

Had this reasoning been flawless, one could have concluded that if g is a UMVUE, then the statistics $1, g, g^2, \dots$ with finite second moment are UMVUEs as well and, similarly, if g_1, g_2 are UMVUEs so is $g_1 g_2$, etc. In other words, the class of UMVUEs would have been a ring.

The problem is that though $\mathbf{E}_\theta g^2 < \infty$ and $\mathbf{E}_\theta \chi^2 < \infty$, $\mathbf{E}_\theta (g\chi)^2$ may be infinite. In [1] an elegant example of a family $\mathcal{P}$ is constructed when the class of UMVUEs is not a ring.

However, in case of categorical data, i.e., when an observation X takes finitely many values that, without loss in generality, may be taken as $1, \dots, N$, all the statistics have finite moments of all orders and the class of UMVUEs is a ring.

Let

$$\mathbf{P}_\theta \{X = i\} = p_i(\theta), \quad i \in I = \{1, \dots, N\}, \quad \theta \in \Theta. \quad (2)$$

Plainly, a function $\gamma(\theta)$ is estimable if and only if it is a linear combination of the probabilities,

$$\gamma(\theta) = \sum_1^N c_i p_i(\theta). \quad (3)$$

In Section 3 it is proved that there exists a partition

$$I = I_1 \cup \dots \cup I_k \quad (4)$$

determined by the probabilities (2) such that $g(X)$ is a UMVUE if and only if it is constant on the elements of partition (4), i.e.,

$$g(x) = g_i, \quad x \in I_i, \quad i = 1, \dots, k.$$

An estimable parameter function $\gamma(\theta)$ possesses a UMVUE if and only if the coefficients in (3) are constant on the elements of (4),

$$c_j = c_i^*, \quad j \in I_i, \quad i = 1, \dots, k.$$

Partition (4), in general, is different from the minimal sufficient partition

$$I = I'_1 \cup \dots \cup I'_l. \quad (5)$$

If one treats functions on Θ as elements of a linear vector space and sets

$$p_j(\theta) = v_j, \quad j = 1, \dots, N,$$

then i_1, i_2 belong to the same element of partition (5) if and only if $v_{i_1} \parallel v_{i_2}$. In other words, if $S(X)$ is the minimal sufficient statistic for θ , then

$$S(i_1) = S(i_2) \iff p_{i_1}(\theta) = c(i_1, i_2) p_{i_2}(\theta), \quad \theta \in \Theta,$$

for a constant (in θ) $c(i_1, i_2)$.

Each element of partition (4) is the union of elements of partition (5).

Looking at $p_j(\theta)$ as vectors translates the problem of describing the UMVUEs into a geometric problem in a finite dimensional linear space that may be of an interest in its own. Geometric lemmas are presented in Section 2.

Here we only illustrate the difference between partitions (4) and (5) with examples.

Example 1. Let

$$p_1(\theta) = \theta, \quad p_2(\theta) = \theta^2, \quad p_3(\theta) = 1 - \theta - \theta^2, \quad \Theta = \left(0, \frac{1}{2}\right).$$

Both partitions are $\{1\} \cup \{2\} \cup \{3\}$. The minimal sufficient statistic $S(X) = X$ is trivial and complete and all estimable functions (i.e., all quadratic polynomials in θ) possess UMVUEs.

Example 2. Let

$$p_1(\theta) = \theta, \quad p_2(\theta) = \theta^2, \quad p_3(\theta) = \theta + \theta^2, \quad p_4(\theta) = 1 - 2(\theta + \theta^2), \quad \Theta = \left(0, \frac{1}{4}\right).$$

The minimal sufficient statistic $S(X)$ is still trivial but plainly is incomplete since for any statistic $\chi(X)$ with

$$\chi(1) = \chi(2) = -\chi(3), \quad \chi(4) = 0;$$

we have $\mathbf{E}_\theta \chi(X) = 0$, $\theta \in \Theta$. Partition (4) is $\{1, 2, 3\} \cup \{4\}$ (see Section 3 for the proof) and the estimable functions possessing UMVUEs are of the form $a + b(\theta + \theta^2)$, $a, b \in \mathbf{R}$, while, as in Example 1, all quadratic polynomials are estimable. A UMVUE satisfies the condition

