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ENTROPY NUMBERS OF SOME ERGODIC AVERAGES

Замечательная оценка чисел покрытия для средних сжатия в гильбертовом пространстве H , полученная недавно Талаграном, обобщается на скользящие средние сжатий. За счет введения второй регуляризации в спектральной регуляризации Талаграна получены нежесткие условия на спектральную меру, связанную с каждым $x \in H$, позволяющие оценить число гильбертовых шаров радиуса $0 < \varepsilon \leq 1$, достаточное для покрытия подмножества в H , определяемого как $\{B_n(x) = \frac{1}{n} \sum_{j=n^2}^{n^2+n-1} U^j x, n \in \mathcal{N}\}$, где U — сжатие в H , а $\mathcal{N}$ — геометрическая прогрессия. Показано, что эти условия на спектральную меру гарантируют существование модуля непрерывности для $\{T^{-1} \int_0^T f \circ U_t dt, T \geq 1\}$, где f есть сжатие в $L^2(\mu)$, а $\{U_t, t \in \mathbf{R}\}$ — поток, сохраняющий меру μ . Приведена оценка чисел покрытия в негильбертовом случае.

Ключевые слова и фразы: спектральная лемма, эргодические средние, энтропийное число.

1. Introduction — Results

In a recent work related to the study of the regularity of Gaussian random processes indexed by product sets, M. Talagrand proved a remarkable estimate of the entropy numbers associated to the sequence of averages of Hilbertian contractions, allowing him to settle affirmatively a question raised in [7] (that question has been also solved however in [6], by means of an entirely different approach: the so-called Stein's randomization technic). More precisely,

Theorem 1.1 ([5]). *Let $U: H \rightarrow H$ be a contraction of an Hilbert space $(H, \|\cdot\|)$. Put for any $f \in H$ and $n \geq 1$,*

$$A_n(f) = \frac{1}{n} \sum_{j=0}^{n-1} U^j(f). \quad (1.1)$$

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Then there exists a universal constant $K > 0$, such that

$$\forall f \in H \text{ with } \|f\| \leq 1, \forall 0 < \varepsilon \leq 1, \quad N(\{A_n(f), n \geq 1\}, \varepsilon) \leq \frac{K}{\varepsilon^2}, \quad (1.2)$$

where $N(\{A_n(f), n \geq 1\}, \varepsilon)$ denotes the cardinality of the minimal covering number of $\{A_n(f), n \geq 1\}$ by Hilbertian balls of radius ε centered in $\{A_n(f), n \geq 1\}$.

This result is optimal in general.

However, we will prove that estimate (1.2) can be improved under mild spectral type assumptions on f :

Proposition 1.1. *Let $U: H \rightarrow H$ be a contraction of an Hilbert space $(H, \|\cdot\|)$. Let us denote by μ_f the spectral measure of f relatively to U . Let also $\beta \geq 0$. Assume that f and β satisfy*

$$\mu_f(\{0\}) = 0 \quad \text{and} \quad K(f, \beta) = \int_{-1/2}^{1/2} \left(\log \frac{1}{|\theta|} \right)^\beta \mu_f(d\theta) < \infty. \quad (1.3)$$

Then (with notation of Theorem 1.1) there exists a constant K depending on f and β only, such that

$$\forall f \in H \text{ with } \|f\| \leq 1, \forall 0 < \varepsilon \leq 1, \quad N(\{A_n(f), n \geq 1\}, \varepsilon) \leq \frac{K}{\varepsilon^2 (\log 1/\varepsilon)^\beta}. \quad (1.4)$$

In this work, we also show that estimate (1.2) can be extended to a wider class of averages. To this effect, introduce the following class $\mathcal{B}$ of moving averages associated to U and defined by:

$$\forall f \in L^2(\mu), \forall n \geq 1, \quad B_n(f) = \frac{1}{n} \sum_{j=n^2}^{n^2+n-1} U^j(f). \quad (1.5)$$

Recall that though these averages are L^2 -converging, they do not converge almost everywhere (see [1]). The same observation can be made about the classical averages A_n defined in Theorem 1.1.

We prove

Theorem 1.2. *Let $U: H \rightarrow H$ be a contraction. For any $f \in H$, let μ_f be the spectral measure of f relatively to U . Assume that f satisfies (1.3) for some real $\beta \geq 1$. Then, for any partial index $\mathcal{N}$ such that*

$$c_{\mathcal{N}} = \sup_{k \geq 1} \text{Card} \left(\{\mathcal{N} \cap [2^k, 2^{k+1}]\} \right) < \infty,$$

there exists a constant C depending on f , $c_{\mathcal{N}}$ and β only, such that

$$\forall 0 < \varepsilon \leq 1, \quad N(\{B_n(f), n \in \mathcal{N}\}, \varepsilon) \leq \frac{C}{\varepsilon^2 (\log(1/\varepsilon))^{\beta-1}},$$

where $N(\{B_n(f), n \in \mathcal{N}\}, \varepsilon)$ is the minimal number of Hilbertian balls of radius ε , centered in $\{B_n(f), n \in \mathcal{N}\}$ and enough to cover $\{B_n(f), n \in \mathcal{N}\}$.

Remark 1. a) The proof of Theorem 1.2 we purpose, will also enable us to show (see Remark 2 in Section 2), under condition (1.3), the same result for slightly more general moving averages:

$$P_n = \frac{1}{n} \sum_{j=p(n)}^{p(n)+n-1} T^j, \quad n \geq 1,$$

where p is a polynomial.

b) We were however able to control the entropy numbers of whole sequence $\{B_n(f), n \geq 1\}$. Some computations we made, let us believe that the associated entropy numbers might be in this case much larger than before, depending on the spectral measure of f .

Assume now that $H = L^2(\mu)$, where $(X, \mathcal{A}, \mu)$ is a probability space. Consider a continuous flow $(U_t)_{t \geq 0}$ of contractions. For any $f \in L^2(\mu)$, there exists a spectral measure μ_f on $\mathbf{R}$ such that, for any value of t

$$\langle U_t f, f \rangle = \int_{-\infty}^{\infty} \exp(2i\pi t x) \mu_f(dx).$$

By means of Proposition 1.1, we obtain in the next theorem, an estimate of the modulus of uniform continuity of the averages

$$\forall T \geq 1, \quad A_T(f) = \frac{1}{T} \int_0^T f \circ U_t dt.$$

Theorem 1.3. *Let $H = L^2(\mu)$, where $(X, \mathcal{A}, \mu)$ is a probability space. Let $U = (U_t)_{t \geq 0}$ be a continuous flow of contractions in H . Let $f \in H$, and assume that condition (1.3) is fulfilled for some real $\beta > 2$. Then*

$$\exists \gamma > 0: \mathbf{E} \sup_{\substack{s, T \geq 1 \\ s \neq T}} \left[\frac{|A_T - A_s|}{(\log \|A_T - A_s\|^{-1})^{-\gamma}} \right]^2 < +\infty. \quad (1.6)$$

We end by indicating an extension to the Wiener space $\mathcal{S}$ of correlated sequences, namely the space consisting in sequences $a = \{a(n), n \in \mathbf{Z}\}$ such that for any integer k , the limit

$$\gamma_a(k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} a(j) a(j+k)$$

exists. We provide $\mathcal{S}$ with the semi-norm

$$\Delta(a) = \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{j=0}^{n-1} a^2(j) \right)^{1/2}$$

Entropy numbers associated to any subset E of $(\mathcal{S}, \Delta)$ are noted $N(E, \Delta, \cdot)$. For any $a = (a_j)_{j \in \mathbf{Z}} \in l_2(\mathbf{Z})$, let us write $\varphi_a(\alpha) = \sum_{j \in \mathbf{Z}} e^{-ij\alpha} a(j)$. Let also T

be the right shift on the space of sequences: $T(b_n, n \in \mathbf{Z}) = (b_{n+1}, n \in \mathbf{Z})$, and note

$$A_N = \frac{I + T + \dots + T^{N-1}}{N}, \quad N = 1, 2, \dots$$

Theorem 1.4. *For any $a \in \mathcal{S}$, there exists a constant $K(a)$ depending on a only, such that*

$$\forall 0 < \varepsilon \leq K(a), \quad N\left(\{A_n^T(a), n \geq 1\}, \Delta, \varepsilon\right) \leq \frac{K(a)}{\varepsilon^2}.$$

Proofs are based on the spectral lemma (see [4, p. 33]).