$$g(1) = g(2) = g(3).$$

The structure of UMVUEs is of interest for foundations of statistics, at the very least. General results for incomplete exponential families were obtained in [3], [4] (see also [7]). An interesting general result is due [2]. Namely, let $\mathcal{P} = \{\mathbf{P}_{\theta, \xi}, (\theta, \xi) \in \Theta \times \Xi\}$ be a family of distributions on a measurable space parameterized by a «bivariate» parameter (θ, ξ) , Θ, Ξ being arbitrary sets. If for each fixed ξ , a statistic T is complete sufficient for θ (but T is not assumed sufficient for (θ, ξ)), then any statistic $g(T)$ with finite second moment is a UMVUE.

The description of the class of all UMVUEs from categorical data presented in the next sections is not related to sufficiency and is based on properties of finite dimensional linear spaces. The structure of the class is different from the known structures.

2. Geometric lemmas. Let L be an abstract linear space with zero element $\mathbf{0}$. Subsets $U_1, \dots, U_n$ of L are called linearly independent if so are the subspaces $\text{span}(U_1), \dots, \text{span}(U_n)$, i.e., the relation

$$u_1 + \dots + u_n = \mathbf{0}, \quad u_i \in \text{span}(U_i), \quad i = 1, \dots, n,$$

implies

$$u_1 = \dots = u_n = \mathbf{0}.$$

The following lemma is a key to the structure of the class of UMVUEs from categorical data.

Lemma 2. Let $V = \{v_1, \dots, v_N\}$ be a finite set of elements of L ; it is assumed that $0 \notin V$. There exists a partition

$$V = V_1 \cup \dots \cup V_k \quad (6)$$

such that (i) $V_1, \dots, V_k$ are linearly independent, (ii) (6) is irreducible, i.e., if for some $j \in \{1, \dots, k\}$

$$V_j = V_{j1} \cup V_{j2}, \quad V_{j1} \cap V_{j2} = \emptyset$$

and $V_1, \dots, V_{j-1}, V_{j1}, V_{j2}, V_{j+1}, \dots, V_k$ are linearly independent, then either $V_{j1} = V_j$ or $V_{j2} = V_j$.

Such a partition is unique up to the order in (6) and within $V_1, \dots, V_k$.

P r o o f. We proceed by induction. For $N = 1$ Lemma 2 is trivially true. Suppose that

$$V' = V'_1 \cup \dots \cup V'_m$$

is the claimed partition of the set $V' = \{v_1, \dots, v_{N-1}\}$. If $V'_1, \dots, V'_m, \{v_N\}$ are linearly independent, the partition

$$V = V'_1 \cup \dots \cup V'_m \cup V'_{m+1}, \quad V'_{m+1} = \{v_N\}$$

satisfies (i) and (ii).

Suppose now that $V'_1, \dots, V'_m, \{v_N\}$ are linearly dependent. Due to the assumed linear independence of $V'_1, \dots, V'_m$, there is exactly one collection $(V'_{i_1}, \dots, V'_{i_l}), i_1, \dots, i_l \in \{1, \dots, m\}$, such that $v_N \in \text{span}(V'_{i_1}, \dots, V'_{i_l})$ and

$$v_N = v_{i_1} + \dots + v_{i_l} \quad \text{with} \quad v_{i_1} \in V_{i_1}, \dots, v_{i_l} \in V_{i_l}, \quad v_{i_1} \neq 0, \dots, v_{i_l} \neq 0. \quad (7)$$

Without loss in generality, one may assume

$$V'_{i_1} = V'_1, \dots, V'_{i_l} = V'_l, \quad l \leq m.$$

The partition

$$V = V_1 \cup \dots \cup V_k$$

with $k = m - l + 1$ and

$$V_1 = V'_1 \cup \dots \cup V'_l \cup \{v_N\}, \quad V_2 = V'_{l+1}, \dots, V_k = V'_m$$

satisfies (i) and (ii). Indeed, linear independence of $V_1, \dots, V_k$ follows from linear independence of $V'_1, \dots, V'_m$ and the fact that $v_N \in \text{span}(V'_1, \dots, V'_l)$.

Suppose now that

$$V_1 = V_{11} \cup V_{12}, \quad V_{11} \cap V_{12} = \emptyset.$$

If $v_N \in \text{span}(V_{11})$ and $V_{12} \neq \emptyset$ or $v_N \in \text{span}(V_{12})$ and $V_{11} \neq \emptyset$, either pair of conditions contradicts (7).

To prove the uniqueness, assume that, besides (6), there is another partition with properties (i) and (ii) of Lemma 2,

$$V = \tilde{V}_1 \cup \dots \cup \tilde{V}_n \quad (8)$$

for some n . Take the intersections

$$V_{i1} = V_i \cap \tilde{V}_1, \dots, V_{in} = V_i \cap \tilde{V}_n, \quad i = 1, \dots, k. \quad (9)$$

They are linearly independent since (8) satisfies (i). But this contradicts irreducibility of (6) unless only one of the sets (9) is nonempty. The same arguments prove that only one of the intersections

$$\tilde{V}_{j1} = \tilde{V}_j \cap V_1, \dots, \tilde{V}_{jk} = \tilde{V}_j \cap V_k, \quad j = 1, \dots, n,$$

is nonempty. This proves that for the partitions (6) and (8), $k = n$ and each $\tilde{V}_j$ is one of V_i . Lemma 2 is proved.

Plainly, collinear vectors belong to the same component of partition (6).

Example 3. Let u_1, u_2, u_3 be linearly independent elements of L and $V = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = u_1, \quad v_2 = u_2, \quad v_3 = u_2 + u_3, \quad v_4 = u_1 + u_2.$$

Then $V' = V'_1 \cup V'_2 \cup V'_3$ with $V'_1 = \{v_1\}$, $V'_2 = \{v_2\}$, $V'_3 = \{v_3\}$, $v_4 \in \text{span}(V'_1, V'_2)$ and $V = V_1 \cup V_2$ with $V_1 = V'_1 \cup V'_2 \cup \{v_4\}$, $V_2 = V'_3$.

Example 4. Add $v_5 = u_3$ and let $V = \{v_1, v_2, v_3, v_4, v_5\}$. Then

$$V' = V'_1 \cup V'_2 \quad \text{with} \quad V'_1 = \{v_1, v_2, v_4\}, \quad V'_2 = \{v_3\}$$

and $v_5 \in \text{span}(V'_1, V'_2)$. In this case, the partition is trivial, $V = V_1$.

The partition (6) generates a partition of the set $I = \{1, \dots, N\}$,

$$I = I_1 \cup \dots \cup I_k, \quad I_j = \{i \in I: v_i \in V_j\}, \quad j = 1, \dots, k. \quad (10)$$

A linear combination $\sum_{i=1}^N c_i v_i$ is called a *contrast* if

$$\sum_{i \in I_1} c_i v_i = \mathbf{0}. \quad (11)$$

A contrast is called *decomposable* if there is a partition $I \neq I' \cup I''$ such that $\sum_{i \in I'} c_i v_i$ and $\sum_{i \in I''} c_i v_i$ are nontrivial contrasts (i.e., not all the coefficients equal zero) and *indecomposable* otherwise.

Corollary 1. If $\sum_{i=1}^N c_i v_i$ is a contrast and

$$\sum_{i=1}^N c_i v_i = \sum_{i \in I_1} c_i v_i + \dots + \sum_{i \in I_k} c_i v_i,$$

then

$$\sum_{i \in I_1} c_i v_i, \dots, \sum_{i \in I_k} c_i v_i \quad (12)$$

are also contrasts.

Proof. Indeed, since

$$\sum_{i \in I_1} c_i v_i = u_1 \in \text{span}(V_1), \dots, \sum_{i \in I_k} c_i v_i = u_k \in \text{span}(V_k)$$

the relation $u_1 + \dots + u_k = \mathbf{0}$ implies $u_1 = \dots = u_k = \mathbf{0}$.

Turn now to contrasts (12).

Lemma 3. Suppose that V_1 contains at least two vectors. For any two vectors $v_i, v_j \in V_1$, there exists an indecomposable contrast $\sum_{i \in I_1} c_i v_i$ with $c_i c_j \neq 0$.

Proof. Re-enumerating the vectors $v_1, \dots, v_N$, if necessary, one may assume that $V_1 = (v_1, v_2, \dots, v_{m+1})$. It suffices to prove Lemma 3 for v_1, v_2 .