Lemma 1.1. *Let T be a contraction of an Hilbert space $(H, \|\cdot\|)$. Let $p(x)$ be a polynomial. For any $f \in H$, let μ_f be the spectral measure of f relatively to T . Then*

$$\|p(T)(f)\|^2 \leq \int_{|x|=1} |p(x)|^2 \mu_f(dx).$$

The following theorem due to Pisier [3] will be used in the proof of Theorem 1.3, and is recalled for the convenience of the reader.

Theorem 1.5. *Let (E, d) be a compact space provided with a continuous pseudo-metric d . Assume that*

$$\int_0^1 N_d(E, \varepsilon)^{1/2} d\varepsilon < \infty,$$

where $N_d(E, \varepsilon)$ is the minimal covering of E by d -open balls of radius ε . Then any stochastic process $X = \{X_t, t \in E\}$ such that:

$$\forall s, t \in E, \quad \|X_s - X_t\|_2 \leq d(s, t)$$

possesses a modification with continuous sample paths. Further,

$$\left[\mathbf{E} \sup_{E \times E} |X_t - X_s|^2 \right]^{1/2} \leq K \int_0^{KD} N_d(E, \varepsilon)^{1/2} d\varepsilon, \quad (1.7)$$

where $D = \sup_{E \times E} d(s, t)$ and K is a numerical constant. Besides, for any continuous increasing function $\phi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$, with $\phi(0) = 0$, such that

$$\int_0^D \frac{N_d(E, \varepsilon)^{1/2}}{\phi(\varepsilon)} d\varepsilon < \infty, \quad (1.8)$$

we have

$$\left(\mathbf{E} \sup_{T \times T} \left| \frac{X_t - X_s}{\phi(d(t, s))} \right|^2 \right)^{1/2} \leq K \int_0^D \frac{N_d(E, \varepsilon)^{1/2}}{\phi(\varepsilon)} d\varepsilon, \quad (1.9)$$

where K is a numerical constant.

2. Proofs

2.1. Proof of Theorem 1.2. Without loss of generality, we can assume $\mu_f(\{0\}) = 0$.

Let indeed $\mathcal{E}_0$ be the space of fixed points of U , then $U(\mathcal{E}_0^\perp) \subset \mathcal{E}_0^\perp$ and $H = \mathcal{E}_0 \oplus \mathcal{E}_0^\perp$. Thus, every $f \in H$ can be written as a sum $f = f_0 + f_1$, where f_0 is the orthogonal projection of f on $\mathcal{E}_0$ and f_1 belongs to $\mathcal{E}_0^\perp$. And so $\mu_f = \mu_{f_0} + \mu_{f_1}$ since for any $k \geq 0$, $(U^k f | f) = (U^k f_0 | f_0) + (U^k f_1 | f_1)$.

As $(N^{-1} \sum_{n=0}^{N-1} U^n(f_1))_{N \geq 1}$ converges to 0 in H , we thus deduce that

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n(f_1) \right\|^2 = \mu_{f_1}(\{0\}) = 0,$$

so that

$$\lim_{A \rightarrow +\infty} \left(\int_{-1/2}^{-1/A} + \int_{1/A}^{1/2} \right) \left(\log \frac{1}{2|\theta|} \right)^\beta \mu_f(d\theta) = \int_{-1/2}^{1/2} \left(\log \frac{1}{2|\theta|} \right)^\beta \mu_f(d\theta).$$

Entropy numbers associated to f and f_1 are identical since averages $A_N(f_1)$ (or $B_N(f_2)$) are obtained by translation of $-f_0$ from averages $A_N(f)$ (or $B_N(f)$).

Assumption $c_{\mathcal{N}} < \infty$ implies that $\mathcal{N}$ can be written as a union of at most $c_{\mathcal{N}}$ partial index $\mathcal{N}_*$ such that $c_{\mathcal{N}_*} = 1$. Thus, it suffices to prove Theorem 1.2 for such an index. Let $\mathcal{N}_* = \{n(j), j \geq 1\}$ be such an index. By considering firstly $\mathcal{N}' = \{n(j): j \text{ is even}\}$, next $\mathcal{N}'' = \{n(j): j \text{ is odd}\}$, we can restrict ourselves for the same reasons to the case, where $\mathcal{N}$ verifies:

$$\text{if } k(j) \text{ denotes the integer } k \text{ such that } n(j) \in [2^{k(j)}, 2^{k(j)+1}[,$$

then

$$\forall j \geq 1, \quad k(j) + 1 < k(j+1). \quad (2.1)$$

Having operated these reductions, now fix an element f of H such that condition (1.3) is realized. Without loss of generality, we can and do assume that $\|f\| = 1$. Define the spectral kernels W_n by

$$\forall n \in \mathcal{N}^*, \quad \forall \theta \in \left] -\frac{1}{2}, \frac{1}{2} \right], \quad W_n(\theta) = \frac{1}{n} \sum_{j=n^2}^{n^2+n-1} e^{2i\pi j\theta}. \quad (2.2)$$

Knowing that $B_n = n^{-1} \sum_{j=n^2}^{n^2+n-1} U^j(f)$, we have from the spectral lemma:

$$\|B_n(f) - B_m(f)\|^2 \leq \int_{-1/2}^{1/2} |W_n(\theta) - W_m(\theta)|^2 \mu_f(d\theta), \quad (2.3)$$

for any $n, m \geq 1$.

Define for any $l \geq 1$:

$$J_l = \left\{ \theta \in \left] -\frac{1}{2}, \frac{1}{2} \right] : 2^{-l} < |\theta| \leq 2^{-l+1} \right\}, \quad a_l = \mu_f(J_l). \quad (2.4)$$

Introduce now the regularized sequence $(b_l)_{l \geq 1}$ of $(a_l)_{l \geq 1}$, according to [5]:

$$\forall l \geq 1, b_l = \sum_{k=1}^{\infty} 2^{-|k-l|} a_k. \quad (2.5)$$

The building of the sequence $(b_l)_{l \geq 1}$ does not appeal to any particular assumption on the spectral measure μ_f . Observe however that if $b_l = 0$ for some index $l \geq 1$, then all the a_k , $k \geq 1$, are null. It follows that $\mu_f([\frac{1}{2^l}, \frac{1}{2^{l-1}}]) = 1 = \|f\|^2$ for any $l \geq 1$. Thus $\mu_f = \delta_{\{0\}}$ (the Dirac measure of mass 1 at point $\{0\}$). This situation appears, if and only if f is a fix point of U . It is indeed clear that $\mu_f = \delta_{\{0\}}$ if $U(f) = f$ (and of course $\|f\| = 1$). Conversely, if $\mu_f = \delta_{\{0\}}$, set $Uf = \lambda f + f'$ with f' orthogonal to f . Then

$$\langle Uf, f \rangle = \int_{-1/2}^{1/2} e^{2i\pi\theta} \mu_f(d\theta) = 1 = \lambda(f | f) = \lambda,$$

hence $\lambda = 1$ and $f' = 0$.