We shall proceed by induction in $|I_1|$, the number of elements in V_1 . For $V_1 = (v_1, v_2)$, the vectors v_1 and v_2 are linearly dependent (otherwise they would have belonged to different components of the partition) and neither is $\mathbf{0}$. Hence $c_1 v_1 + c_2 v_2 = \mathbf{0}$ for some c_1, c_2 with $c_1 c_2 \neq 0$.

Now assume that Lemma 3 holds for $|I_1| \leq m$ and prove it for $|I_1| = m + 1$. If the set $(v_1, v_2, \dots, v_m)$ is irreducible, the claim holds true due to the induction assumption.

Suppose now that $(v_1, v_2, \dots, v_m)$ is reducible (though V_1 is irreducible). By virtue of Lemma 2, there is a partition

$$V_1 = V_{11} \cup V_{12} \cup \dots \cup V_{1l} \quad (13)$$

similar to (6). If v_1, v_2 belong to the same component of the partition (13), the claim of Lemma 3 again holds due to the induction assumption.

Consider the case when v_1, v_2 belong to different components of (13), $v_1 \in V_{11}$, $v_2 \in V_{12}$, say. There is no loss in generality in assuming that a basis of $\text{span}(V_{1i})$, $i = 1, \dots, l$, consists of elements of V_{1i} ,

$$v_{1i}, v_{2i}, \dots, v_{m_i i}, \quad v_{11} = v_1, \quad v_{21} = v_2.$$

Since

$$v_{11}, \dots, v_{1m_1}; v_{21}, \dots, v_{2m_2}; \dots; v_{l1}, \dots, v_{lm_l}$$

is a basis of $\text{span}(V_{11}, V_{12}, \dots, V_{1l})$ and V_1 is irreducible, there exists a unique representation

$$v_{m+1} = \sum_{i=1}^{m_1} c_{1i} v_{1i} + \sum_{i=1}^{m_2} c_{2i} v_{2i} + \dots + \sum_{i=1}^{m_l} c_{li} v_{li}. \quad (14)$$

One has $c_{11}c_{21} \neq 0$. Indeed, were $c_{11} = 0$ it would imply that $v_1 = v_{11}$ is linearly independent of $v_2, \dots, v_{m+1}$ contradicting irreducibility of V_1 . Similarly, $c_{21} = 0$ implies the linear independence of v_2 and $v_1, v_3, \dots, v_{m+1}$. Thus,

$$\sum_{i=1}^{m_1} c_{1i} v_{1i} + \sum_{i=1}^{m_2} c_{2i} v_{2i} + \dots + \sum_{i=1}^{m_l} c_{li} v_{li} - v_{m+1}$$

is the required contrast. It is indecomposable due to the uniqueness of the representation (14). Lemma 3 is proved.

Let $c = (c_1, \dots, c_N)^T$, $g = (g(1), \dots, g(N))^T$ be vectors in $\mathbf{R}^N$. The Hadamard product $h = g \odot c$ of g and c is the vector with components $h_1 = g(1)c_1, \dots, h_N = g(N)c_N$.

Lemma 4. Suppose g is such that for any contrast $\sum_1^N c_i v_i$, $\sum_1^N h_i v_i$ is also a contrast. Then g is constant on each I_l , $l = 1, \dots, k$, i.e.,

$$g(x) = g_1, \quad x \in I_1, \dots, g(x) = g_k, \quad x \in I_k. \quad (15)$$

P r o o f. The claim of Lemma 4 is nontrivial only for the components of (10) containing at least two elements. Assuming that I_1 is such and $v_1, v_2 \in V_1$, we shall prove that $g(1) = g(2)$. Dividing $g(x)$ by $\max\{|g(x)|, x \in I_1\}$, one may assume that $|g(x)| \leq 1$, $x \in I_1$. Let $\sum_{i \in I_1} c_i v_i$ be an indecomposable contrast with $c_1 c_2 \neq 0$. Such a contrast exists due to Lemma 3.

Using the condition of Lemma 4 repeatedly, we get that the linear combinations

$$\sum_{i \in I_1} g(i) c_i v_i, \quad \sum_{i \in I_1} [g(i)]^2 c_i v_i, \quad \dots, \quad \sum_{i \in I_1} [g(i)]^n c_i v_i \quad (16)$$

are all contrasts.

Let $I_1 = I'_1 \cup I''_1$, where

$$I'_1 = \{i \in I_1: |g(i)| = 1\}, \quad I''_1 = \{i \in I_1: |g(i)| < 1\}.$$

Letting $n = 2, 4, 6, \dots \rightarrow \infty$ in (16) leads to that $\sum_{i \in I'_1} c_i v_i$ is a contrast which contradicts indecomposability of $\sum_{i \in I_1} c_i v_i$ unless $I''_1 = \emptyset$.

Let now $I_1^+ = \{i \in I_1: g(i) = +1\}$, $I_1^- = \{i \in I_1: g(i) = -1\}$. Since

$$\sum_{i \in I_1} g(i) c_i v_i = \sum_{i \in I_1^+} c_i v_i - \sum_{i \in I_1^-} c_i v_i,$$

we have $\sum_{i \in I_1^+} c_i v_i = \sum_{i \in I_1^-} c_i v_i$ and thus $\sum_{i \in I_1^+} c_i v_i$ and $\sum_{i \in I_1^-} c_i v_i$ are contrasts again contradicting indecomposability of the contrast $\sum_{i \in I_1} c_i v_i$ unless one of the sets I_1^+ , I_1^- is empty. Since $c_1 c_2 \neq 0$, we have $g(1) = g(2)$. Lemma 4 is proved.

3. Main result. Return to the statistical setup when an observation X takes values $1, \dots, N$ with probabilities (2).

Theorem 1. An estimable parameter function (3) has a UMVUE if and only if

$$c_i = c_j^*, \quad i \in I_j, \quad j = 1, \dots, k, \quad (17)$$

with the sets $I_1, \dots, I_k$ defined in (10).

A statistic $g(X)$ is a UMVUE if and only if

$$g(x) = g_j^*, \quad x \in I_j, \quad j = 1, \dots, k. \quad (18)$$

P r o o f. Let $\chi(X)$ be an unbiased estimator of zero,

$$\mathbf{E}_{\theta}\chi(X) = \sum_1^N \chi(i) p_i(\theta) = 0, \quad \theta \in \Theta. \quad (19)$$

On looking at $p_j(\theta)$ as an element v_j of the linear space L of functions on Θ , the unbiased estimator of zero from (19) becomes a contrast $\sum_1^N c_i v_i$ with $c_i = \chi(i)$.

By Lemma 1, a statistic $g(X)$ is a UMVUE if for any unbiased estimator of zero $\chi(X)$,

$$\mathbf{E}_{\theta}\{g(X) \chi(X)\} = \sum_1^N g(i) \chi(i) p_i(\theta) = 0, \quad \theta \in \Theta. \quad (20)$$

In terms of L , (20) means that $g = (g(1), \dots, g(N))^T \in \mathbf{R}^N$ is such a vector that for any contrast $\sum_1^N c_i v_i$, $\sum_1^N h_i v_i$ with $h_i = g(i) c_i$, $i = 1, \dots, N$, is also a contrast.

By Lemma 4, such a g must be constant on each I_j thus proving (18).

Conversely, suppose that (18) holds. By virtue of Corollary 1, $\sum_{i \in I_j} c_i v_i$ is a contrast for $j = 1, \dots, k$, i.e.,

$$\sum_{i \in I_j} \chi(i) p_i(\theta) = 0, \quad \theta \in \Theta, \quad j = 1, \dots, k,$$

and thus

$$\mathbf{E}_{\theta}\{g(X) \chi(X)\} = \sum_1^N g(i) \chi(i) p_i(\theta) = \sum_{j=1}^k g_j^* \left[\sum_{i \in I_j} \chi(i) p_i(\theta) \right] = 0, \quad \theta \in \Theta.$$

This proves the second claim of Theorem 1.

For $g(X)$ satisfying (18),

$$\mathbf{E}_{\theta} g(X) = \gamma(\theta) = \sum_{j=1}^k g_j^* \left[\sum_{i \in I_j} p_i(\theta) \right]$$

proving (17). Theorem 1 is proved.

Corollary 2. *If for some $i \in I$,*

$$p_i(\theta) = p_i > 0$$

does not depend on θ , any UMVUE is a constant.