We keep away the case $Uf = f$ which corresponds to a trivial situation.

We collect in the next lemma the essential properties of that regularization.

Lemma 2.1.

$$\sum_{l \geq 1} a_l = 1, \quad \sum_{l \geq 1} b_l \leq 3, \quad \forall l \geq 1, \quad 0 \leq a_l \leq b_l \leq 1, \quad (2.6)$$

$$\forall l \geq 1, \quad b_l > 0 \quad \text{and} \quad \frac{1}{2} \leq \frac{b_{l+1}}{b_l} \leq 2, \quad (2.7)$$

$$\forall l \geq 1, \quad b_{l+1} 2^{2(l+1)} \geq 2b_l 2^{2l}. \quad (2.8)$$

From (2.7) and (2.8) follows that the sequence $(b_l 2^{2l})_{l \geq 1}$ is strictly increasing and unbounded.

Regularize a second time the sequence $(a_l)_{l \geq 1}$ by letting

$$\forall l \geq 1, \quad c_l = \sum_{l \leq j \leq 2l} b_j.$$

Define $g: \mathbf{N} \setminus \{0, 1\} \mapsto \mathbf{R}^+$ as follows. For any $n \geq 2$, let $k \geq 1$ be the integer such that $2^k \leq n < 2^{k+1}$. Put

$$g(n) = (2^{-k}n - 1)c_k + \sum_{l=1}^{k-1} c_l.$$

Observe that $g(2^k) = \sum_{l < k} c_l$; then g increases by constant jumps equal to $2^{-k}c_k$ until the value $g(2^{k+1} - 1) = \sum_{l \leq k} c_l - 2^{-k}c_k$. Thus

g is strictly increasing and

$$\forall k \geq 1, \quad \forall n \in [2^k + 1, 2^{k+1}], \quad g(n) - g(n-1) = 2^{-k}c_k.$$

We will see that g is bounded, under condition (1.3). Fix $0 < \varepsilon \leq 1$. For proving Theorem 1.2, it is thus enough to exhibit a subset F of $\mathcal{N}$ such that

$$\text{Card}(F) \leq \frac{C}{\varepsilon^2 (\log(1/\varepsilon))^{\beta-1}}, \quad \text{where } \beta \geq 1, \quad (2.9)$$

$$\forall n \in \mathcal{N}, \exists n' \in F: \|B_n(f) - B_{n'}(f)\| \leq C_1 \varepsilon, \quad (2.10)$$

where C depends on f , $c_{\mathcal{N}}$ and condition (1.3), and C_1 is a universal constant. Build F as follows: for each $\varepsilon \in [0, 1[$, put

$$k_\varepsilon = \left\lceil \frac{1}{\log 2} \log \frac{2}{\varepsilon} \right\rceil.$$

We set $[1, 2^{k_\varepsilon+1}] \cap \mathcal{N}$ in F . Next, for any element n of $\mathcal{N}$, $n > 2^{k_\varepsilon+1}$, we put n in F whenever for an integer $p \geq 1$,

$$g(n) \geq p\varepsilon^2 \quad \text{and for any } m < n, m \in \mathcal{N}, \quad g(m) < p\varepsilon^2.$$

We first show (2.9). Write $n_\varepsilon = 2^{k_\varepsilon+1}$. The contribution of $[1, 2^{k_\varepsilon+1}] \cap \mathcal{N}$ in F intervenes for at most $k_\varepsilon + 1 \leq 4/\varepsilon$. Examine the contribution of $]2^{k_\varepsilon+1}, \infty[\cap \mathcal{N}$. The sequence $(g(n))_{n \geq 2}$ being strictly increasing, to each integer $p \geq 1$ such that $g(n_\varepsilon) < p\varepsilon^2 \leq \|g\|_\infty$ there corresponds at most one integer n in F such that $g(n-1) < p\varepsilon^2 \leq g(n)$. The number of elements of $F \cap]2^{k_\varepsilon+1}, \infty[\cap \mathcal{N}$ is thus bounded from above by L/ε^2 , where L is the length of the interval $[g(n_\varepsilon), \|g\|_\infty]$. The definition of g thus provides the bound

$$L = \sum_{l \geq 1} c_l - \sum_{l \leq k_\varepsilon} c_l \leq \sum_{l \geq k_\varepsilon} c_l.$$

Estimate $\sum_{l \geq k_\varepsilon} c_l$. We have

$$\begin{aligned} \sum_{l \geq k_\varepsilon} c_l &= \sum_{l \geq k_\varepsilon} \sum_{l \leq j \leq 2l} b_j \leq \sum_{j \geq k_\varepsilon} j b_j \leq \frac{1}{k_\varepsilon^{\beta-1}} \sum_{j \geq k_\varepsilon} j^\beta b_j \\ &\leq \frac{C_0}{(\log(1/\varepsilon))^{\beta-1}} \sum_{j \geq 1} j^\beta b_j, \end{aligned} \quad (2.11)$$

where $C_0 = C_0(\beta)$ is a constant depending on β , $C_0(\beta) = (\log 4)^{\beta-1}$. Estimate now $S = \sum_{j \geq 1} j^\beta b_j$. We have

$$S = \sum_{j=1}^{\infty} j^\beta \sum_{k=1}^{\infty} 2^{-|k-j|} a_k = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &= \sum_{j=1}^{\infty} j^\beta \sum_{k=1}^j 2^{-j+k} a_k = \sum_{k=1}^{\infty} 2^k a_k \left(\sum_{j=k}^{\infty} 2^{-j} j^\beta \right), \\ S_2 &= \sum_{j=1}^{\infty} j^\beta \sum_{k=j+1}^{\infty} 2^{-k+j} a_k. \end{aligned}$$

Consider first S_1 . Observe that the function $\phi(x) = x^\beta 2^{-x}$, $x > 0$, is monotonic and nonincreasing on the interval $[\sqrt{\beta/\log 2}, \infty[$. Moreover,

$$x \geq \frac{2\beta}{\log 2} \implies -\phi'(x) \geq \frac{\log 2}{2} \phi(x).$$

Thus, there exists a real k_1 depending on β only such that $S_1 \leq S'_1 + S''_1$, with

$$S'_1 = \sum_{k=1}^{k_1} a_k 2^k \sum_{j \geq k} 2^{-j} j^\beta \quad \text{and} \quad S''_1 = \sum_{k=k_1+1}^{\infty} a_k 2^k \int_{k-1}^{\infty} \phi(x) dx.$$

On the one hand,

$$S'_1 = \left(\sum_{k=1}^{k_1} a_k 2^k \right) \left(\sum_{j=1}^{\infty} 2^{-j} j^\beta \right) \leq 2^{k_1+1} \sum_{j=1}^{\infty} 2^{-j} j^\beta = C_1(\beta).$$

The sum S'_1 is thus bounded from above by an expression which is independent of the sequence $(a_k)_{k \geq 1}$, and thus of μ_f . On the other hand,

$$\begin{aligned} S''_1 &\leq \frac{2}{\log 2} \sum_{k=k_1+1}^{\infty} a_k 2^k \int_{k-1}^{\infty} -\phi'(x) dx \\ &= \frac{2}{\log 2} \sum_{k=k_1+1}^{\infty} a_k 2^k (k-1)^\beta 2^{-k+1} \leq \frac{4}{\log 2} \sum_{k=1}^{\infty} a_k k^\beta. \end{aligned}$$

This shows that $S_1 \leq C_2(\beta) \sum_{k=1}^{\infty} a_k k^\beta$, where $C_2(\beta) = C_1(\beta) + 4/\log 2$. Consider S_2 :

$$S_2 = \sum_{j=1}^{\infty} j^\beta \sum_{k=j+1}^{\infty} 2^{-k+j} a_k = \sum_{k=2}^{\infty} a_k \left(\sum_{j \leq k-1} j^\beta 2^{j-k} \right) \leq \sum_{k=2}^{\infty} a_k k^\beta.$$

Thus, $S \leq C_3 \sum_{k=1}^{\infty} a_k k^\beta$, with $C_3 = C_3(\beta) = 1 + C_2(\beta)$. Since $a_k = \mu_f\{\theta: 2^{-k-1} < |\theta| \leq 2^{-k}\}$, we deduce

$$a_k k^\beta = \int_{-1/2}^{1/2} 1_{J_k} k^\beta \mu_f(d\theta) \leq \int 1_{J_k} (-\log 2|\theta|)^\beta \mu_f(d\theta),$$

hence

$$S \leq C_3 \int_{-1/2}^{1/2} \left(\log \frac{1}{2|\theta|} \right)^\beta \mu_f(d\theta), \quad (2.12)$$

and since, by (2.11),

$$L \leq \frac{C_0(\beta)S}{(\log(1/\varepsilon))^{\beta-1}},$$

we have

$$\text{Card}(F) \leq \frac{C}{\varepsilon^2 (\log(1/\varepsilon))^{\beta-1}},$$

where $C = C_0(\beta)S$ depends on β and μ_f only. It remains to prove (2.10). The integer $n \in \mathcal{N}$, $n \geq 2$, being fixed, determines n' as follows: let $p \geq 1$ be the greatest integer such that $g(n) \geq p\varepsilon^2$. Let then n' be the smallest integer such that $n' \in \mathcal{N}$ and $g(n') \geq p\varepsilon^2$. Let k and k' be defined by $n \in [2^k, 2^{k+1}[$ and $n' \in [2^{k'}, 2^{k'+1}[$. If $n = n'$, then estimate (2.10) is fulfilled.

If $n \neq n'$, then by (2.1), we have $k' + 1 < k$. For estimating $\|B_n(f) - B_{n'}(f)\|$, we will use the following lemmas.

Lemma 2.2. *For any integers a, b such that $a < b$ and any $\theta \in]-\frac{1}{2}, \frac{1}{2}]$, we have*

$$\left| \frac{1}{b-a} \sum_{a \leq n < b} e^{2i\pi n\theta} \right| \leq \frac{1}{\pi n|\theta|} \wedge 1. \quad (2.13)$$

As concerns the sums W_n , there exists a constant K_0 such that

$$|W_n(\theta) - W_m(\theta)| \leq K_0 |\theta| |n^2 - m^2|, \quad (2.14)$$

for every $\theta \in]-\frac{1}{2}, \frac{1}{2}]$ and every $n, m \geq 1$.

P r o o f. The estimate (2.13) follows from the classical minoration

$$|e^{i\theta} - 1| \geq K|\theta|, \quad \theta \in \left] -\frac{1}{2}, \frac{1}{2} \right]. \quad (2.15)$$

We show (2.14). Put $V_n(\theta) = n^{-1} \sum_{j=0}^{n-1} e^{2i\pi j\theta}$ for every $n \geq 1$, every $\theta \in]-\frac{1}{2}, \frac{1}{2}]$, and write

$$\phi(x) = \frac{1}{x} (e^{2i\pi x\theta} - 1), \quad x > 0.$$

Then, by (2.15), for every $\theta \in]-\frac{1}{2}, \frac{1}{2}]$, $\theta \neq 0$, and every $m, n \geq 1$,

$$\begin{aligned} |V_n(\theta) - V_m(\theta)| &= \frac{|\phi(n) - \phi(m)|}{|e^{i\theta} - 1|} \leq |n - m| \frac{\sup_{n \wedge m \leq x \leq n \vee m} |\phi'(x)|}{|e^{i\theta} - 1|} \\ &\leq \frac{\pi |n - m|}{2|\theta|} \sup_{n \wedge m \leq x \leq n \vee m} |\phi'(x)|. \end{aligned} \quad (2.16)$$

But $\phi'(x) = -(1 - 2i\pi x\theta) e^{2i\pi x\theta} - 1/x^2$, for every $x > 0$ and $\theta \in]-\frac{1}{2}, \frac{1}{2}]$. We easily verify (by applying Schwartz's lemma for instance) that for every $z \in \mathbb{C}$ with $|z| \leq 1$,

$$|(1 - z)e^z - 1| \leq K|z|^2, \quad (2.17)$$

where K is a numerical constant ($K = \sup_{|z| \leq 1} |(1 - z)e^z - 1|$). By combining (2.16) with (2.17), we obtain for every $n, m \geq 1$ and $|\theta| \leq (2\pi n \vee m)^{-1}$, $|V_n(\theta) - V_m(\theta)| \leq K'|\theta| |n - m|$, where $K' = K\pi/2$. We have for $n > m$ and $\theta \in]-\pi, \pi]$,

$$\begin{aligned} |V_n(\theta) - V_m(\theta)| &= \left| \left(\frac{1}{n} - \frac{1}{m} \right) \sum_{j=0}^{m-1} e^{2i\pi j\theta} + \frac{1}{n} \sum_{j=m}^{n-1} e^{2i\pi j\theta} \right| \\ &\leq \left| \frac{1}{n} - \frac{1}{m} \right| m + \frac{1}{n} |n - m| = \frac{2|n - m|}{n}, \end{aligned}$$

and thus for every $n, m \geq 1$, and $\theta \in] - \frac{1}{2}, \frac{1}{2}]$,

$$|V_n(\theta) - V_m(\theta)| \leq 2K' \left(|\theta| \wedge \frac{1}{n \vee m} \right) |n - m|. \quad (2.18)$$

Observe now that for every $n, m \geq 1$, $\theta \in] - \frac{1}{2}, \frac{1}{2}]$,

$$W_n(\theta) - W_m(\theta) = (V_n(\theta) - V_m(\theta)) e^{2i\pi n^2 \theta} + V_m(\theta) (e^{2i\pi n^2 \theta} - e^{2i\pi m^2 \theta}).$$

Thus,

$$|W_n(\theta) - W_m(\theta)| \leq |V_n(\theta) - V_m(\theta)| + |e^{2i\pi n^2 \theta} - e^{2i\pi m^2 \theta}|.$$

Further

$$|e^{2i\pi n^2 \theta} - e^{2i\pi m^2 \theta}|^2 = 2 \left| \sin \pi \theta (m^2 - n^2) \frac{\theta}{2} \right| \leq (1 \wedge \pi \theta |m^2 - n^2|). \quad (2.19)$$

Thus, by combining (2.18) with (2.19), we easily obtain (2.14). Lemma 2.2 is proved.

The next lemma will be the key tool for proving (2.10). In what follows, we put for any integer $k \geq 1$

$$I_k = \bigcup_{l \leq k} J_l \quad \text{and} \quad \hat{I}_k = \bigcup_{l > k} J_l.$$

Thus $I_k =] - \frac{1}{2}, -\frac{1}{2^{k+1}}[\cup] \frac{1}{2^{k+1}}, \frac{1}{2}]$.