P r o o f. Since $p_1(\theta) + \dots + p_N(\theta) = 1 = \lambda p_i$ for some $\lambda > 0$, partition (6) is trivial, $V = V_1$, and so is partition (8), $I = I_1$.

E x a m p l e 5. A pair of coins, one fair, the other with probability θ of falling head as a parameter is tossed independently n times. In the i -th tossing, only the number of heads X_i is recorded (but not which coin falls what). An observation X is the vector $(X_1, \dots, X_n)$. There is no UMVUE of θ from X (neither of any other nonconstant parameter function) since

$$\mathbf{P}_{\theta}\{X = (1, \dots, 1)\} = 2^{-n}$$

does not depend on θ .

Note in passing that Corollary 2 holds in a general setup (not necessarily of categorical data). Namely, let $\mathcal{P} = \{\mathbf{P}_{\theta}, \theta \in \Theta\}$ be a dominated family of distributions of a random element X taking values in a measurable space $(\mathcal{X}, \mathcal{A})$ with $d\mathbf{P}_{\theta}/d\mu = p(x; \theta)$ for some measure μ .

If $p(x; \theta) = p(x) > 0$, $x \in A$ and $\mu(A) > 0$, then any UMVUE from X is a constant.

Let us look at Examples 1, 2 from Section 1.

In Example 1 the vectors $v_1 = \theta$, $v_2 = \theta^2$, $v_3 = 1 - (\theta + \theta^2)$, $\theta \in (0, \frac{1}{2})$, are linearly independent and (6) holds with $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$. Correspondingly, $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$ and any statistic is a UMVUE.

In Example 2 the vectors are $v_1 = \theta$, $v_2 = \theta^2$, $v_3 = \theta + \theta^2$, and $v_4 = 1 - 2(\theta + \theta^2)$, $\theta \in (0, \frac{1}{4})$, so that v_1, v_2, v_3 are linearly dependent (but any two of them are linearly

independent) and v_4 is linearly independent of $\text{span}(v_1, v_2, v_3)$. Partition (6) holds with $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4\}$ and $I_1 = \{1, 2, 3\}$, $I_2 = \{4\}$. A statistic $g(X)$ is a UMVUE if and only if $g(1) = g(2) = g(3)$ and only the quadratic polynomials $a + b(\theta + \theta^2)$ possess UMVUEs.

Example 6. Let X be a binomial random variable $\text{Bin}(n, \theta)$,

$$\mathbf{P}_\theta\{X = j\} = p_j(\theta) = \binom{n}{j} \theta^j (1 - \theta)^{n-j}, \quad j = 0, 1, \dots, n.$$

As is well known, $v_0 = p_0(\theta)$, $v_1 = p_1(\theta), \dots, v_n = p_n(\theta)$ are linearly independent so that (6) holds with

$$V_1 = \{v_0\}, \dots, V_{n+1} = \{v_n\}.$$

Any statistic $g(X)$ is a UMVUE and every polynomial of degree $\leq n$ in θ possesses a UMVUE.

Suppose now that X is additively perturbed by an independent binary random variable Y with a known

$$\mathbf{P}\{Y = 1\} = 1 - \mathbf{P}\{Y = 0\} = \varepsilon$$

and what is observed is

$$\tilde{X} = X + Y.$$

The distribution of $\tilde{X}$, $\mathbf{P}_\theta\{\tilde{X} = j\} = \tilde{p}_j(\theta)$, $j = 0, 1, \dots, n+1$, is

$$\tilde{p}_0(\theta) = (1 - \varepsilon)p_0(\theta), \quad \tilde{p}_1(\theta) = \varepsilon p_0(\theta) + (1 - \varepsilon)p_1(\theta), \quad \dots, \quad \tilde{p}_{n+1}(\theta) = \varepsilon p_n(\theta)$$

and the vectors $\tilde{v}_j$, $j = 0, 1, \dots, n+1$, are linearly dependent ($n+2$ polynomials of degree $\leq n$) so that partition (6) of $\tilde{V} = \{\tilde{v}_0, \dots, \tilde{v}_{n+1}\}$ is trivial,

$$\tilde{V} = \tilde{V}_1,$$

and only constants are UMVUEs from $\tilde{X}$.

The last example is an illustration of a general principle that, as a rule, the optimal procedures are not robust with respect to perturbations of the model.

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Поступила в редакцию
16.III.2005