Lemma 2.3. For any integers n and $k \geq 1$ satisfying $n \geq 2^k$, we have

$$\int_{I_k} |W_n(\theta)|^2 \mu_f(d\theta) \leq \frac{8}{\pi^2} b_k. \quad (2.20a)$$

Assume moreover that $n \in \mathcal{N}$ and let n' be the integer which is associated to it. (As before, n' is the smallest integer of $\mathcal{N}$ such that $g(n') \geq [g(n)/\varepsilon^2]$.) Then

$$\int_{I_k} |W_n(\theta)|^2 \mu_f(d\theta) \leq \frac{8}{\pi^2} \varepsilon^2, \quad (2.20b)$$

and if k' is the integer defined by $2^{k'+1} > n' \geq 2^{k'}$, then

$$\int_{I_{k'}} |W_{n'}(\theta)|^2 \mu_f(d\theta) \leq \frac{16}{\pi^2} \varepsilon^2. \quad (2.20c)$$

Proof. Observe that for $l \leq k$ and $\theta \in J_l$, we have $(\pi n \theta)^{-1} \leq 2/\pi < 1$. Estimate (2.13) implies

$$\begin{aligned} \int_{I_k} |W_n(\theta)|^2 \mu_f(d\theta) &\leq \frac{1}{\pi^2 n^2} \sum_{l \leq k} \int_{J_l} \frac{1}{|\theta|^2} \mu_f(d\theta) \leq \frac{4}{\pi^2 n^2} \sum_{l \leq k} 2^{2l} a_l \\ &\leq \frac{4}{\pi^2 n^2} \sum_{l \leq k} 2^{2l} b_l \leq \frac{8}{\pi^2 n^2} b_k 2^{2k}, \end{aligned}$$

since by (2.8), we have $\sum_{l \leq k} 2^{2l} b_l \leq 2 \cdot 2^{2k} b_k$. By taking into account the fact that $n \geq 2^k$, we obtain (2.20a).

Observe now, by construction of n'

$$\varepsilon^2 \geq g(n) - g(n') = (2^{-k}n - 1) c_k + \sum_{k' < l < k} c_l + (2 - 2^{-k'}n') c_{k'} \geq \sum_{k' < l < k} c_l,$$

the latter summation being nonempty since $k' + 1 < k$. Hence,

$$\sum_{k' < l < k} \sum_{l \leq j \leq 2l} b_j \leq \varepsilon^2, \quad (2.21)$$

in particular $b_k \leq \varepsilon^2$. Since $I_l \subset I_k$ if $l \leq k$, inequality (2.20b) follows. Besides, by estimate (2.21), we also have $b_{k'+1} \leq \varepsilon^2$ and (2.7) provides $b_{k'} \leq 2b_{k'+1}$, hence inequality (2.20c) by using (2.20a).

Lemma 2.4. For every $k \geq 1$ and every $n, m \geq 1$,

$$\int_{\hat{I}_k} |W_n(\theta) - W_m(\theta)|^2 \mu_f(d\theta) \leq K_1 |n^2 - m^2|^2 b_k 2^{-2k},$$

where K_1 is a universal constant.

Proof. By means of estimate (2.14), we have:

$$\begin{aligned} \int_{\hat{I}_k} |W_n(\theta) - W_m(\theta)|^2 \mu_f(d\theta) &\leq K_0 |n^2 - m^2|^2 \sum_{l > k} \int_{J_l} |\theta^2| \mu_f(d\theta) \\ &\leq K_0 |n^2 - m^2|^2 \sum_{l > k} 2 \cdot 2^{-2l} a_l \leq K_0 |n^2 - m^2|^2 \sum_{l > k} 2 \cdot 2^{-2l} b_l. \end{aligned}$$

As in the above proof, relation (2.8) implies $\sum_{l > k} 2^{-2l} b_l < b_k 2^{-2k}$. Hence

$$\int_{\hat{I}_k} |W_n(\theta) - W_m(\theta)|^2 \mu_f(d\theta) \leq 2K_0^2 |n^2 - m^2|^2 b_k 2^{-2k},$$

which completes the proof of Lemma 2.4.

Estimate now $\|B_n(f) - B_{n'}(f)\|^2$. Recall that $n \in \mathcal{N}$, $n' \in F$, $n \neq n'$, where $2^{k+1} > n \geq 2^k$ and $2^{k'+1} > n' \geq 2^{k'}$. We have

$$\begin{aligned} \|B_n(f) - B_{n'}(f)\|^2 &\leq \int_{-1/2}^{1/2} |W_n(\theta) - W_{n'}(\theta)|^2 \mu_f(d\theta) \\ &= \left(\int_{I_{k'}} + \int_{I_{2k} \setminus I_{k'}} + \int_{\hat{I}_{2k}} \right) |W_n(\theta) - W_{n'}(\theta)|^2 \mu_f(d\theta). \end{aligned}$$

According to Lemma 2.3,

$$\int_{I_{k'}} |W_n(\theta) - W_{n'}(\theta)|^2 \mu_f(d\theta) \leq 2 \left(\frac{8}{\pi^2} + \frac{16}{\pi^2} \right) \varepsilon^2. \quad (2.22)$$

By (2.21), we have

$$\sum_{k'+1 \leq j \leq 2(k-1)} b_j \leq \varepsilon^2. \quad (2.23)$$

Using now (2.7), we deduce from (2.23)

$$\sum_{k'+1 \leq j \leq 2k} b_j \leq \varepsilon^2 + b_{2k-1} + b_{2k} \leq \varepsilon^2 + 4b_{2k-3} + 4b_{2k-2} \leq 5\varepsilon^2. \quad (2.24)$$

Since $|W_n(\theta)| \leq 1$, (2.24) allows one to get

$$\int_{I_{2k} \setminus I_{k'}} |W_n(\theta) - W_{n'}(\theta)|^2 \mu_f(d\theta) \leq 20\varepsilon^2. \quad (2.25)$$

Examine now the last integral. By means of Lemma 2.4, we have

$$\begin{aligned} \int_{\widehat{I}_{2k}} |W_n(\theta) - W_{n'}(\theta)|^2 \mu_f(d\theta) &\leq K_1 |n^2 - (n')^2|^2 b_{2k} 2^{-4k} \leq K_1 (2^{k+1})^4 b_{2k} 2^{-4k} \\ &\leq 16K_1 b_{2k} \leq 64K_1 b_{2k-2} \leq 64K_1 \varepsilon^2 \quad \text{by (2.21)}. \end{aligned} \quad (2.26)$$

Putting together (2.22), (2.25) and (2.26) leads to

$$\|B_n(f) - B_{n'}(f)\|^2 \leq \varepsilon^2 \left(\frac{48}{\pi^2} + 20 + 64K_1 \right),$$

which proves (2.10) (with $C_1 = (48/\pi^2 + 60 + 128K_1^2)$), and consequently completes the proof of Theorem 1.2.

R e m a r k 2. By modifying somewhat the above proof, it is possible to treat slightly more general moving averages

$$P_n = \frac{1}{n} \sum_{j=p(n)}^{p(n)+n-1} U^j, \quad n \geq 1,$$

where p is a polynomial. We give the modification in the case, where $p(x) = x^r$, r is a positive integer for instance. Let

$$C_n = \frac{1}{n} \sum_{j=n^\alpha}^{n^\alpha+n-1} T^j, \quad n \geq 1, \quad \text{where } \alpha \in \mathcal{N} \setminus \{0, 1\}.$$

Consider

$$\forall n \in \mathcal{N}^*, \quad \forall \theta \in]-\pi, \pi], \quad W_{\alpha, n}(\theta) = \frac{1}{n} \sum_{j=n^\alpha}^{n^\alpha+n-1} e^{ij\theta}.$$

Then, we regularize the sequence $(b_j, j \geq 1)$ as follows: we put

$$\forall l \geq 1, \quad \delta_l = \sum_{m(l) \leq j \leq m(\alpha l)} b_j.$$

For any $n \geq 1$, let $k \geq 1$ be the integer such that $2^k \leq n < 2^{k+1}$. Put

$$h(n) = \sum_{l < k} \delta_l + (2^{-k}n - 1) \delta_k.$$

Estimate (2.9) becomes

$$\text{Card}(F) \leq \frac{\alpha C}{\varepsilon^2 (\log(1/\varepsilon))^{\beta-1}}, \quad (2.9')$$

and we prove

$$\forall n \in \mathcal{N}, \quad \exists n' \in F: \|C_n(f) - C_{n'}(f)\| \leq C\varepsilon, \quad (2.10')$$

by modifying (2.14):

$$|W_{\alpha,n}(\theta) - W_{\alpha,m}(\theta)| \leq K|\theta| |n^\alpha - m^\alpha|, \quad (2.14')$$

for any $\theta \in]-\pi, \pi]$ and any $n, m \geq 1$; next by integrating $|W_{\alpha,n}(\theta) - W_{\alpha,m}(\theta)|$ over $I_{k'}$, $I_{\alpha k} \setminus I_{k'}$ and $\hat{I}_{\alpha k}$.

2.2. Proof of Proposition 1.1. We examine the particular cases, where $p(x) \equiv 0$ and $p(x)$ is of degree 1, but this time with the total index $\mathcal{N} = \mathbb{N}$. We show that if $\mu_f(\{0\}) = 0$ and $K(f, \beta) < \infty$ for some $\beta \geq 0$, then for every fixed $\varepsilon \in]0, 1[$

$$N(A_n(f), n \geq 1) \leq \frac{C(\beta)K(f, \beta)}{\varepsilon^2 (\log(1/\varepsilon))^\beta},$$

when $p = 0$ and

$$N(B_n(f), n \geq 1) \leq \frac{C(\beta)K(f, \beta)}{\varepsilon^2 (\log(1/\varepsilon))^\beta},$$

when p is of degree 1. Proofs corresponding to the cases $p(x) \equiv 0$ and $p(x)$ is of degree 1 are identical. Thus, we only give the proof for the case $p(x) \equiv 0$.

If $\beta = 0$, this is Talagrand's result [5]. The case $\beta > 0$ can be treated analogously, but some technical modifications are necessary.

Assume $\|f\| = 1$ and $\mu_f(\{0\}) = 0$. The sequences $(a_l)_{l \geq 1}$ and $(b_l)_{l \geq 1}$ are the same as in the proof of Theorem 1.2.

Fix now $0 < \varepsilon \leq 1$. For any $k \geq 1$, let $m(k)$ be the smallest positive integer such that $b_{m(k)} 2^{2m(k)} \geq 2^{2k} \varepsilon^2$. Let

$$k^* = \sup\{r \mid m(r) = 1\}.$$

It is easy to estimate k^* by $k^* \leq \log_2 2\sqrt{b_1}/\varepsilon$. The next lemma shows that the sequences $\{m(k), k \geq 1\}$ and $\{b_{m(k)}, k \geq 1\}$ are remarkably regular.

Lemma 2.5.

$$\forall k \geq 1, \quad m(k) \leq m(k+1), \quad \forall k \geq k^*, \quad m(k) + 1 \leq m(k+2),$$

$$\forall k \geq 1, \quad m(k+1) \leq m(k) + 2,$$

$$\forall k \geq 1, \quad \frac{1}{4} \leq \frac{b_{m(k+1)}}{b_{m(k)}} \leq 4,$$

$$\forall k \geq k^*, \quad \frac{1}{2} (k - k^*) \leq m(k) - 1 \leq 2 (k - k^*).$$

Now, we build a set $F = F(\varepsilon, f)$ having the following properties: for each $n \in \mathbb{N}$, there exists $n' \in F$ such that $\|A_n(f) - A_{n'}(f)\| \leq C'\varepsilon$, C' is an absolute constant, and

$$\text{Card}(F) \leq \frac{C(\beta, f)}{\varepsilon^2 (\log(2/\varepsilon))^\beta},$$

where $C(\beta, f)$ is a constant depending on β and μ_f only. For this, we follow Talagrand's method and replace the sequence $(g(n))$ by the following:

$$\gamma(n) = \sum_{1 \leq l < k} b_{m(l)} + (2^{-k}n - 1)b_{m(l)}, \quad \forall n \in [2^k, 2^{k+1}[.$$

Put $k_\varepsilon = 1 + [\frac{3}{2} \log_2(2/\varepsilon)]$ (so that $k_\varepsilon - k^* \geq \frac{1}{2} \log_2(2/\varepsilon)$), and notice also that $(2/\varepsilon)^{3/2} \leq 2^{k_\varepsilon} \leq 2(2/\varepsilon)^{3/2}$. Define F as follows:

$$F = \{1, \dots, 2^{k_\varepsilon}\} \cup \{n > 2^{k_\varepsilon} : \exists p \in \mathcal{N}^*, \gamma(n-1) < p\varepsilon^2 \leq \gamma(n)\}.$$

Then

$$\text{Card}(F) \leq 2 \left(\frac{2}{\varepsilon}\right)^{3/2} + \frac{1}{\varepsilon^2} (\|\gamma\|_\infty - \gamma(2^{k_\varepsilon})).$$

The difference $\|\gamma\|_\infty - \gamma(2^{k_\varepsilon})$ is easy to estimate. Knowing first that $k_\varepsilon \geq \frac{3}{2} k^*$, then the monotonic properties of the sequence $(m(k))_{k \geq 1}$ and two last inequalities of Lemma 2.5, lead to

$$\begin{aligned} \|\gamma\|_\infty - \gamma(2^{k_\varepsilon}) &= \sum_{l \geq k_\varepsilon} b_{m(l)} \leq \frac{1}{(m(k_\varepsilon))^\beta} \sum_{l \geq k_\varepsilon} b_{m(l)} m(l)^\beta \\ &\leq \left(\frac{4}{\log_2(2/\varepsilon)}\right)^\beta \sum_{l \geq k_\varepsilon} b_{m(l)} m(l)^\beta \leq \left(\frac{4}{\log_2(2/\varepsilon)}\right)^\beta \sum_{j \geq 1} j^\beta b_j. \end{aligned}$$

By taking now into account the previous evaluations of $\sum_{j \geq 1} j^\beta b_j$ (see (2.12)), we get:

$$\text{Card}(F) \leq \frac{C(\beta, f)}{\varepsilon^2 (\log(2/\varepsilon))^\beta}.$$

It remains to find an absolute constant C' such that

$$\forall n > 2^{k_\varepsilon}, \quad \exists n' \in F \quad \text{such that} \quad \|A_n(f) - A_{n'}(f)\| \leq C'\varepsilon.$$

The integer n being given, let us consider integers $k (\geq k_\varepsilon)$ and $p \geq 1$ such that

$$n \in [2^k, 2^{k+1}[\quad \text{and} \quad p\varepsilon^2 \leq \gamma(n) < (p+1)\varepsilon^2.$$

We say that a number n' is associated with n if

$$n' = \min\{l \geq 1, p\varepsilon^2 \leq \gamma(l)\} \quad \text{and} \quad n' \in [2^{k'}, 2^{k'+1}[.$$

It is clear that $n' \in F$ because $\gamma(n'-1) < p\varepsilon^2 \leq \gamma(n')$; and we have $k' \leq k$.

Lemma 2.6. Let $n > 2^{k_\epsilon}$, $n \in [2^k, 2^{k+1}[$, and let $n' \in F$ be an associated number $n' \in [2^{k'}, 2^{k'+1}[$, $k' \leq k$.

1) If $k' = k$ or $k' = k - 1$, then

$$n - n' \leq \frac{5 \cdot 2^k \varepsilon^2}{b_{m(k)}}.$$

2) If $k' = k - 1$, then

$$n - n' \leq 2^{k+1} \quad \text{and} \quad \sum_{m(k') < l \leq m(k)} b_l \leq 3\varepsilon^2.$$

P r o o f. Equality $k = k'$ provides

$$\varepsilon^2 \geq \gamma(n) - \gamma(n') = 2^{-k}(n - n') b_{m(k)}, \quad (2.27)$$

so that $n - n' \leq 2^k \varepsilon^2 / b_{m(k)}$. If $k' = k - 1$, we have similarly, $n - n' \leq 5 \cdot 2^k \varepsilon^2 / b_{m(k)}$, $n - n' \leq 2^{k+1}$. And also,

$$\begin{aligned} \varepsilon^2 &\geq \gamma(2^k) - \gamma(n') = (2 - 2^{-k+1} n') b_{m(k)-1} = 2^{-k+1} (2^k - n') b_{m(k)-1} \\ &\geq \sum_{k' < l < k} b_{m(l)} \geq \frac{1}{3} \sum_{m(k') < l \leq m(k)} b_l. \end{aligned}$$

Hence, $\sum_{m(k') < l \leq m(k)} b_l \leq 3\varepsilon^2$, which proves Lemma 2.6.

Now, return to the proof of Proposition 1.1. As before, we have

$$\int_{I_{m(k)}} |V_n(\theta)|^2 \mu_f(d\theta) \leq \frac{8}{\pi^2 n^2} b_{m(k)} 2^{2m(k)} \leq \frac{16}{\pi^2 n^2} 2^{2(k+1)} \varepsilon^2.$$

We use this bound in order to obtain

$$\int_{I_{m(k)}} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq C'_1 \varepsilon^2$$

when $k = k'$ or $k' = k - 1$. Indeed, for $k = k'$, we get a bound with

$$2 \int_{I_{m(k)}} (|V_n(\theta)|^2 + |V_{n'}(\theta)|^2) \mu_f(d\theta) \leq \frac{128}{\pi^2} \varepsilon^2,$$

and for $k' = k - 1$, constant 128 above is multiplied by 4.

Return to $k' = k$ for estimating the integration over $\hat{I}_{m(k)}$. We have

$$\begin{aligned} \int_{\hat{I}_{m(k)}} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) &\leq K' |n - n'|^2 \sum_{l \geq m(k)} 2^{-2l} a_l \\ &\leq 2K' |n - n'|^2 2^{-2m(k)} b_{m(k)}, \end{aligned}$$

so that by Lemma 2.6

$$(n - n')^2 2^{-2m(k)} b_{m(k)} \leq \varepsilon^4 2^{2k} \frac{1}{b_{m(k)} 2^{2m(k)}} \leq \varepsilon^2,$$

and thus

$$\int_{-1/2}^{1/2} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq C'_1 \varepsilon^2,$$

where C'_1 is an absolute constant.

If $k' = k - 1$, we get by using (2.27)

$$\int_{\widehat{I}_{m(k)}} |V_n(\theta) - V_{2^k}(\theta)|^2 \mu_f(d\theta) \leq K' \varepsilon^2.$$

Besides, we also have

$$\varepsilon^2 \geq \gamma(2^k) - \gamma(n') = (2 - 2^{-k+1}n') b_{m(k)-1} = 2^{-k+1}(2^k - n') b_{m(k)-1}.$$

Hence

$$\int_{\widehat{I}_{m(k)}} |V_{n'}(\theta) - V_{2^k}(\theta)|^2 \mu_f(d\theta) \leq K' \varepsilon^2,$$

and finally

$$\int_{\widehat{I}_{m(k)}} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq 4K' \varepsilon^2.$$

Thus, for $k' = k - 1$, we also get

$$\int_{-1/2}^{1/2} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq C'_2 \varepsilon^2,$$

where C'_2 is an absolute constant.

It remains to examine the case where $k' < k - 1$. We cut the domain of integration $] -\frac{1}{2}, \frac{1}{2} [\setminus \{0\}$ in three intervals $I_{m(k')}, I_{m(k)} \setminus I_{m(k')}, \widehat{I}_{m(k)}$.

Concerning $I_{m(k')}$, we have

$$\int_{I_{m(k')}} |V_n(\theta)|^2 \mu_f(d\theta) \leq \int_{I_{m(k)}} |V_n(\theta)|^2 \mu_f(d\theta).$$

As for $I_{m(k)} \setminus I_{m(k')}$, we bound $|V_n|$ and $V_{n'}$ by 1, which provides

$$\int_{I_{m(k)} \setminus I_{m(k')}} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq 4 \sum_{m(k') < l \leq m(k)} b_l.$$

But

$$\begin{aligned} \varepsilon^2 &\geq \gamma(n) - \gamma(n') = \sum_{k' < l < k} b_{m(l)} + (n2^{-k} - 1) b_{m(k)} + (2 - 2^{-k'}n') b_{m(k')} \\ &\geq \sum_{k' < l < k} b_{m(l)} \geq \frac{1}{3} \sum_{m(k') < l \leq m(k)} b_l, \end{aligned}$$

thus

$$\int_{I_{m(k)} \setminus I_{m(k')}} |V_n(\theta) - V_{n'}(\theta)|^2 \mu_f(d\theta) \leq 12 \varepsilon^2.$$

Finally

$$\int_{\widehat{I}_{m(k)}} \left| V_n(\theta) - V_{n'}(\theta) \right|^2 \mu_f(d\theta) \leq K' 2^{2k-2m(k)} b_{m(k)} \leq K' b_{m(k)}^2 \varepsilon^{-2} \leq 9K' \varepsilon^2,$$

which provides

$$\int_{-1/2}^{1/2} \left| V_n(\theta) - V_{n'}(\theta) \right|^2 \mu_f(d\theta) \leq C'_3 \varepsilon^2,$$

where C'_3 is an absolute constant. The proof of Proposition 1.1 is now complete.

R e m a r k 3. Theorem 1.1 of Talagrand, as well as Theorem 1.2 are naturally connected with Von Neumann's Theorem. By definition, the map $n \mapsto n'$ is nondecreasing. Let indeed N_ε be the smallest integer of $\mathcal{N}$ such that $N_\varepsilon \geq 2^{k_\varepsilon}$ and such that $g(N_\varepsilon) > \|g\|_\infty - \varepsilon^2$ and let also K_ε be such that $2^{K_\varepsilon+1} > N_\varepsilon \geq 2^{K_\varepsilon}$; then for any $n > 4N_\varepsilon$, we have $n' = N_\varepsilon$ (where $k > k' + 1 = K_\varepsilon + 1$), without assuming $n \in \mathcal{N}$. Thus $\|B_n(f) - B_{n'}(f)\| \leq C'(\alpha) \varepsilon$. This shows immediately that $\{B_n, n \in \mathcal{N}\}$ is a Cauchy sequence in H .

2.3. Proof of Theorem 1.3. Fix $\varepsilon \in [0, 1[$, and $f \in H$ with $\|f\| = 1$ and $\mu_f(\{0\}) = 0$. We will see later how to suppress this restriction. Remark first that if N is an integer, the continuity assumption on the flow $\{U_t, t \in \mathbf{R}\}$ allows us to write:

$$\begin{aligned} A_N(f) &= \frac{1}{N} \int_0^N U_t(f) dt = \frac{1}{N} \sum_{k=0}^{N-1} \int_k^{k+1} U_t(f) dt \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \int_0^1 U_t(U_1^k(f)) dt = \int_0^1 U_t \left(\frac{1}{N} \sum_{k=0}^{N-1} U_1^k(f) \right) dt \\ &= A_1 \left(\frac{1}{N} \sum_{k=0}^{N-1} U_1^k(f) \right), \end{aligned}$$

hence for any integer N and N'

$$\|A_N(f) - A_{N'}(f)\| \leq \|A_1\| \cdot \|A_N^{(1)}(f) - A_{N'}^{(1)}(f)\| \leq \|A_N^{(1)}(f) - A_{N'}^{(1)}(f)\|,$$

where we have set

$$A_N^{(1)}(f) = \frac{1}{N} \sum_{k=0}^{N-1} U_1^k(f).$$

Moreover, we have for any $1 \leq T' \leq T$

$$\begin{aligned} \|A_T(f) - A_{T'}(f)\| &\leq \left\| \left(\frac{1}{T} - \frac{1}{T'} \right) \int_0^{T'} U_t(f) dt \right\| + \left\| \frac{1}{T} \int_{T'}^T U_t(f) dt \right\| \\ &\leq 2 \frac{T - T'}{T} \|f\|. \end{aligned}$$

Introduce now the set $F(f, \varepsilon)$ built in the proof of Proposition 1.1 with $U = U_1$. It contains all integers less than 2^{k_ε} . For $T > 2^{k_\varepsilon}$, we have

$$\|A_T(f) - A_{[T]'}(f)\| \leq \|A_T(f) - A_{[T]}(f)\| + \|A_{[T]}(f) - A_{[T]'}(f)\|.$$

Here $[T]'$ denotes the integer in the set F which is associated to $[T]$ (integer part of T) by:

$$\|A_T^{(1)}(f) - A_{[T]'}^{(1)}(f)\| \leq C'_1 \varepsilon,$$

where C'_1 is an absolute constant. Then

$$\begin{aligned} \|A_T(f) - A_{[T]'}(f)\| &\leq 2 \frac{T - [T]}{T} + C'_1 \varepsilon \leq \frac{2}{2^{k_\varepsilon}} + C'_1 \varepsilon \\ &\leq \left(\frac{\varepsilon}{2}\right)^{3/2} + C'_1 \varepsilon \leq \left(\left(\frac{1}{2}\right)^{3/2} + C'_1\right) \varepsilon. \end{aligned}$$

It remains to treat the case where $1 \leq T \leq 2^{k_\varepsilon} \leq 2((2/\varepsilon))^{3/2}$. For this, add to F the points $t_k = (1+\varepsilon)^k$ for $0 \leq k \leq 2 \log 4/\varepsilon / (\varepsilon \log 2)$. The cardinality of the new set is thus bounded from above by $K'(\beta, f)/[\varepsilon^2(\log(2/\varepsilon))^\beta]$, where the constant $K'(\beta, f)$ depends on β and f only. Further, for $1 \leq T \leq 2(2/\varepsilon)^{3/2}$, choose k such that $t_k \leq T \leq t_{k+1}$, then

$$\|A_T(f) - A_{t_k}(f)\| \leq 2 \frac{T - t_k}{T} \leq 2 \frac{t_{k+1} - t_k}{t_k} = 2\varepsilon.$$

We now show (1.6) with $\beta > 2$. For this, we will apply Theorem 1.5 with the choice $E = [1, +\infty]$. Introduce the process $X = \{X_t, t \in E\}$ with basic probability space $(X, \mathcal{A}, \mu)$, defined by

$$X_t = \begin{cases} A_t(f) & \text{if } t \in [1, \infty[, \\ P(f) & \text{if } t = \infty, \end{cases}$$

where $P(f)$ is the orthogonal projection of f on the space H_U of fixed points of the flow $\{U_t, t \in \mathbf{R}\}$. According to Von Neumann's theorem for weakly continuous flows, we have:

$$\lim_{T \rightarrow \infty} \|A_T(f) - P(f)\| = 0.$$

Consider the pseudo-metric on E defined by $\forall s, t \in E, \|X_s - X_t\|_2 = d(s, t)$. Since

$$d(s, t) \leq \frac{|s - t|}{s \vee t} \|f\| \quad (1 \leq s, t < +\infty),$$

$$|d(s, t) - d(s', t')| \leq d(s, s') + d(t, t') \quad (1 \leq s, s', t, t' \leq +\infty),$$

$d(\cdot, \cdot)$ is thus continuous.

By Proposition 1.1, we deduce that $\forall \varepsilon \in]0, 1]$,

$$N(E \setminus \{\infty\}, d, \varepsilon) \leq \frac{K'}{\varepsilon^2(\log(1/\varepsilon))^\beta}.$$

But, $N(E, d, \varepsilon) \leq 1 + N(E \setminus \{\infty\}, d, \varepsilon)$, hence

$$N(E, d, \varepsilon) \leq \frac{K''}{\varepsilon^2 (\log(1/\varepsilon))^\beta},$$

where K'' is a constant depending on β and μ_f . The assumption $\beta > 2$ ensures that (1.3) holds true. Let α be such that $2 < \alpha < \beta$ and define $\phi: \mathbf{R}^+ \mapsto \mathbf{R}^+$ by

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \left(\log \frac{1}{t}\right)^{(\alpha-\beta)/2} & \text{if } 0 < t \leq \frac{1}{2}, \\ (\log 2)^{(\alpha-\beta)/2} & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Then ϕ satisfies conditions of Theorem 1.5. Observe now $D = \sup_{E \times E} d(s, t) \leq 2\|f\| = 2$, hence

$$\int_0^D \frac{N(E, d, \varepsilon)^{1/2}}{\phi(\varepsilon)} d\varepsilon \leq \frac{N(E, d, \frac{1}{2})^{1/2}}{(\log 2)^{(\alpha-\beta)/2}} + K'' \int_0^{1/2} \frac{d\varepsilon}{\varepsilon (\log(1/\varepsilon))^{\alpha/2}} < \infty.$$

From Theorem 1.5, we get

$$\mathbf{E} \left(\sup_{1 \leq s, t \leq \infty} \left(\|X_t - X_s\| \left| \log \|X_t - X_s\| \right| \right)^\gamma \right)^2 < \infty,$$

with $\gamma = (\beta - \alpha)/2$, which completes the proof of Theorem 1.3.

2.4. Proof of Theorem 1.5. By Bessel-Parseval's equality,

$$\begin{aligned} \forall N, M \in \mathbf{N}, \quad \sum_{n \in \mathbf{Z}} \left| (A_N - A_M)(a)(n) \right|^2 \\ = \frac{1}{2\pi} \int_0^{2\pi} |\varphi_a(\alpha)|^2 \left| V_N(\alpha) - V_M(\alpha) \right|^2 d\alpha. \end{aligned}$$

Hence $\forall J \geq 1$, $\forall N, M$ such that $N \vee M < J$,

$$\begin{aligned} \frac{1}{J} \sum_{0 \leq n < J - N \vee M} \left| (A_N - A_M)(a)(n) \right|^2 \\ \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{J} \left| \sum_{0 \leq j < J} e^{-ij\alpha} a(j) \right|^2 \left| V_N(\alpha) - V_M(\alpha) \right|^2 d\alpha. \end{aligned}$$

We can view the right integral as an integration with respect to the measure

$$\Lambda_{J,a}(d\alpha) = \frac{1}{J} \left| \sum_{0 \leq j < J} e^{-ij\alpha} a(j) \right|^2 d\alpha$$

Assume now that $a \in \mathcal{S}$. By means of Herglotz's theorem, there exists a unique nonnegative bounded measure Λ_a on $[-\pi, \pi[$, the *spectral measure* of the sequence a , such that

$$\forall m \in \mathbf{Z}, \quad \gamma_a(m) = \int_{-\pi}^{\pi} e^{im\alpha} \Lambda_a(d\alpha).$$

From Theorem 1 of [2], we know that the family of measures $\Lambda_{J,a}$ converges weakly to Λ_a . We thus deduce

$$\begin{aligned} \forall N, M \geq 1, \quad \limsup_{J \rightarrow \infty} \frac{1}{J} \sum_{0 \leq n < J} \left| (A_N - A_M)(a)(n) \right|^2 \\ \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| V_N(\alpha) - V_M(\alpha) \right|^2 \Lambda_a(d\alpha). \end{aligned}$$

The result is now easily deduced from Theorem 1.1.

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